Level – 1	Daily Tutorial Sheet - 5

61.(A) $A = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$\therefore \quad \log A = \lim\limits_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)\right] = \lim\limits_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$

$\therefore \quad \log A = \int_0^1 \log\left(1+x^2\right)dx$ (Apply by parts)

$=\left[x\,.\,\log\left(1+x^2\right)\right]_0^1 - \int_0^1 x\,.\,\frac{1}{\left(1+x^2\right)}\,.2x\,dx$

$=\left[x\,.\,\log\left(1+x^2\right)\right]_0^1 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx = \log 2 - 2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx = \log 2 - 2\left[x-\tan^{-1}x\right]_0^1$

or $\quad \log A = \log 2 - 2 + \frac{2\pi}{4} \quad \Rightarrow \quad \log A - \log 2 = \frac{\pi-4}{2}$

or $\quad \log\frac{A}{2} = \frac{\pi-4}{2} \quad \Rightarrow \quad \frac{A}{2} = e^{(\pi-4)/2}$ or $\quad A = 2e^{(\pi-4)/2}$

62.(B)

63.(C) $\lim\limits_{n\to\infty}\frac{1}{r+\sqrt{rn}} = \Sigma\lim\limits_{n\to\infty}\frac{1}{n\left[\frac{r}{n}+\sqrt{\frac{r}{n}}\right]} = \int_0^1\frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)} = 2\left[\log\left(1+\sqrt{x}\right)\right]_0^1 = 2\log 2$

64.(D) $I = \int_0^1\sqrt{\left(\frac{1+x}{1-x}\right)} = \int_0^1\frac{1+x}{\sqrt{\left(1-x^2\right)}}\,dx = \left[\sin^{-1}x - \sqrt{\left(1-x^2\right)}\right]_0^1 = \frac{\pi}{2}+1$

65.(A)

66.(B) $I = \lim\limits_{n\to\infty}\Sigma\frac{r}{n^2}\sec^2\left(\frac{r}{n}\right)^2 = \lim\limits_{n\to\infty}\frac{1}{n}\Sigma\frac{r}{n}\sec^2\left(\frac{r}{n}\right)^2 = \int_0^1 x\sec^2 x^2 dx = \frac{1}{2}\int_0^1\sec^2 t\,dt = \frac{1}{2}\left[\tan 1\right]_0^1 = \frac{1}{2}\tan 1$

67.(B) $I = \lim\limits_{n\to\infty}\Sigma\frac{1}{n}\tan\frac{\pi}{4}\frac{r}{n} = \int_0^1\tan\frac{\pi}{4}x\,dx = \frac{4}{\pi}\int_0^{\pi/4}\tan t\,dt = \frac{4}{\pi}\left[\log\sec t\right]_0^{\pi/4} = \frac{4}{\pi}\log\sqrt{2} = \frac{2}{\pi}\log 2$

68.(C) $\frac{d}{dx}(f(x)) = f'(x) = g(x)$; $I = \int_a^b\left(f(x)\,.\,g(x)\right)dx = \int_a^b f(x)\,.\,f'(x)dx = \left[\frac{f^2(x)}{2}\right]_a^b = \frac{1}{2}[f^2(b)-f^2(a)]$

69.(A) $\int_1^x\frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6} \Rightarrow [sec^{-1}(t)]_1^x = \frac{\pi}{6} \Rightarrow sec^{-1}(x) = sec^{-1}(1)+\frac{\pi}{6} \Rightarrow sec^{-1}(x) = \frac{\pi}{6} \Rightarrow x = \frac{2}{\sqrt{3}}$

70.(B) $g(x) = \int_2^x\frac{dt}{1+t^4} \Rightarrow g(2) = \int_2^2\frac{dt}{1+t^4} = 0$; $g'(x) = \frac{1}{1+x^4} \Rightarrow g'(2) = \frac{1}{1+2^4} = \frac{1}{17}$

and $f(x) = e^{g(x)} \Rightarrow f'(x) = e^{g(x)}\,.\,g'(x) \Rightarrow f(2) = e^0\,.\,\frac{1}{17} = \frac{1}{17}$

71.(C) $\frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+9y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{9.2.y}{2\sqrt{1+9y^2}} \cdot \frac{dy}{dx}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{9y}{\sqrt{1+9y^2}} \cdot \frac{\sqrt{1+9y^2}}{1} = 9y \Rightarrow a = 9$

72.(A) Differentiate w.r.t. $x \Rightarrow f(x) - 0 = 1 + 0 - x f(x) \times 1$; Put $x = 1 \Rightarrow f(1) = 1 - f(1) \Rightarrow f(1) = \frac{1}{2}$

73.(D) $f(x) = \cos x - \int_0^t (x-t) f(t)\, dt \Rightarrow f'(x) = -\sin x - \int_0^t \frac{d}{dx}[(x-t) f(t)] dt \Rightarrow f'(x) = -\sin x - \int_0^t [f(t) - 0] dt$

$\Rightarrow \quad f''(x) = -\cos x - 0 \quad \Rightarrow \quad f(x) + f''(x) = \cos x - \int_0^t (x-t) f(t) dt + (-\cos x)$

74.(B) If $f(x)$ is an odd function then $F(x)$ is an even function,

but if $f(x)$ is an even function then $F(x)$ is an odd function only if $F(0) = 0$ $\left(F(x) = \int f(x) dx\right)$

75.(A) $I = \int_0^{2a} f(x) dx = \int_0^a f(x) + f(2a - x)\, dx = \int_0^a 2 f(x) dx$.