Level - 3	Daily Tutorial Sheet-14

162.

(i)(D) $\int \frac{e^x(x-1)(x-\ln x)}{x^2}dx = \int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)\left(\ln\frac{e^x}{x}\right)dx = \int \ln\frac{e^x}{x}\,d\left(\frac{e^x}{x}\right)$

$= \ln\left(\frac{e^x}{x}\right)\frac{e^x}{x} - \frac{e^x}{x} + c = \frac{e^x}{x}(x-\ln x-1)+c$

(ii)(A) Let $I = \int \frac{x^m-1}{x^{m+1}\sqrt{1-2x^m+mx^{2m}}}dx$

Dividing numerator and denominator with x^{2m+1} we get

$$I = \int \frac{\frac{1}{x^{m+1}}-\frac{1}{x^{2m+1}}}{\sqrt{\frac{1}{x^{2m}}-\frac{2}{x^m}+m}}dx$$

Put $t^2 = \frac{1}{x^{2m}}-\frac{2}{x^m}+m$

Therefore $2t\,dt = \left(\frac{-2m}{x^{2m+1}}+\frac{2m}{x^{m+1}}\right)dx$

$\Rightarrow \quad t\ dt = m\left(\frac{1}{x^{m+1}}-\frac{1}{x^{2m+1}}\right)dx$

Hence $I = \int \frac{1}{t}\left(\frac{t}{m}\right)dt = \frac{t}{m}+c = \frac{1}{m}\sqrt{\frac{1}{x^{2m}}-\frac{2}{x^m}+m}+c = \frac{1}{mx^m}\sqrt{1-2x^m+mx^{2m}}+c$

163.

(i)(A) $I = \int(\sin(100x+x)\cdot(\sin x)^{99}dx = \int[(\sin(100x)\cos x+\cos100x\cdot\sin x)(\sin x)^{99}]dx$

$= \int \underbrace{\sin(100x)}_{I}\underbrace{\cos x\cdot(\sin x)^{99}}_{II}dx + \int\cos(100x)\cdot(\sin x)^{100}dx$

$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100}\int\cos(100x)(\sin x)^{100}dx + \int\cos(100x)(\sin x)^{100}dx = \frac{\sin(100x)(\sin x)^{100}}{100}+C$

(ii)(A) Let $I = \int \frac{a+b\sin x}{(b+a\sin x)^2}dx \quad = \frac{b}{a}\frac{\frac{a^2}{b}-b+(b+a\sin x)}{(b+a\sin x)^2}dx$

$= \frac{a^2-b^2}{a}\int\frac{dx}{(b+a\sin x)^2}+\frac{b}{a}\int\frac{dx}{b+a\sin x} \qquad \ldots(i)$

Let $f(x) = \frac{a\cos x}{b+a\sin x}$

So that $f'(x) = \dfrac{-a - b\sin x}{(b + a\sin x)^2} = -\dfrac{b}{a}\dfrac{\left(a\sin x + b + \dfrac{a^2}{b} - b\right)}{(b + a\sin x)^2}$

$= -\dfrac{b}{a}\left(\dfrac{1}{b + a\sin x}\right) - \dfrac{a^2 - b^2}{a}\left(\dfrac{1}{(b + a\sin x)^2}\right)$ ...(ii)

Therefore, $f(x) = \int f'(x) + c$

$= \dfrac{-b}{a}\int \dfrac{dx}{b + a\sin x} - \dfrac{a^2 - b^2}{a}\int \dfrac{dx}{(b + a\sin x)^2} + c$ [By Eq. (ii)

Hence from Eq. (i)

$I = -f(x) + c = -\left(\dfrac{a\cos x}{b + a\sin x}\right) + c$

164.(B) Let $I_n = \int (\sin x + \cos x)^{n-1}(\sin x + \cos x)\,dx$

$u = (\sin x + \cos x)^{n-1}$ and $dv = (\sin x + \cos x)dx$

So that $v = \sin x - \cos x$

Therefore, using integration by parts we have

$I_n = (\sin x + \cos x)^{n-1}(\sin x - \cos x) - \int (n-1)(\sin x + \cos x)^{n-2} \times (\cos x - \sin x)(\sin x - \cos x)$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)\int (\sin x + \cos x)^{n-2}(\cos x - \sin x)^2 dx$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + (n-1)\int (\sin x + \cos x)^{n-2}[2 - (\sin x + \cos x)^2]dx$

$= (\sin x + \cos x)^{n-1}(\sin x - \cos x) + 2(n-1)I_{n-2} - (n-1)I_n$

Therefore, $I_n + (n-1)I_n - 2(n-1)I_{n-2} = (\sin x + \cos x)^{n-1}(\sin x - \cos x)$

$\Rightarrow nI_n - 2(n-1)I_{n-2} = (\sin x + \cos x)^{n-1}(\sin x - \cos x)$

165.(D) $\int \dfrac{\operatorname{cosec}^2 x - 2005}{\cos^{2005} x}dx = \int (\cos x)^{-2005}\operatorname{cosec}^2 x\,dx - 2005\int \dfrac{dx}{\cos^{2005} x} = (\cos x)^{-2005}(-\cot x)$

$-\int (-2005)(\cos x)^{-2006}(-\sin x)(-\cot x)dx - 2005\int \dfrac{dx}{\cos^{2005} x} = -\dfrac{\cot x}{(\cos x)^{2005}} + C$

166. Here, $I = \int \dfrac{(1 - x\sin x)dx}{x\left(1 - \left(xe^{\cos x}\right)^3\right)}$

Put $xe^{\cos x} = t$ so that $(xe^{\cos x}(-\sin x) + e^{\cos x})dx = dt$ $\therefore$ $I = \int \dfrac{dt}{t(1 - t^3)}$, $= \int \dfrac{dt}{t^4\left(\dfrac{1}{t^3} - 1\right)}$

Let $\dfrac{1}{t^3} - 1 = y$ or $dy = \dfrac{-3}{t^4}dt$

$\therefore \quad I = -\frac{1}{3}\int \frac{dy}{y} = -\frac{1}{3}\log|y| + C = -\frac{1}{3}\log\left|\frac{1}{t^3} - 1\right| + C \;= \int \frac{dt}{t} + \frac{1}{3}\int \frac{dt}{1-t} + \int \frac{\left(-\frac{2}{3}t - \frac{1}{3}\right)}{1+t+t^2}dt$

$= \log|t| - \frac{1}{3}\log|1-t| - \frac{1}{3}\log|1+t+t^2|$

where, $t = xe^{\cos x}$

167. $I = \int \frac{x^2\left(1 - \frac{1}{x^2}\right)dx}{x^2\left[\sqrt{\left(x + \frac{1}{x} + \alpha\right)}\sqrt{x + \frac{1}{x} + \beta}\right]}$

Put $x + \frac{1}{x} = z$ or $\left(1 - \frac{1}{x^2}\right)dx = dz$

$\therefore \quad I = \int \frac{dz}{\sqrt{(z+\alpha)(z+\beta)}} = \int \frac{dz}{\sqrt{z^2 + (\alpha+\beta)z + \alpha\beta}} = \int \frac{dz}{\sqrt{\left(z + \frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2}}$

$= \log\left|z + \frac{\alpha+\beta}{2} - \sqrt{(z+\alpha)(z+\beta)}\right| + c = \log\left|x + \frac{1}{x} + \frac{\alpha+\beta}{2} - \sqrt{(z+\alpha)(z+\beta)}\right| + c$

$= \log\left|x + \frac{1}{x} + \frac{\alpha+\beta}{2} - \sqrt{\left(x + \frac{1}{x} + \alpha\right)\left(x + \frac{1}{x} + \beta\right)}\right| + c \;= \log\left[\frac{2x^2 + (\alpha+\beta)x + 2}{2x} - \frac{\sqrt{(x^2+\alpha x+1)(x^2+\beta x+1)}}{x}\right] + c$

$= \log\frac{1}{2}\left(\frac{\sqrt{x^2+\alpha x+1} - \sqrt{x^2+\beta x+1}}{\sqrt{x}}\right)^2 + c \;= 2\log\left(\frac{\sqrt{x^2+\alpha x+1} - \sqrt{x^2+\beta x+1}}{\sqrt{x}}\right)^2 + c$