Integral Calculus-1

Level - 3	Daily Tutorial Sheet-13

156.(C) We have $x^2y = a\int \frac{x^3}{y}dx + b\int \frac{x^2}{y}dx + c\int \frac{x}{y}dx$

Differentiating both sides with respect x, we have $2xy + x^2\frac{dy}{dx} = \frac{ax^3}{y} + \frac{bx^2}{y} + \frac{cx}{y}$

$$2xy^2 + x^2y\left(\frac{dy}{dx}\right) = ax^3 + bx^2 + cx \qquad \ldots \text{(i)}$$

But $y^2 = x^2 - x + 1 \;\Rightarrow\; y\frac{dy}{dx} = \frac{1}{2}(2x-1)$

Substituting the value of $y^2 = x^2 - x + 1$ and $y\frac{dy}{dx} = \frac{1}{2}(2x-1)$ in Eq. (i) we obtain

$$2x(x^2 - x + 1) + x^2\frac{(2x-1)}{2} = ax^3 + bx^2 + cx \qquad \Rightarrow \qquad 3x^3 - \frac{5}{2}x^2 + 2x = ax^3 + bx^2 + cx$$

Therefore $a = 3, b = -\frac{5}{2}, c = 2$

157.(C) Given that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{3}$$

Substituting $y = 0$ and replacing x with $3x$, we have

$$f(x) = \frac{f(3x) + f(0)}{3} = \frac{f(3x)}{3} \qquad [\because f(0) = 0]$$

$$3f(x) = f(3x) \qquad \ldots \text{(i)}$$

Now using Eq. (i) we have

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f\left(3\frac{(x+h)}{3}\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{3f\left(\frac{(x+h)}{3}\right) - f(x)}{h} = \lim_{h\to 0}\frac{3\left(\frac{f(x) + f(h)}{3}\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f(h) - f(0)}{h} \qquad [\because f(0) = 0]$$

$$= f'(0) = 3$$

Therefore f is differentiable at any real x and $f'(x) = f'(0) = 3.$ Hence

$f(x) = 3x + c$ and $f(0) = 0 \Rightarrow c = 0$

So $f(x) = 3x$ and hence $\int f(x)dx = \frac{3}{2}x^2 + c$

158.(A) $I = \int \dfrac{x^2+1}{x\sqrt{x^2+2x-1}\sqrt{1-x^2-x}}dx = \int \dfrac{1+\dfrac{1}{x^2}}{\sqrt{x+2-\dfrac{1}{x}}\sqrt{\dfrac{1}{x}-x-1}}dx$

Let $t = \sqrt{x+2-\dfrac{1}{x}}$ $\quad\therefore\quad$ $dt = \dfrac{1}{2\sqrt{x+2-\dfrac{1}{x}}}\left(1+\dfrac{1}{x^2}\right)dx$

$\therefore \quad I = \int \dfrac{2dt}{\sqrt{1-t^2}} = 2\sin^{-1}t + c = 2\sin^{-1}\sqrt{x-\dfrac{1}{x}+2}+c$

159.

(i)(C) $I = \int \dfrac{3(\tan x-1)\sec^2 x}{(\tan x+1)\sqrt{\tan^3 x+\tan^2 x+\tan x}}dx$

Put $\tan x = t$ $\quad\Rightarrow\quad$ $I = 3\int \dfrac{1-\dfrac{1}{t^2}}{\left(t+2+\dfrac{1}{t}\right)\sqrt{t+\dfrac{1}{t}+1}}dt$

Put $t+\dfrac{1}{t}+1 = z^2$ $\quad\Rightarrow\quad$ $I = 3.2\tan^{-1}(\sqrt{\cot x+\tan x+1})+C$ $\quad\Rightarrow\quad$ $K = 6$

(ii)(C) Let $I = \int \dfrac{1-(\cot x)^{2010}}{\tan x+(\cot x)^{2011}}dx$

$= \int \dfrac{(\sin x)^{2010}-(\cos x)^{2010}}{(\sin x)^{2010}} \cdot \dfrac{(\sin x)^{2011}\ \cos x}{(\sin x)^{2012}+(\cos x)^{2012}}dx$

$= \int \dfrac{[(\sin x)^{2010}-(\cos x)^{2010}]\sin x\cos x}{(\sin x)^{2012}+(\cos x)^{2012}}$

Put $t = (\sin x)^{2012}+(\cos x)^{2012}$. Then

$dt = [2012(\sin x)^{2011}\cos x+2012(\cos x)^{2011}(-\sin x)]dx$

$= [2012(\sin x)^{2010}-(\cos x)^{2010}](\sin x\cos x)dx$

Therefore, $I = \int \dfrac{1}{t}\left(\dfrac{1}{2012}\right)dt = \dfrac{1}{2012}\log_e|t|+c$

$= \dfrac{1}{2012}\log_e\left|(\sin x)^{2012}+(\cos x)^{2012}\right|+c$

160.

(i)(B) $I = \int x^{24}(1+x+x^2)^6\{x^3(4+5x+6x^2)\}dx = \int (x^4+x^5+x^6)^6(6x^5+5x^4+4x^3)dx = \dfrac{x^{28}(1+x+x^2)^7}{7}+C$

(ii)(A) We have $\lim\limits_{x\to 0}\left(1+x+\dfrac{f(x)}{x}\right)^{1/x} = e^3$ $\quad\Rightarrow\quad$ $\lim\limits_{x\to 0}\left(1+x\left(1+\dfrac{f(x)}{x^2}\right)\right)^{1/x} = e^3$

$$\Rightarrow \quad \lim_{x\to 0}\left[\left(1+\frac{x^2+f(x)}{x}\right)^{\frac{x}{x^2+f(x)}}\right]^{\frac{x^2+f(x)}{x}\frac{1}{x}} = e^3 \quad \Rightarrow \quad \frac{x^2+f(x)}{x^2}=3 \quad \Rightarrow \quad f(x)=2x^2$$

Therefore, $\int f(x)\log_e x dx = 2\int x^2 \log_e x dx$

$$=2\left[\frac{x^3}{3}\log_e x-\int \frac{x^3}{3}\cdot\frac{1}{x}dx\right] \qquad \text{(By Parts)}$$

$$=\frac{2}{3}x^3\log_e x-\frac{2}{9}x^3+c=\frac{2}{3}x^3\left(\log_e x-\frac{1}{3}\right)+c$$

161.(B) We have $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}}dx = \int \frac{x(x^2-1)}{(x^2+1)(x+1)^2\sqrt{x^3+x^2+x}}dx$

$$=\int \frac{x^3\left(1-\frac{1}{x^2}\right)}{x^3\left(x+\frac{1}{x}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\sqrt{x+\frac{1}{x}+1}}dx = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}dx$$

$$I=\int \frac{2t}{(t^2-1)(t^2+1)\sqrt{t^2}}dt$$

Where $x+\frac{1}{x}+1=t^2$ $=\int \frac{2}{(t^2-1)(t^2+1)}dt = \int \frac{1}{t^2-1}dt-\int\frac{1}{t^2+1}dt = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right|-\tan^{-1}t+c$

$$=\frac{1}{2}\log\left|\frac{\sqrt{x+\frac{1}{x}+1}-1}{\sqrt{x+\frac{1}{x}+1}+1}\right|-\tan^{-1}\sqrt{x+\frac{1}{x}+1}+c$$