Level – 2	Daily Tutorial Sheet - 14

171.(B) $\int_a^b \frac{x^{n-1}\left\{nx^2-2x^2+n(a+b)x-(a+b)x+nab\right\}}{(x+a)^2(x+b)^2}dx$

$$= \int_a^b \frac{x^{n-1}\left\{n(x+a)(x+b)-x(2x+a+b)\right\}}{(x+a)^2(x+b)^2}dx = \int_a^b \frac{nx^{n-1}}{(x+a)(x+b)}dx - \int_a^b \frac{x^n(x+a+x+b)}{(x+a)^2(x+b)^2}dx$$

$$= \int_a^b \frac{d}{dx}\left(\frac{x^n}{(x+a)(x+b)}\right)dx = \left[\frac{x^n}{(x+a)(x+b)}\right]_a^b = \frac{b^{n-1}-a^{n-1}}{2(a+b)}$$

172.(B) $\int_{-1}^{\pi}\left(8t^2+\frac{28}{3}t+4\right)dt = \frac{\frac{3}{2}x+1}{\frac{1}{2}\log_{(x+1)}(x+1)} \quad \Rightarrow \quad \frac{8x^3}{3}+\frac{28}{3}x^2+4x+2 = 2\left(\frac{3}{2}x+1\right)$

Only $x = 0$ is satisfied – only one solution.

173.(D) $I = \int_0^1 \prod_{r=1}^{n}(x+r)\sum_{k=1}^{n}\frac{1}{x+k}dx$ [Let $\ell n\prod_{r=1}^{n}(x+r) = t \Rightarrow \sum_{k=1}^{n}\frac{1}{x+k}dx = dt$]

$$I = \int_{\ell n(n!)}^{\ell n(n+1)!} e^t dt = \left[e^t\right]_{\ell n(n!)}^{\ell n(n+1)!} = (n+1)!-n! = n\ .\ n!$$

174.(D) Applying Newton-Leibnitz formula find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ then solve.

175.(D) $A_{n+1}-A_n = \int_0^{\pi/2}\frac{\sin(2n+1)x-\sin(2n-1)x\,dx}{\sin x} = \int_0^{\pi/2}\frac{2\cos 2nx\sin x}{\sin x}dx = \left(\frac{2\sin 2nx}{2n}\right)_0^{\pi/2} = 0$ [$\therefore\ A_{n+1} = A_n$]

Now $B_{n+1}-B_n = \int_0^{\pi/2}\left(\frac{\sin^2(n+1)x-\sin^2 nx}{(\sin^2 x)}\right)dx = \int_0^{\pi/2}\frac{\sin(2n+1)}{\sin x}dx = A_{n+1}$

176.(B) $u = \int_0^1 \frac{\tan(x+1)}{(x^2+1)}dx$ [Put $x = \tan\theta$]

$$dx = \sec^2\theta\, d\theta = \int_0^{\pi/4}\frac{\ell n(1+\tan\theta)}{\sec^2\theta}(\sec^2\theta)d\theta = \int_0^{\pi/4}\ell n\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta = \int_0^{\pi/4}\ell n\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$u = \int_0^{\pi/4}\ell n\, 2d\theta - u \quad \Rightarrow \quad u = \frac{\pi}{8}\ell n 2 \quad \Rightarrow \quad u = \frac{-v}{4} \quad \Rightarrow \quad 4u+v = 0$$

177.(D) Let $P = \int_0^{\pi} x^2\ .\ \frac{\cos x}{(1+\sin x)^2}dx$. Integrate by parts.

$\therefore \quad P = \left[x^2\left\{-\dfrac{1}{1+\sin x}\right\}\right]_0^{\pi} + 2\int_0^{\pi} x\,\dfrac{1}{1+\sin x}\,dx \;= -\pi^2 + 2I \qquad \ldots \text{(i)}$

Now $\quad I = \int_0^{\pi} x\,\dfrac{1}{1+\sin x}dx \qquad = \int_0^{\pi}(\pi - x)\,.\,\dfrac{1}{1+\sin x}\,dx$

$\therefore \quad 2I = \pi\int_0^{\pi}\dfrac{1}{1+\sin x}\,dx = 2\pi\int_0^{\pi/2}\dfrac{1}{1+\sin x}\,dx = 2\pi I \quad \therefore \qquad P = -\pi^2 + 2\pi I$ by (i)

178.(B) Let $A = \lim\limits_{n\to\infty}\left[\left(1+\dfrac{1}{n^2}\right)^{2/n^2}\left(1+\dfrac{2^2}{n^2}\right)^{4/n^2}\ldots\ldots\left(1+\dfrac{n^2}{n^2}\right)^{2^n/n^2}\right]$

$\log A = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{2r}{n^2}\log\left(1+\dfrac{r^2}{n^2}\right) = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{2r}{n^2}\log\left(1+\dfrac{r^2}{n^2}\right).\dfrac{1}{n} = \int_0^1 2x\log\left(1+x^2\right)dx$ (apply by parts)

$\log A = \left[x^2\log\left(1+x^2\right)\right]_0^1 - \int\dfrac{x^2}{1+x^2}\,.\,2x\,dx \;= \log 2 - \int_0^1\left(1-\dfrac{1}{1+x^2}\right)2x\,dx$

$\log A = \log 2 - \left[x^2 - \log\left(1+x^2\right)\right]_0^1 = \log 2 - 1 + \log 2$

$\log A = 2\log 2 - \log 1 = \log\dfrac{4}{e} \qquad \Rightarrow \quad A = \dfrac{4}{e}$

179. Let $I = \int_0^{2\pi}\left\{\cos x\cos 2x\cos 2^2x\ldots.\cos\left(2^{n-1}\right)x\right\}\cos\left(2^n - 1\right)x\,dx$

$I = \int_0^{2\pi}\dfrac{\sin 2^n x}{2^n\sin x}\cos\left(2^n-1\right)x\,dx \;= \dfrac{1}{2^{n+1}}\int_0^{2\pi}\dfrac{(e^{2^n xi} - e^{-2^n xi})(e^{(2^n-1)xi} + e^{-(2^n-1)xi})}{(e^{ix} - e^{-ix})}dx$

[Using $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$]

$= \dfrac{1}{2^{n+1}}\int_0^{2\pi}\dfrac{\left\{e^{\left(2^{n+1}-1\right)xi} + e^{xi} - e^{-ix} - e^{-\left(2^{n+1}-1\right)xi}\right\}dx}{e^{ix}-e^{-ix}} \;= \dfrac{1}{2^{n+1}}\int_0^{2\pi}\left\{1 + \dfrac{(e^{(2^{n+1}-1)xi} - e^{-(2^{n+1}-1)xi}}{e^{ix}-e^{-ix}}\right\}dx$

$= \dfrac{1}{2^{n+1}}\int_0^{2\pi}\{1 + e^{(2^{n+1}-2)xi} + e^{(2^{n+1}-3)xi} + \ldots + e^{ix} + 1 + e^{-ix} + \ldots + e^{-(2^{n+1}-3)xi} + e^{-(2^{n+1}-2)^{xi}}\}dx \quad \ldots \text{(i)}$

[Using $\int_0^{2\pi}e^{imx}dx = 0$, whenever $m \neq 0$] $\qquad \therefore \qquad I = \dfrac{(2\pi).2}{2^{n+1}} = \dfrac{\pi}{2^{n-1}}$

180.(C) Let $A = \left\{\lim\limits_{n\to\infty}\sin\dfrac{\pi}{2n}.\sin\dfrac{2\pi}{2n}.\sin\dfrac{3\pi}{2n}\ldots\sin\dfrac{2(n-1)\pi}{2n}\right\}^{1/n}$

$\log A = \lim\limits_{n\to\infty}\dfrac{1}{n}\log\left(\sin\dfrac{\pi}{2n}.\sin\dfrac{2\pi}{2n}\ldots.\sin\dfrac{2(n-1)\pi}{2n}\right) = \lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{r=1}^{2(n-1)}\log\sin\dfrac{r\pi}{2n}$

$\int_0^2\log\sin\left(\dfrac{\pi x}{2}\right)dx \qquad \left[\text{using, }\lim\limits_{n\to\infty}\sum\dfrac{1}{n}f\left(\dfrac{r}{n}\right) = \int_a^b f(x)\,dx\right]$

$= \int_0^{\pi} \log(\sin t) . \frac{2}{\pi} dt$ Putting, $\frac{\pi x}{2} = t$

$= \frac{2.2}{\pi} \int_0^{\pi/2} \log(\sin t) dt$ [using $\int_0^{2a} f(x) dx = 2\int_0^{a} f(x) dx$, if $f(2a - x) = f(x)$]

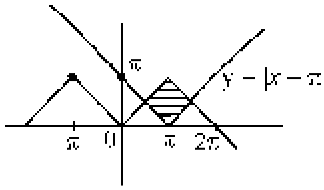
$= \frac{4}{3}\left\{-\frac{\pi}{2}\log 2\right\}\left[\text{using} \int_0^{\pi/2} \log(\sin x) dx = -\frac{\pi}{2}\log 2\right] = -2\log 2$

$\therefore$ $\log A = \log(1/4) \Rightarrow A = 1/4$. Hence, (C) is the correct answer.

181.(C) $y = cos^{-1}(cosx)$ and $y = | x - \pi |$

Required Region is a square with diagonal = π units

$\Rightarrow$ Area = $\frac{1}{2}\pi^2 = \frac{\pi^2}{2}$



182.(D) $[\,|x|\,] + [\,|y|\,] = 1$

No. of squares = 8

Area = 8 Sq. units.