Integral Calculus – 2

Level – 1	Daily Tutorial Sheet - 1

1. $\displaystyle\int_{\ln 2}^{\ln 3}\frac{e^x\,dx}{1+e^x}=\int_{\ln 2}^{\ln 3}\frac{d(1+e^x)}{1+e^x}\,dx=\ln\left|1+e^x\right|_{\ln 2}^{\ln 3}=\ln\frac{4}{3}$

2. $\displaystyle\int_0^2 f(x)dx=\int_0^1 x^2dx+\int_1^2\sqrt{x}\,dx=\left.\frac{x^3}{3}\right|_0^1+\left.\frac{2}{3}x\sqrt{x}\right|_1^2=\frac{1}{3}\left(4\sqrt{2}-1\right).$

3. $\displaystyle I=\int_0^2|x-1|\,dx=\int_0^1(1-x)dx+\int_1^2(x-1)\,dx=\left.\left(x-\frac{x^2}{2}\right)\right|_0^1+\left.\left(\frac{x^2}{2}-x\right)\right|_1^2=1.$

4. $\displaystyle I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx=\int_0^{\pi/2}\frac{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}dx=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx$

On adding, $2I=\displaystyle\int_0^{\pi/2}dx$ $\Rightarrow$ $I=\dfrac{\pi}{4}$

5. $\displaystyle I=\int_0^{\pi}x\,f(\sin x)\,dx=\int_0^{\pi}(\pi-x)\,f(\sin(\pi-x))dx=\pi\int_0^{\pi}f(\sin x)\,dx-I$

$\displaystyle\Rightarrow\quad 2I=\pi\int_0^{\pi}f(\sin x)\,dx\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx.$

6. $\displaystyle I=\int_0^{\pi}\frac{(\pi-x)\,dx}{a^2\cos^2x+b^2\sin^2x}=\pi\int_0^{\pi}\frac{dx}{a^2\cos^2x+b^2\sin^2x}-\mathrm{I}.$

$\displaystyle\Rightarrow\quad I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2x+b^2\sin^2x}=\pi\int_0^{\pi/2}\frac{dx}{a^2\cos^2x+b^2\sin^2x}=\pi\int_0^{\infty}\frac{dt}{a^2+b^2t^2}\ (\tan x=t)$

$\displaystyle=\frac{\pi}{b^2}\int_0^{\infty}\frac{dt}{\left(\frac{a}{b}\right)^2+t^2}=\frac{\pi}{ab}\tan^{-1}\left.\frac{bt}{a}\right|_0^{\infty}=\frac{\pi}{ab}\left[\frac{\pi}{2}\right]=\frac{\pi^2}{2ab}$

7. $\displaystyle I=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\cos^4\left(\frac{\pi}{2}-x\right)+\sin^4\left(\frac{\pi}{2}-x\right)}dx=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\sin^4x+\cos^4x}\cdot dx$

$\displaystyle\Rightarrow\quad I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\cos x\sin x\,dx}{\sin^4x+\cos^4x}-I\Rightarrow 2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\cos x\sin x\,dx}{\sin^4x+\cos^4x}$

$\displaystyle\Rightarrow\quad I=\frac{\pi}{4}\int_0^{\pi/2}\frac{\tan x\sec^2x\,dx}{1+\tan^4x}\cdot$ let $\tan^2x=t\Rightarrow 2\tan x\sec^2x\,dx=dt$

$\displaystyle\Rightarrow\quad I=\frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}=\frac{\pi}{8}\tan^{-1}t\Big|_0^{\infty}=\frac{\pi}{8}\left[\frac{\pi}{2}\right]=\frac{\pi^2}{16}$

8. LHS. $= \int_a^b f(x)\,dx$ Put $x = (b-a)\,t + a \Rightarrow dx = (b-a)\,dt$

$\Rightarrow$ L.H.S. $= \int_0^1 f((b-a)\,t + a)\,(b-a)\,dt = (b-a)\int_0^1 f\left(\left(b-a\right)t + a\right) = (b-a)$

$\int_0^1 f\left(\left(b-a\right)x + a\right)dx$. = RHS. Hence proved.

9. $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7 - 10x^5 - 7x^3 + x}{x^2+2}\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^6 - 12x^2 + 1}{x^2+2}\,dx$

$= 0 + 2\int_0^{\sqrt{2}} \frac{3x^2(x^4-4)}{x^2+2}\,dx + 2\int_0^{\sqrt{2}} \frac{dx}{x^2+2} = 0 + 2\int_0^{\sqrt{2}} 3x^2(x^2-2)\,dx + 2\int_0^{\sqrt{2}} \frac{dx}{x^2+2} = \frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$

10. $I = \int_{-2}^{2} \frac{dx}{x^2+4} = \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{1}(-1)\right] = \frac{\pi}{4}$ Alternatively , Put $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2}dt$

$I = \int_{-2}^{2} \frac{dx}{4+x^2} = \int_{-1/2}^{1/2} \frac{\left(-1\right)dt}{t^2\,(4+\frac{1}{t^2})} = -\int_{-1/2}^{1/2} \frac{dt}{4t^2+1} = -\left[\frac{1}{2}\tan^{-1}(2t)\right]_{-1/2}^{1/2}$

$= -\frac{1}{2}\tan^{-1}(1) - \left(-\frac{1}{2}\tan^{-1}(-1)\right) = -\frac{\pi}{4}$

The second method is wrong because the substitution $x = \frac{1}{t}$ is discontinuous at $t = 0$.

Hence substitution $x = \frac{1}{t}$ is wrong.

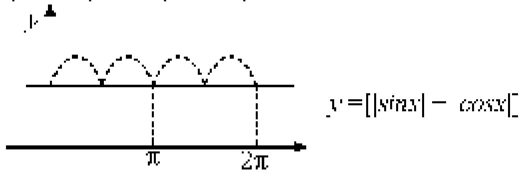
11.(A) $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx = \ell_1 + \ell_2$

12.(A) $\int_{ac}^{bc} f(x)dx$; Put $x = ct \Rightarrow dx = c\,dt \Rightarrow I = \int_a^b f(ct)(c\,dt) = c\int_a^b f(ct)dt$

13.(C) $I = \int_0^{\pi/2n} \frac{dx}{1+(\tan nx)^n} = \int_0^{\pi/2n} \frac{dx}{1+\left(\tan n\left(\frac{\pi}{2n} - x\right)\right)^n} = \int_0^{\pi/2n} \frac{\left(\tan nx\right)^n}{1+\left(\tan nx\right)^n} \Rightarrow 2I = \int_0^{\pi/2n} dx \Rightarrow I = \frac{\pi}{4n}$

14.(C) $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx = \int_0^{\pi/2}\left(\frac{\cos x}{1+e^x} + \frac{\cos(-x)}{1+e^{-x}}\right)dx = \int_0^{\pi/2} \cos x\left(\frac{1+e^x}{1+e^x}\right)dx = \int_0^{\pi/2} \cos x\,dx = -1$

15.(B) $|\,sin\,x|$ and $|\,cos\,x|$ and add :



$\Rightarrow \quad I = \int_0^{2\pi}\left[|\sin x| + |\cos x|\right]dx = \int_0^{2\pi} 1\,.\,dx = 2\pi$