Level – 1	Daily Tutorial Sheet - 4

46.(A) $I_1 = \int_0^{\pi/2} \ell n(\sin x)dx = \frac{-\pi}{2}\ell n 2$

$I_2 = \int_{-\pi/4}^{\pi/4} \ell n\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]dx = \int_{-\pi/4}^{\pi/4} \ell n\sqrt{2}dx + \int_0^{\pi/2} \ell n(\sin x)dx = \frac{\pi}{2}\ell n\sqrt{2} - \frac{\pi}{2}\ell n 2 = \frac{-\pi}{4}\ell n 2 \Rightarrow I_2 = \frac{I_1}{2}$

47.(B) $\int_0^1 \ell n\left(\sin\frac{\pi x}{2}\right)dx \Rightarrow$ Let $\frac{\pi x}{2} = X \Rightarrow dx = \frac{2}{\pi}dX \quad \Rightarrow \quad I = \frac{2}{\pi}\int_0^{\pi/2} \ell n \sin X dX = \frac{2}{\pi}\left(-\frac{\pi}{2}\ell n 2\right) = -\ell n 2$

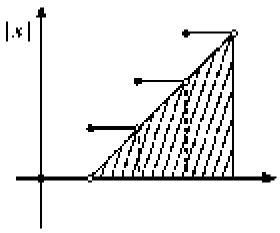
48.(C) $I = \int_{1/2}^{2} \frac{1}{x}.\operatorname{cosec}^{101}\left(x-\frac{1}{x}\right)dx = \int_2^{1/2} t\operatorname{cosec}^{101}\left(\frac{1}{t}-t\right)\left(-\frac{dt}{t^2}\right) \qquad \left[x=\frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt\right]$

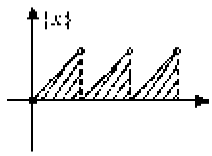
$\Rightarrow \quad I = \int_2^{1/2} \frac{1}{t}.\operatorname{cosse}^{101}\left(t-\frac{1}{t}\right)dt = -\int_{1/2}^{2} \frac{1}{x}\operatorname{cosec}^{101}\left(x-\frac{1}{x}\right)dx = -I \quad \Rightarrow \quad I = 0$

49.(C) $I = \int_0^{16\pi/3} |\sin x|\, dx = 5\int_0^{\pi}|\sin x|dx + \int_0^{\pi/3}|\sin x|dx = 5(2) + \left(1-\frac{1}{2}\right) = \frac{21}{2}$

50.(B) $I = \int_0^{1000} e^{\{x\}}dx = 1000\int_0^1 e^{\{x\}}dx = 1000\,(e-1)$

51.(D) $I = \frac{\int_0^n [x]dx}{\int_0^n \{x\}dx} = \frac{0+1+2+......+(n-1)}{\frac{1}{2}+\frac{1}{2}+.........n\ time} = n-1$





52. **(a)** $\int_a^b e^x dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\sum_{r=1}^{n} e^{a+rh} = \lim_{\substack{n\to\infty \\ h\to 0}} h\left[e^a\,(e^h + e^{2h} + e^{3h} + ...+e^{nh})\right]$

$= \lim_{\substack{n\to\infty \\ h\to 0}} e^a h\, e^h\,\frac{(1-e^{nh})}{1-e^h} = \lim_{\substack{n\to\infty \\ h\to 0}} \frac{h}{e^h-1}\bullet \lim_{\substack{n\to\infty \\ h\to 0}} e^h\,.\,e^a \lim_{\substack{n\to\infty \\ h\to 0}} (e^{nh}-1)$

$= 1.1\, e^a\,(e^{b-a}-1) \qquad (\because nh = b-a)$

$= e^b - e^a.$

(b) Let $b - a = nh$

$$\therefore \quad I = \lim_{\substack{h\to 0\\ n\to\infty}} h\left[\sin a + \sin(a+h) + \ldots.\sin(a+(n-1)h)\right]$$

$$\lim_{\substack{h\to 0\\ n\to\infty}} h\,\frac{\sin n\frac{h}{2}}{\sin\frac{h}{2}}\sin\left(\frac{2a+nh-h}{2}\right) \;;\; \lim_{\substack{h\to 0\\ n\to\infty}} 2\frac{h}{2}\frac{\sin n\frac{h}{2}}{\sin\frac{h}{2}}\sin\left(\frac{a+b}{2}\right)$$

$$\lim_{\substack{h\to 0\\ n\to\infty}} 2\frac{\frac{h}{2}}{\sin\frac{h}{2}}\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{a+b}{2}\right) \;;\; = 2\sin\left(\frac{b-a}{2}\right)\sin\left(\frac{a+b}{2}\right) = \cos a - \cos b$$

53.(B) $\underset{n\to\infty}{lt}\dfrac{1}{n}\left[\dfrac{1}{\sqrt{1+\frac{0}{n}}}+\sqrt{\dfrac{1}{1+\frac{1}{n}}}+\ldots+\sqrt{\dfrac{1}{1+\frac{3n-3}{n}}}\right] = \underset{n\to\infty}{lt}\dfrac{1}{n}\sum_{r=0}^{3n-3}\sqrt{\dfrac{1}{1+\frac{r}{n}}} = \int_0^3\dfrac{1}{\sqrt{1+x}}\,dx = 2\sqrt{1+x}\Big|_0^3 = 2(2-1) = 2$

54.(A) $\displaystyle\int_1^2 \log(1+x)\,dx = (x+1)\log(1+x) - x\Big|_0 = \log\frac{27}{4} - 1 = \log\left(\frac{27}{4e}\right).$

55.(B) Required limit $= \displaystyle\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{(r/n)^2}{1+(r/n)^3} = \int_0^1\frac{x^2}{1+x^3}\,dx = \frac{1}{3}\log(2)$

56.(A) Let $p = \displaystyle\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\left(3+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{2}\left(3\sqrt{2}+4\sqrt{n}\right)^2}+\ldots+\frac{\sqrt{n}}{\sqrt{n}\left(3\sqrt{n}+4\sqrt{n}\right)^2}\right]$

Analyzing the expression with the view of increasing integral value we get the expression in terms of r as

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\sqrt{\frac{r}{n}}\left(3\sqrt{\frac{r}{n}}+4\right)^2} = \int_0^1\frac{dx}{\sqrt{x}\left(3\sqrt{x}+4\right)^2}$$

Put $3\sqrt{x}+4 = t$, $\quad\therefore\quad \dfrac{3}{2\sqrt{x}}dx = dt \quad$ Hence $\quad p = \dfrac{2}{3}\displaystyle\int_4^7\frac{dt}{t^2} = \frac{2}{3}\left[-\frac{1}{t}\right]_4^7 = \frac{2}{3}\left(-\frac{1}{7}+\frac{1}{4}\right) = \frac{1}{14}$

57.(A) Let $P = \displaystyle\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+\ldots+\frac{1}{n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{1(2n)-1^2}}+\frac{1}{\sqrt{2(2n)-2^2}}+\frac{1}{\sqrt{3(2n)-3^2}}+\ldots+\frac{1}{\sqrt{n(2n)-n^2}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{r(2n)-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\,.\sqrt{2\frac{r}{n}-\left(\frac{r}{n}\right)^2}} = \int_0^1\frac{dx}{\sqrt{\left(2x-x^2\right)}}\quad [\text{Put } x = t^2 \Rightarrow dx = 2t\,dt\,]$$

$$P = \int_0^1\frac{2t\,dt}{t\sqrt{2-t^2}} = \left[2\sin^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\left(\frac{\pi}{4}\right)\quad \text{Hence}\quad P = \frac{\pi}{2}$$

58.(B) Now $\lim\limits_{n\to\infty} \frac{n}{n^2}\left[\frac{1}{1+\left(\frac{0}{n}\right)^2}+\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\ldots+\frac{1}{1+\left(\frac{n-1}{n}\right)^2}\right]$

$= \lim\limits_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+(r/n)^2} \quad \Rightarrow \quad \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1-\tan^{-1}0 = \frac{\pi}{4}-0=\frac{\pi}{4}$

59.(B) Here $\frac{1}{2n}=\frac{n^2}{n^3+n^3}$ $\quad\therefore\quad$ The given expression is $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}=A$

$\therefore \quad A = \lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3\left[1+\left(\frac{r}{n}\right)^3\right]} = \lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{(r/n)^2}{1+(r/n)^3}$

$A = \int_0^1 \frac{x^2dx}{1+x^3} = \frac{1}{3}\int_0^1\frac{3x^2dx}{1+x^3} = \frac{1}{3}\left[\log\left(1+x^3\right)\right]_0^1 = \frac{1}{3}\log 2$

60.(D) Here the last term $\frac{1}{n}=\frac{n+n}{n^2+n^2}$

The given expression $= \lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{n+r}{n^2+r^2}=A \quad \therefore \quad A = \lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{n(1+r/n)}{n^2\left(1+r^2/n^2\right)} = \lim\limits_{n\to\infty}\frac{1}{n}\left(\frac{1+r/n}{1+r^2/n^2}\right)$

$\therefore \quad A = \int_0^1\frac{1+x}{1+x^2}\,dx = \int_0^1\frac{1}{1+x^2}dx+\frac{1}{2}\int_0^1\frac{2x}{1+x^2}dx$

$=\left[\tan^{-1}x\right]_0^1+\frac{1}{2}\left[\log\left(1+x^2\right)\right]_0^1 = \left(\tan^{-1}1-\tan^{-1}0\right)+\frac{1}{2}\left[\log 2-\log 1\right] = \frac{\pi}{4}+\frac{1}{2}\log 2$