183.(36) $$\int_0^1 \frac{\ln(1+x)}{x}dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)dx$$

$$= \int_0^1\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^4}{4} + \ldots\infty\right)dx \quad = \left(x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\infty\right)_0^1 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$= \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\infty\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\infty\right) = \frac{\pi^2}{6} - 2.\frac{1}{2^2}.\frac{\pi^2}{6} = \frac{\pi^2}{12} = \frac{3\pi^2}{36}$$

$\therefore \quad k = 36$

184.(6) Let $u = \frac{c}{x}$, so $du = \frac{-c}{x^2}dx$,

So $$\int_1^{\sqrt{c}} \frac{f(x)}{x}dx = \int_c^{\sqrt{c}} \frac{uf(u)}{c}\left(\frac{-x^2}{c}\right)du = \int_{\sqrt{c}}^{c} \frac{f(u)}{u}du$$

Therefore, $$\int_1^c \frac{f(x)}{x}dx = \int_1^{\sqrt{c}} \frac{f(x)}{x}dx + \int_{\sqrt{c}}^{c} \frac{f(u)}{u}du = 3 + 3 = 6$$

185.(0.25) $$x\int_0^x g(t)dt + \int_0^x (1-t)\,g(t)dt = x^4 + x^2$$

Differentiate w.r.t x $\quad x\,g(x) + \int_0^x g(t)\,dt + (1-x)\,g(x) = 4x^3 + 2x$

Again differentiate w.r.t x $\quad g'(x) + g(x) = 12x^2 + 2$

Now, $$\int_0^1 \frac{12}{g' + g + 10}dx = \int_0^1 \frac{dx}{x^2+1} = \tan^{-1}x]_0^1 = \frac{\pi}{4}$$

186.(0.5) $$\int_k^{k+1} x.\ln\left[(x-k)(k+1-x)\right]dx$$

$$= \frac{(2k+1)}{2}\int_k^{k+1}\ln(x-k)(k+1-x)dx \qquad [\text{using } (a+b-x) \text{ property}]$$

$$= \frac{(2k+1)}{2}\int_0^1 \ln t(1-t)\,dt = -(2k+1)$$

Limit $$= \lim_{n\to\infty}\frac{-1}{n^4}\sum_{k=1}^{n}(2k+1)k^2 = \lim_{n\to\infty} -\left[\frac{2\frac{n^2(n+1)^2}{4}}{n^4} + \frac{n(n+1)(2n+1)}{6n^4}\right] = \frac{-1}{2}$$

187.(2) $$\cot^{-1}\left(1-x+x^2\right) = \tan^{-1}\frac{1}{1-x+x^2} = \tan^{-1}\frac{x-(x-1)}{1+x(x-1)^2} = \tan^{-1}x - \tan^{-1}(x-1)$$

Also, $$\int_0^1 \tan^{-1}(x-1)dx = -\int_0^1 \tan^{-1}x\,dx$$

Therefore, $\int_0^1 \cot^{-1}\left(1-x+x^2\right)dx = \int_0^1\left(\tan^{-1}x - \tan^{-1}(x-1)\right)dx = 2\int_0^1 \tan^{-1}x\ dx$

Hence, $\dfrac{\int_0^1 \cot^{-1}\left(1-x+x^2\right)dx}{\int_0^1 \tan^{-1}x\ dx} = 2$

188.(0.06) $I = \int_0^1\left(f(x).x^2 - x\left(f(x)\right)^2\right)dx$

$I = \int_0^1 -x\left(-xf(x)+\left(f(x)\right)^2\right)dx$

$I = -\int_0^1 -x\left[f(x)^2 - 2.\frac{x}{2}f(x)+\frac{x^2}{4}-\frac{x^2}{4}\right]dx$

$I = \int_0^1\left(\frac{x^3}{4} - x\left(f(x)-\frac{x}{2}\right)^2\right)dx$, which is maximum when $f(x)=\frac{x}{2}$

i.e., $I = \int_0^1 \frac{x^3}{4}dx = \frac{1}{16}$

189.(16) Given $\int_0^x t^2 \sin(x-t)\,dt = x^2$

$\Rightarrow \quad \int_0^x (x-t)^2 \sin t\,dt = x^2$

$\Rightarrow \quad x^2\int_0^x \sin t\,dt - 2x\int_0^x t\sin t + \int_0^x \sin t = x^2$

$\Rightarrow \quad x^2(1-\cos x) - 2x(-x\cos x + \sin x) + \left(-x^2\cos x + 2x\sin x + \cos x - 1\right) = x^2$

$\Rightarrow \quad \cos x = 1$

So, $x = 2n\pi,\ n \in I$

Hence, number of values of $0, 2\pi, 4\pi, \ldots, 30\pi$ equals 16.

190.(2) Let $I_1 = \int_0^{\pi/2}(\cos x)^{\sqrt{2}+1}dx$ ...(i)

And $I_2 = \int_0^{\pi/2}(\cos x)^{\sqrt{2}-1}dx$ ...(ii)

Now, $I_1 = \int_0^{\pi/2}\underset{\mathrm{I}}{(\cos x)^{\sqrt{2}}}.\underbrace{(\cos x)}_{\mathrm{II}}dx$

$= \left((\cos x)^{\sqrt{2}}.\sin x\right)_0^{\pi/2} + \int_0^{\pi/2}\sqrt{2}.(\cos x)^{\sqrt{2}-1}.\sin^2 x\ dx$

$= \sqrt{2}\int_0^{\pi/2}(\cos x)^{\sqrt{2}-1}.\left(1-\cos^2 x\right)dx$

$\therefore \quad I_1 = \sqrt{2}I_2 - \sqrt{2}I_1$

$\Rightarrow \quad \dfrac{\left(\sqrt{2}+1\right)I_1}{\sqrt{2}I_2}=1$

$\therefore \quad \dfrac{I_1}{I_2}=\left(\dfrac{\sqrt{2}}{\sqrt{2}+1}\right)\dfrac{\left(\sqrt{2}-1\right)}{\left(\sqrt{2}-1\right)} \quad =(2-\sqrt{2})\equiv(n-\sqrt{n})$

$\Rightarrow \quad n = 2$

Note: $\dfrac{\int_0^{\pi/2}(\cos x)^{\sqrt{2}+1}dx}{\int_0^{\pi/2}(\cos x)^{\sqrt{2}-1}dx}=\dfrac{\int_0^{\pi/2}(\sin x)^{\sqrt{2}+1}dx}{\int_0^{\pi/2}(\sin x)^{\sqrt{2}-1}dx}$ [using $(a-x)$ property]

191.(58) $I=\int_0^2 \dfrac{ax+b}{\left(x^2+5x+6\right)^2}dx=\dfrac{7}{30}(\text{Given})$

Now, for some value of k

$$\frac{ax+b}{\left(x^2+5x+6\right)^2}=\frac{d}{dx}\left(\frac{k}{x^2+5x+6}\right)$$

$$\frac{ax+b}{\left(x^2+5x+6\right)^2}=\frac{-(2x+5)k}{\left(x^2+5x+6\right)^2}$$

Hence, $a=-2k; b=-5k$

Now, $I=\left.\dfrac{k}{\left(x^2+5x+6\right)}\right]_0^2 \Rightarrow k\left(\dfrac{1}{20}-\dfrac{1}{6}\right)=\dfrac{7}{30} \Rightarrow \dfrac{-14k}{120}=\dfrac{7}{30} \Rightarrow k=-2$

Hence, $a = 4$; $b = 10$ $\quad\therefore\quad a^2+b^2=16+100=116$

192.(10) Let $I=\int_1^{100}\dfrac{f(x)}{x}dx=\int_1^{10}\dfrac{f(x)}{x}dx+\int_{10}^{100}\dfrac{f(x)}{x}dx$

$=\int_1^{10}\dfrac{f(x)}{x}dx+\int_{10}^{100}\dfrac{f\left(\dfrac{100}{x}\right)}{x}dx$

Put, $\dfrac{100}{x}=t\Rightarrow\dfrac{-100}{x^2}dx=dt\Rightarrow -t\dfrac{dx}{x}dt\Rightarrow\dfrac{dx}{x}=\dfrac{-dt}{t}$

$I=\int_1^{10}\dfrac{f(x)}{x}dx+\int_{10}^{1}\dfrac{-f(t)}{t}dt=2\int_1^{10}\dfrac{f(x)}{x}dx=2\times5=10$

193.(31) $y^2=4x;16y^2=5(x-1)^3$

Solving, $64x=5(x-1)^3\Rightarrow(x-5)\left(5x^2+10x+1\right)$

$\boldsymbol{x}=5$ and $y=2\sqrt{5}$ $\quad$ Hence, $A=2\left[\int_0^5 2\sqrt{x}\,dx-\int_1^5\dfrac{5}{16}(x-1)^{\frac{3}{2}}dx\right]$

Which on evaluating gives.

$$A = \frac{104\sqrt{5}}{15}$$

$\therefore \quad L = 104;\ M = 5;\ N = 15$

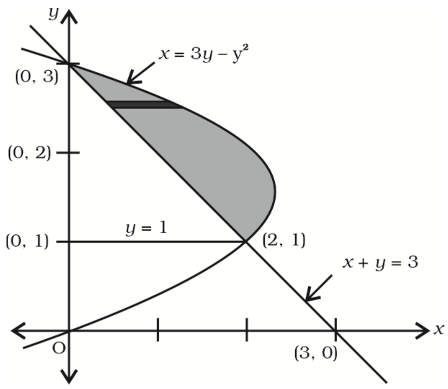
$\therefore \quad L + M + N = 124$

194.(1) Area between $x = 3y - y^2$ and $x + y = 3$

$$A = \int_1^3 \left(3y - y^2\right) - \left(3 - y\right) dy$$

$$= 4.\frac{y^2}{2} - \frac{y^3}{3} - 3y\Bigg]_1^3 = \left(2.9 - 9 - 9\right) - \left(2 - \frac{1}{3} - 3\right)$$

$$A = \frac{4}{3}$$



Now area bounded by f(x) above the x-axis = $\int_0^1 \left(x - x^2\right) dx = \frac{1}{6}$, hence $a < 0$ (think)

Given, $\int_0^{1-a} \left(\left(x - x^2\right) - \left(a\,x\right)\right) dx = \frac{4}{3}$

Solving $\frac{1}{6}\left(1 - a\right)^3 = \frac{4}{3}$

$\left(1 - a\right)^3 = 8$

$1 - a = 2 \Rightarrow a = -1$

$\Rightarrow \quad \left|[a]\right| = 1$

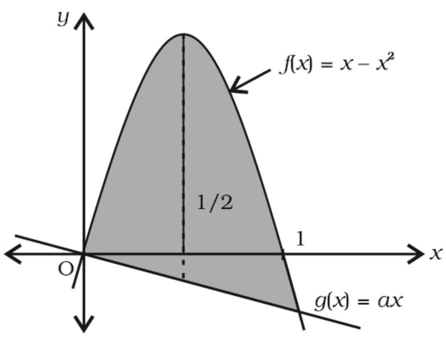
Note : $x - x^2 = ax$

$x^2 + \left(a - 1\right)x = 0$

$x_1 + x_2 = 1 - a$

But $x_1 = 0$

$\Rightarrow \quad x_2 = 1 - a$



195.(2) $A_n = \int_0^{1/2} x^n\,dx \Rightarrow A_n = \frac{1}{2^{n+1}\left(n+1\right)} \Rightarrow 2^n A_n = \frac{1}{2\left(n+1\right)}$

$$\Rightarrow \quad \frac{2^n A_n}{n} = \frac{1}{2\left(n+1\right)n}$$

Hence, $\sum_{n=1}^{n} \frac{2^n A_n}{n} = \frac{1}{2}\sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{n+1}\right)$

Given, $\frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{3} \Rightarrow 1 - \frac{2}{3} = \frac{1}{n+1} \Rightarrow \frac{1}{3} = \frac{1}{n+1} \Rightarrow n = 2$

196.(16) We have $f(x) = e + (1-x)\ln\left(\frac{x}{e}\right) + \int_1^x f(t)\,dt$

Differentiate both sides with respect to x, we get

$f'(x) = (1-x)\frac{1}{x} + \ln\left(\frac{x}{e}\right)(-1) + f(x) \Rightarrow f'(x) - f(x) = \left(\frac{1}{x} - 1 - \ln\left(\frac{x}{e}\right)\right)$

$\therefore$ Multiplying both sides with e^{-x}, we get

$$e^{-x} f'(x) - e^{-x} f(x) = e^{-x}\left(\frac{1}{x} - 1 - \ln\left(\frac{x}{e}\right)\right)$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x}.f(x)\right) = e^{-x}\left(\frac{1}{x} - 1 - \ln\left(\frac{x}{e}\right)\right)$$

$\therefore$ On integrating both sides with respect to x, we get $e^{-x} f(x) = e^{-x} - \int e^{-x}\left(\ln\left(\frac{x}{e}\right) - \frac{1}{x}\right)dx$

$$= e^{-x} + e^{-x}.\ln\left(\frac{x}{e}\right) + \lambda \Rightarrow f(x) = 1 + \ln\left(\frac{x}{e}\right) + \lambda e^{x}$$

Now putting $x = 1$ in equation (i), we get f(1) = e.

$\therefore$ using (iii) and (ii) we get $\lambda = 1$

Thus, $f(x) = 1 + e^{x} + \ln\left(\frac{x}{e}\right)$ *or* $f(x) = e^{x} + \ln x$

Hence, $g(x) = x \ln x$: now $A = \int_0^1 x \ln x \, dx = \frac{1}{4}$

197.(7.5) $\int_0^1 \left(4x^6 + \left(4x^3 - f(x)\right) f(x) - 4x^6\right) dx = \frac{4}{7}$

$$\int_0^1 \left(f(x) - 2x^3\right)^2 dx = 0$$

$\therefore$ $f(x) = 2x^3$

Area = $\int_1^2 2x^3 dx = 8 = \frac{15}{2}$