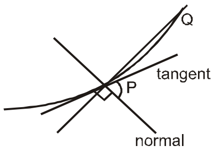
APPLICATION OF DERIVATIVES

Tangent and Normal

Let y = f(x) be function with graph as shown in figure. Consider secant PQ. If Q tends to P along the curve passing through the points Q_1, Q_2, ...I.e. $Q \to P$, secant PQ will become tangent at P. A line through P



Figure

perpendicular to tangent is called normal at P.

Geometrical Meaning of $\frac{dy}{dx}$

As $Q \to P$, $h \to 0$ and slope of chord PQ tends to slope of tangent at P (see figure).

Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{Q\to P}$ slope of chord PQ = $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\Rightarrow$ slope of tangent at P = $f'(x) = \frac{dy}{dx}$

Equation of tangent and normal

$\left.\frac{dy}{dx}\right]_{(x_1, y_1)} = f'(x_1)$ denotes the slope of tangent at point (x_1, y_1) on the curve y = f(x). Hence the equation of tangent at (x_1, y_1) is given by

$(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

Also, since normal is a line perpendicular to tangent at (x_1, y_1) so its equation is given by

$(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

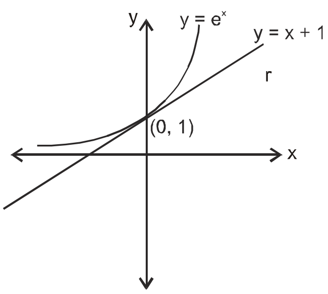
If $f'(x_1) = 0$, then tangent is the line $y = y_1$ and normal is the line $x = x_1$.

If $\lim\limits_{h\to 0} \frac{f(x_1+h)-f(x_1)}{h} = \infty$ or $-\infty$, then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

Example # 1 Find equation of tangent to $y = e^x$ at x = 0. Hence draw graph

Solution : At x = 0 $\Rightarrow$ $y = e^0 = 1$

$\frac{dy}{dx} = e^x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 1$



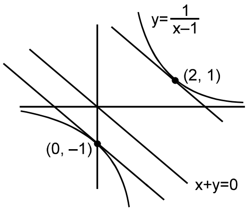
Hence equation of tangent is

$1(x - 0) = (y - 1)$

$\Rightarrow$ y = x + 1

Example # 2 Find the equation of all straight lines which are tangent to curve $y = \frac{1}{x-1}$ and which are parallel to the line x + y = 0.

Solution : Suppose the tangent is at (x_1, y_1) and it has slope – 1.



$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x_1\,,\,y_1)} = -1.$

$\Rightarrow \quad -\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow \quad x_1 = 0 \quad$ or $\quad 2$

$\Rightarrow \quad y_1 = -1 \quad$ or $\quad 1$

Hence tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$-1(x-0) = (y+1)$ and $-1\,(x-2) = (y-1)$

$\Rightarrow \quad x + y + 1 = 0$ and $y + x = 3$

Example # 3 Find equation of normal to the curve $y = |x^2 - |x||$ at x = – 2.

Solution : In the neighborhood of x = – 2, $y = x^2 + x$.

Hence the point of contact is (– 2, 2)

$\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3.$

So the slope of normal at (– 2, 2) is .

Hence equation of normal is

$\frac{1}{3}(x+2) = y - 2 \quad \Rightarrow \quad 3y = x + 8$

Example # 4 Prove that sum of intercepts of the tangent at any point to the curve represented by $x = 3\cos^4\theta$ & $y = 3\sin^4\theta$ on the coordinate axis is constant.

Solution : Let $P(3\cos^4\theta\ , 3\sin^4\theta)$ be a variable point on the given curve.

$\Rightarrow \quad m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3.4\sin^3\theta.\cos\theta}{-3.4\cos^3\theta\sin\theta} = -\tan^2\theta$

$\Rightarrow$ equation of tangent at point P is

$y - 3\sin^4\theta = -\tan^2\theta\,(x - 3\cos^4\theta)$

$\Rightarrow \quad \frac{x}{3\cos^2\theta} + \frac{y}{3\sin^2\theta} = 1$

$\Rightarrow$ sum of x-axis intercept and y-axis intercept = $3\cos^2\theta + 3\sin^2\theta = 3$ (which is constant)

Self Practice Problems :

(1) Find the slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$.

(2) Find the equation of the tangent and normal to the given curves at the given points.

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(ii) $y^2 = \frac{x^3}{4 - x}$ at (2, – 2).

(3) Prove that area of the triangle formed by any tangent to the curve $xy = c^2$ and coordinate axes is constant.

(4) A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q . Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

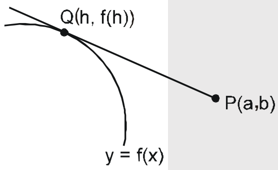
Ans. (1) $-\frac{a}{2b}$ (2) (i) Tangent : $y = 2x + 1$, Normal : $x + 2y = 7$

(ii) Tangent : $2x + y = 2$, Normal : $x - 2y = 6$

Tangent and Normal from an external point

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h) - b}{h - a}$$

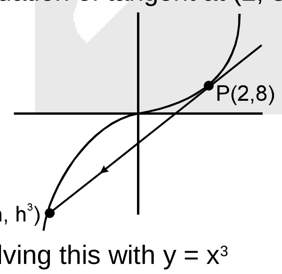


Figure

And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

Example # 5 Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Solution : Equation of tangent at (2, 8) is $y = 12x - 16$



Solving this with $y = x^3$

$x^3 - 12x + 16 = 0$

This cubic will give all points of intersection of line and curve $y = x^3$ i.e., point P and Q. (see figure)

But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations :

sum of roots $\Rightarrow$ $2 + 2 + h = 0$ $\Rightarrow$ $h = -4$

Hence coordinates of Q are (– 4, – 64)

Self Practice Problems :

(5) How many tangents are possible from (1, 1) to the curve $y - 1 = x^3$. Also find the equation of these tangents.

(6) Find the equation of tangent to the hyperbola $y = \frac{x+9}{x+5}$ which passes through (0, 0) origin

Ans. (5) $y = 1$, $4y = 27x - 23$ (6) $x + y = 0$; $25y + x = 0$

Derivative as rate of change

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of charge of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition : The average rate of change of a function f(x) with respect to x over an interval [a, a + h] is defined as $\frac{f(a+h)-f(a)}{h}$

Definition : The instantaneous rate of change of f(x) with respect to x is defined as $f'(x) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, provided the limit exists.

Note : To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

Example # 6 How fast the area of a circle increases when its radius is 5cm;
(i) with respect to radius (ii) with respect to diameter

Solution : (i) $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$

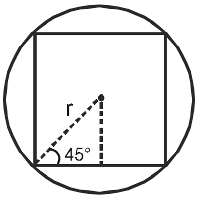
$\therefore \left.\frac{dA}{dr}\right]_{r=5} = 10\pi$ cm²/cm.

(ii) $A = \frac{\pi}{4} D^2$, $\frac{dA}{dD} = \frac{\pi}{2} D$

$\therefore \left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2} . 10 = 5\pi$ cm²/cm.

Example # 7 If area of circle increases at a rate of 2cm²/sec, then find the rate at which area of the inscribed square increases.

Solution : Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$ (see figure)



$\frac{dA_1}{dt} = 2\pi r \frac{dr}{dt}$, $\frac{dA_2}{dt} = 4r . \frac{dr}{dt}$

$\because \quad 2 = 2\pi r . \frac{dr}{dt} \quad \Rightarrow \quad r \frac{dr}{dt} = \frac{1}{\pi}$

$\therefore \quad \frac{dA_2}{dt} = 4 . \frac{1}{\pi} = \frac{4}{\pi}$ cm²/sec

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm²/sec.

Example # 8 The volume of a cube is increasing at a rate of 7 cm³/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Solution. Let at some time t, the length of edge is x cm.

$v = x^3 \quad \Rightarrow \quad \frac{dv}{dt} = 3x^2 \frac{dx}{dt}$ (but $\frac{dv}{dt} = 7$)

$\Rightarrow \quad \frac{dx}{dt} = \frac{7}{3x^2}$ cm/sec.

Now $S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt} \quad \Rightarrow \quad \frac{dS}{dt} = 12x. \frac{7}{3x^2} = \frac{28}{x}$

when x = 4 cm, $\frac{dS}{dt} = 7$ cm²/sec.

Example # 9 Sand is pouring from pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Solution. $V = \frac{1}{3} \pi r^2 h$

but $h = \frac{r}{6}$

$\Rightarrow \quad V = \frac{1}{3} \pi (6h)^2 h$

$\Rightarrow \quad V = 12\pi h^3$

$\frac{dV}{dt} = 36\pi h^2. \frac{dh}{dt}$

when, $\frac{dV}{dt} = 12$ cm³/s and h = 4 cm

$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}$ cm/sec.

Self Practice Problems :

(7) Radius of a circle is increasing at rate of 3 cm/sec. Find the rate at which the area of circle is increasing at the instant when radius is 10 cm.

(8) A ladder of length 5 m is leaning against a wall. The bottom of ladder is being pulled along the ground away from wall at rate of 2cm/sec. How fast is the top part of ladder sliding on the wall when foot of ladder is 4 m away from wall.

(9) Water is dripping out of a conical funnel of semi-vertical angle 45° at rate of 2cm³/s. Find the rate at which slant height of water is decreasing when the height of water is $\sqrt{2}$ cm.

(10) A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment.

Ans. (7) 60π cm^2/sec (8) $\frac{8}{3}$ cm/sec (9) $\frac{1}{\sqrt{2}\pi}$ cm/sec. (10) 140 ft/min.

Error and Approximation :

Let y = f(x) be a function. If these is an error δx in x then corresponding error in y is $\delta y = f(x + \delta x) - f(x)$.

We have $\lim\limits_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{dy}{dx} = f'(x)$

We define the differential of y, at point x, corresponding to the increment δx as $f'(x)\,\delta x$ and denote it by dy.

i.e. $dy = f'(x)\,\delta x$.

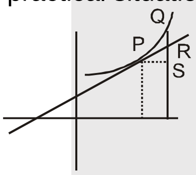
Let P(x, f(x)), Q((x + δx), f(x + δx)) (as shown in figure)

δy = QS,

δx = PS,

dy = RS

In many practical situations, it is easier to evaluate dy but not δy.



Example # 10. Find the approximate value of $25^{1/3}$.

Sol. Let $y = x^{1/3}$

Let x = 27 and $\Delta x = -2$

Now $\Delta y = (x + \Delta x)^{1/3} - x^{1/3} = (25)^{1/3} - 3$

$\frac{dy}{dx}\,\Delta x = 25^{1/3} - 3$

At x = 27, $25^{1/3} = 3 - 0.074 = 2.926$

Monotonicity of a function :

Let f be a real valued function having domain D(DR) and S be a subset of D. f is said to be monotonically increasing (non decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. f is said to be monotonically decreasing (non increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Notes : (i) f is strictly increasing $\Rightarrow$ f is monotonically increasing (non decreasing). But converse need not be true.

(ii) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing (non increasing). Again, converse need not be true.

(iii) If f(x) = constant in S, then f is increasing as well as decreasing in S

(iv) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(v) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

Application of differentiation for detecting monotonicity :

Let I be an interval (open or closed or semi open and semi closed)

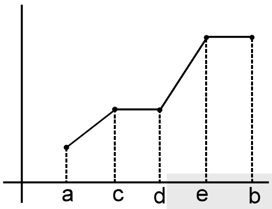
(i) If $f'(x) > 0 \ \forall \ x \in I$, then f is strictly increasing in I

(ii) If $f'(x) < 0 \ \forall \ x \in I$, then f is strictly decreasing in I

Note : Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0, \ \forall \ x \in I$, except for countably many points where $f'(x) = 0$, then f(x) is strictly increasing in I.

{$f'(x) = 0$ at countably many points $\Rightarrow$ $f'(x) = 0$ does not occur on an interval which is a subset of I }

Let us consider another function whose graph is shown below for $x \in (a, b)$.

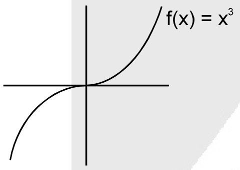


Here also $f'(x) \geq 0$ for all $x \in (a, b)$. But, note that in this case, $f'(x) = 0$ holds for all $x \in (c, d)$ and (e,b). Thus the given function is increasing (monotonically increasing) in (a, b), but not strictly increasing.

Example # 11 : Let $f(x) = x^3$. Find the intervals of monotonicity.

Solution : $f'(x) = 3x^2$

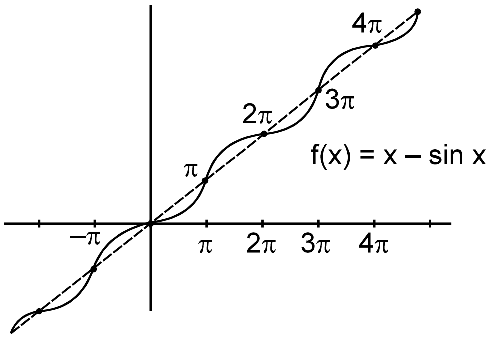
$f'(x) > 0$ everywhere except at $x = 0$. Hence f(x) will be strictly increasing function for $x \in R$ {see figure}



Example # 12 : Let $f(x) = x - \sin x$. Find the intervals of monotonicity.

Solution : $f'(x) = 1 - \cos x$

Now, $f'(x) > 0$ every where, except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. But all these points are discrete (countable) and do not form an interval. Hence we can conclude that f(x) is strictly increasing in R. In fact we can also see it graphically.



Example # 13 : Find the intervals in which $f(x) = x^3 - 2x^2 - 4x + 7$ is increasing.

Solution : $f(x) = x^3 - 2x^2 - 4x + 7$

$f'(x) = 3x^2 - 4x - 4$

$f'(x) = (x - 2)(3x + 2)$

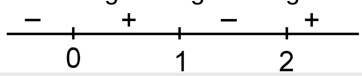
for M.I. $f'(x) \geq 0 \quad \Rightarrow \quad x \in \left(-\infty, -\frac{2}{3}\right] \cup [2, \infty)$

Example # 14 : Find the intervals of monotonicity of the following functions.

(i) $f(x) = x^2 (x - 2)^2$ (ii) $f(x) = x \ell n x$

Solution : (i) $f(x) = x^2 (x - 2)^2 \Rightarrow f'(x) = 4x (x - 1)(x - 2)$

observing the sign change of $f'(x)$



Hence increasing in [0, 1] and in [2, ∞)

and decreasing for $x \in (-\infty, 0]$ and [1, 2]

(ii) $f(x) = x \ell n x$

$f'(x) = 1 + \ell n x$

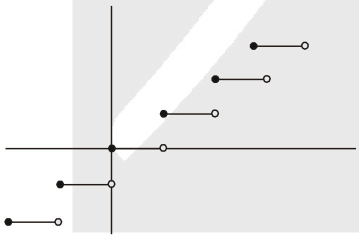
$f'(x) \geq 0 \quad \Rightarrow \quad \ell n x \geq -1 \quad \Rightarrow \quad x \geq \frac{1}{e}$

$\Rightarrow$ increasing for $x \in \left[\frac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \frac{1}{e}\right]$.

Note : If a function f(x) is increasing in (a, b) and f(x) is continuous in [a, b], then f(x) is increasing on [a, b]

Example # 15 : f(x) = [x] is a step up function. Is it a strictly increasing function for $x \in R$.

Solution : No, f(x) = [x] is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.



Example # 16 : If $f(x) = \sin^4x + \cos^4x + bx + c$, then find possible values of b and c such that f(x) is monotonic for all $x \in R$

Solution : $f(x) = \sin^4x + \cos^4x + bx + c$

$f'(x) = 4 \sin^3x \cos x - 4\cos^3x \sin x + b = -\sin 4x + b.$

<u>Case - (i)</u> : for M.I. $f'(x) \geq 0$ for all $x \in R$

$\Leftrightarrow \quad b \geq \sin 4x$ for all $x \in R \quad \Leftrightarrow \quad b \geq 1$

<u>Case - (ii)</u> : for M.D. $f'(x) \leq 0$ for all $x \in R$

$\Leftrightarrow \quad b \leq \sin 4x$ for all $x \in R \quad \Leftrightarrow \quad b \leq -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Example # 17 : Find possible values of 'a' such that $f(x) = e^{2x} - 2(a^2 - 21) e^x + 8x + 5$ is monotonically increasing for $x \in R$

Solution : $f(x) = e^{2x} - 2(a^2 - 21) e^x + 8x + 5$

$f'(x) = 2e^{2x} - 2(a^2 - 21)\, e^x + 8 \geq 0$; $\forall$ c $x \in R$

$\Rightarrow \quad e^x + \dfrac{4}{e^x} \geq a^2 - 21$

$4 \geq a^2 - 21 \qquad \left(\because\ e^x + \dfrac{4}{e^x} \geq 4\right)$

$\Rightarrow \quad a \in [-5, 5]$

Self Practice Problems :

(11) Find the intervals of monotonicity of the following functions.

(i) $f(x) = -x^3 + 6x^2 - 9x - 2$

(ii) $f(x) = x + \dfrac{1}{x+1}$

(iii) $f(x) = x\,.\ e^{x-x^2}$

(iv) $f(x) = x - \cos x$

(12) Let $f(x) = x - \tan^{-1}x$. Prove that f(x) is monotonically increasing for $x \in R$.

(13) If $f(x) = 2e^x - ae^{-x} + (2a + 1)\,x - 3$ monotonically increases for $\forall\ x \in R$, then find range of values of a

(14) Let $f(x) = e^{2x} - ae^x + 1$. Prove that f(x) cannot be monotonically decreasing for $\forall\ x \in R$ for any value of 'a'.

(15) The values of 'a' for which function $f(x) = (a + 2)\,x^3 - ax^2 + 9ax - 1$ monotonically decreasing for $\forall\ x \in R$.

Ans. (11) (i) I in [1, 3] ; D in $(-\infty, 1] \cup (3, \infty)$

(ii) I in $(-\infty, -2] \cup [0, \infty)$; D in $[-2, -1) \cup (-1, 0]$

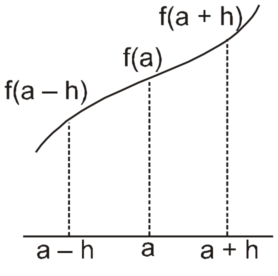
(iii) I in $\left[-\dfrac{1}{2}, 1\right]$; D in $\left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$

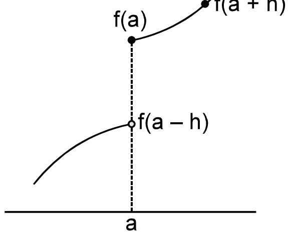
(iv) I for $x \in R$

(13) $a \geq 0$ (15) $-\infty < a \leq -3$

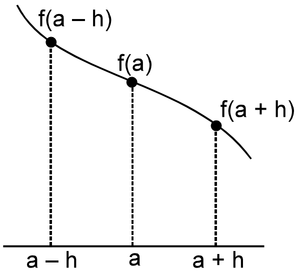
Monotonicity of function about a point :

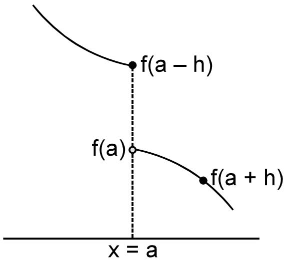
1. A function f(x) is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



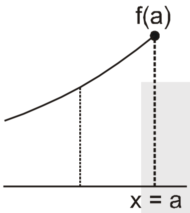


2. A function f(x) is called a strictly decreasing function about a point x = a, if it is strictly decreasing in an open interval containing a (as shown in figure).

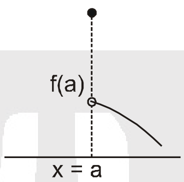




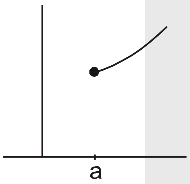
Note : If x = a is a boundary point then use the appropriate one sided inequality to test monotonicity of f(x).



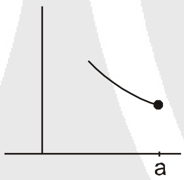
f is increasing at x = a



f is decreasing at x = a



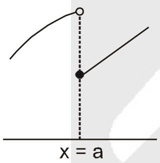
f is increasing at x = a



f is decreasing at x = a

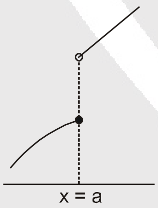
e.g. : Which of the following functions (as shown in figure) is increasing, decreasing or neither increasing nor decreasing at x = a.

(i)



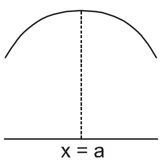
neither increasing
nor decreasing at x = a

(ii)



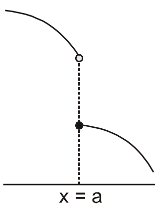
Increasing at x = a

(iii)



neither increasing
nor decreasing at x = a

(iv)



decreasing at x = a

Test for increasing and decreasing functions about a point

Let f(x) be differentiable.

(1) If $f'(a) > 0$ then f(x) is increasing at x = a.

(2) If $f'(a) < 0$ then f(x) is decreasing at x = a.

(3) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

(i) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at x = a.

(ii) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at x = a.

(iii) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at x = a.

Example # 18 : Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points x = 0, 1, 2.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3 \Rightarrow$ decreasing at x = 0

(ii) $f'(1) = 0$

also, $f'(x)$ is positive on left neighbourhood and $f'(x)$ is negative in right neighbourhood.

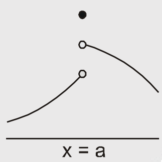
$\Rightarrow$ neither increasing nor decreasing at x = 1.

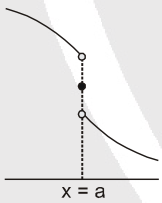
(iii) $f'(2) = 9 \Rightarrow$ increasing at x = 2

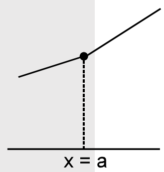
Note : Above method is applicable only for functions those are continuous at x = a.

Self Practice Problems :

(16) For each of the following graph comment on monotonicity of f(x) at x = a.

(i) 

(ii) 

(iii) 

(17) Let $f(x) = x^3 - 3x^2 + 3x + 4$, comment on the monotonic behaviour of f(x) at (i) x = 0 (ii) x = 1.

(18) Draw the graph of function $f(x) = \begin{cases} x & 0 \le x \le 1 \\ [x] & 1 \le x \le 2 \end{cases}$. Graphically comment on the monotonic behaviour of f(x) at x = 1. Is f(x) M.I. for $x \in [0, 2]$?

Ans. (16) (i) neither M.I. nor M.D. (ii) M.D. (iii) M.I

(17) M.I. both at x = 0 and x = 1.

(18) M.I. at x = 1; f(x) is M.I. for $x \in [0, 2]$.

Global Maximum :

A function f(x) is said to have global maximum on a set E if there exists at least one $c \in E$ such that $f(x) \le f(c)$ for all $x \in E$.

We say global maximum occurs at x = c and global maximum (or global maximum value) is f(c).

Local Maxima :

A function f(x) is said to have a local maximum at x = c if f(c) is the greatest value of the function in a small neighbourhood (c – h, c + h), h > 0 of c.

i.e. for all $x \in (c - h, c + h)$, $x \ne c$, we have $f(x) \le f(c)$.

Global Minimum :

A function f(x) is said to have a global minimum on a set E if there exists at least one $c \in E$ such that $f(x) \geq f(c)$ for all $x \in E$.

Local Minima :

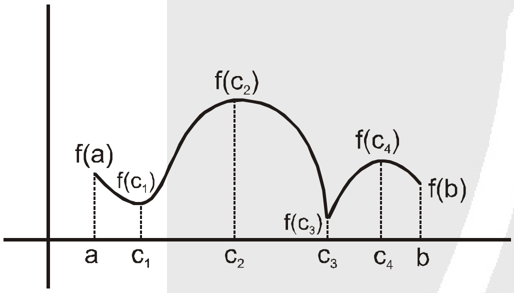
A function f(x) is said to have a local minimum at x = c if f(c) is the least value of the function in a small neighbourhood (c – h, c + h), h > 0 of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \geq f(c)$.

Extrema :

A maxima or a minima is called an extrema.

Explanation : Consider graph of y = f(x), $x \in [a, b]$



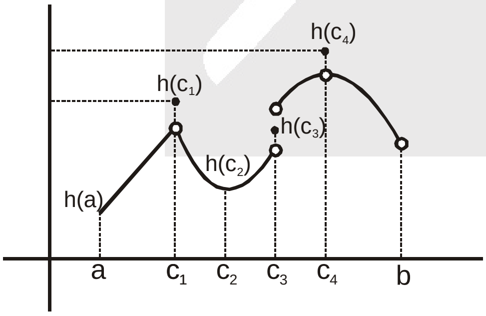
$x = c_2$, $x = c_4$ are points of local maxima, with maximum values $f(c_2)$, $f(c_4)$ respectively.

$x = c_1$, $x = c_3$ are points of local minima, with minimum values $f(c_1)$, $f(c_3)$ respectively

$x = c_2$ is a point of global maximum

$x = c_3$ is a point of global minimum

Consider the graph of y = h(x), $x \in [a, b)$



$x = c_1$, $x = c_4$ are points of local maxima, with maximum values $h(c_1)$, $h(c_4)$ respectively.

$x = c_2$ are points of local minima, with minimum values $h(c_2)$ respectively.

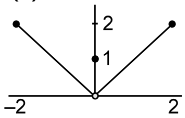
$x = c_3$ is neither a point of maxima nor a minima.

Global maximum is $h(c_4)$

Global minimum is h(a)

Example # 19 : Let f(x) = $\begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

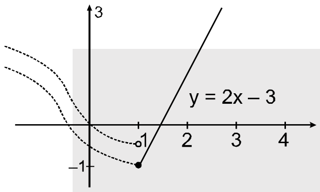
Solution : f(x) has local maxima at x = 0 (see figure).



Example # 20 : Let f(x) = $\begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Find all possible values of b such that f(x) has the smallest value at x = 1.

Solution. Such problems can easily be solved by graphical approach (as in figure).



Hence the limiting value of f(x) from left of x = 1 should be either greater or equal to the value of function at x = 1.

$$\lim_{x \to 1^-} f(x) \ge f(1)$$

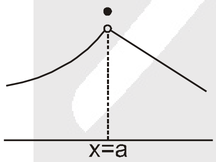
$$\Rightarrow \quad -1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \ge -1$$

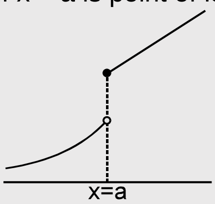
$$\Rightarrow \quad \frac{(b^2 + 1)(b - 1)}{(b + 1)\ (b + 2)} \ge 0$$

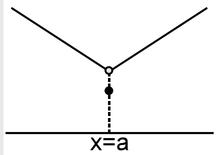
$$\Rightarrow \quad b \in (-2, -1) \cup [1, +\infty)$$

Self Practice Problems :

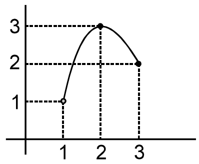
(19) In each of following graphs identify if x = a is point of local maxima, minima or neither

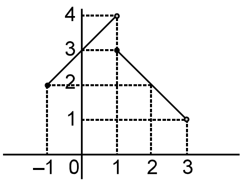
(i) 

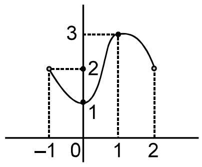
(ii) 

(iii) 

(20) Examine the graph of following functions in each case identify the points of global maximum/minimum and local maximum / minimum.

(i) 

(ii) 

(iii) 

Ans. (19) (i) Maxima (ii) Neither maxima nor minima
(iii) Minima

(20) (i) Local maxima at x = 2, Local minima at x = 3, Global maximum at x = 2. No global minimum
(ii) Local minima at x = – 1, No point of Global minimum, no point of local or Global maxima
(iii) Local & Global maximum at x = 1, Local & Global minimum at x = 0.

Maxima, Minima for differentiable functions :

Mere definition of maxima, minima becomes tedious in solving problems. We use derivative as a tool to overcome this difficulty.

A necessary condition for an extrema :

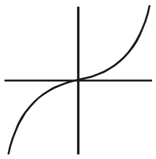
Let f(x) be differentiable at x = c.

Theorem : A necessary condition for f(c) to be an extremum of f(x) is that $f'(c) = 0$.

i.e. f(c) is extremum $\Rightarrow$ $f'(c) = 0$

Note : $f'(c) = 0$ is only a necessary condition but not sufficient

i.e. $f'(c) = 0$ f(c) is extremum.



Consider $f(x) = x^3$

$f'(0) = 0$

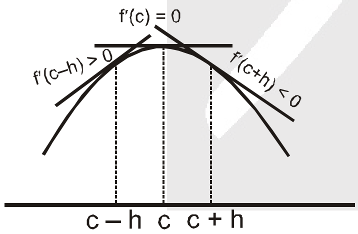
but f(0) is not an extremum (see figure).

Sufficient condition for an extrema :

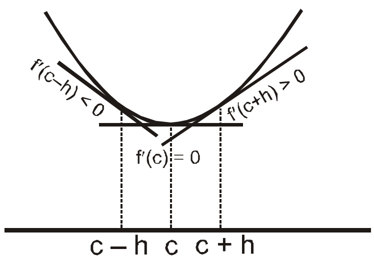
Let f(x) be a differentiable function.

Theorem : A sufficient condition for f(c) to be an extremum of f(x) is that $f'(x)$ changes sign as x passes through c.

i.e. f(c) is an extrema (see figure) $\Leftrightarrow$ $f'(x)$ changes sign as x passes through c.



x = c is a point of maxima. $f'(x)$ changes sign from positive to negative.



x = c is a point of local minima (see figure), $f'(x)$ changes sign from negative to positive.

Stationary points :

The points on graph of function f(x) where $f'(x) = 0$ are called stationary points.
Rate of change of f(x) is zero at a stationary point.

Example # 21 : Find stationary points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$.

Solution : $f'(x) = 12x^2 - 12x - 24$

$f'(x) = 0 \quad \Rightarrow \quad x = -1, 2$

$f(-1) = 23, f(2) = -31$

$(-1, 23), (2, -31)$ are stationary points

Example # 22 : If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at $x = -1$ and $x = 3$. Find a, b, c.

Solution. Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(3)$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 27 + 6a + b = 0$

$f'(-1) = 3 - 2a + b = 0$

$\Rightarrow \quad a = -3, b = -9, c \in R$

First Derivative Test :

Let f(x) be continuous and differentiable function.

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$, let $x = c$ be a solution. (i.e. Find stationary points)

Step - III. Observe change of sign

(i) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then $x = c$ is a point of local minima

(ii) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then $x = c$ is a point of local maxima.

(iii) If $f'(x)$ does not changes sign as x crosses c then $x = c$ is neither a point of maxima nor minima.

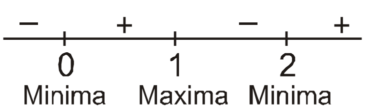
Example # 23 : Find the points of maxima or minima of $f(x) = x^2 (x - 2)^2$.

Solution. $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1) (x - 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 2$

examining the sign change of $f'(x)$



Hence $x = 1$ is point of maxima, $x = 0, 2$ are points of minima.

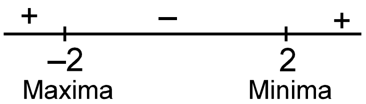
Note : In case of continuous functions points of maxima and minima are alternate.

Example # 24 : Find the points of maxima, minima of $f(x) = x^3 - 12x$. Also draw the graph of this functions.

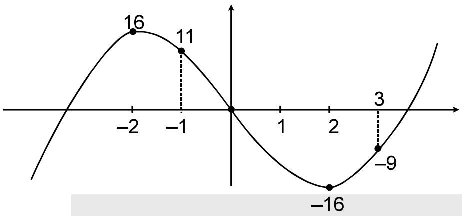
Solution. $f(x) = x^3 - 12x$

$f'(x) = 3(x^2 - 4) = 3(x - 2)(x + 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = \pm 2$



For tracing the graph let us find maximum and minimum values of f(x).



x	f(x)
2	–16
–2	+16

Example # 25 : Show that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not have any point of local maxima or minima. Hence draw graph

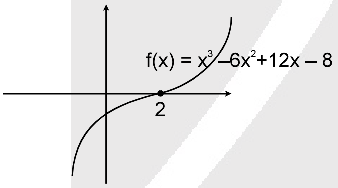
Solution. $f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3(x^2 - 4x + 4)$

$f'(x) = 3(x - 2)^2$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$

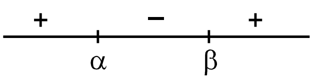
but clearly $f'(x)$ does not change sign about x = 2. $f'(2^+) > 0$ and $f'(2^-) > 0$. So f(x) has no point of maxima or minima. In fact f(x) is a monotonically increasing function for $x \in R$.



Example # 26 : Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) has positive point of maxima, then find possible values of 'a'.

Solution. $f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)]$

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.



Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \quad \Rightarrow \quad a < \frac{29}{7}$

(ii) $-\frac{b}{2a} > 0 \quad \Rightarrow \quad a < 7$

(iii) $f'(0) > 0 \quad \Rightarrow \quad a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii) $\quad \Rightarrow \quad a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

Self Practice Problems :

(21) Find the points of local maxima or minima of following functions
(i) $f(x) = (x - 1)^3 (x + 2)^2$
(ii) $f(x) = x^3 + x^2 + x + 1$.

Ans. (i) Maxima at $x = -2$, Minima at $x = -\frac{4}{5}$
(ii) No point of local maxima or minima.

Maxima, Minima for continuous functions :

Let f(x) be a continuous function.

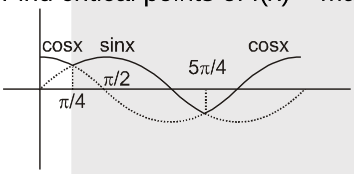
Critical points :

The points where $f'(x) = 0$ or f(x) is not differentiable are called critical points.

Stationary points $\subseteq$ Critical points.

Example # 27 : Find critical points of $f(x) = \max(\sin x, \cos x)\ \forall,\ x \in (0, 2\pi)$.

Solution :



From the figure it is clear that f(x) has three critical points $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$.

Important Note :

For f(x) defined on a subset of R, points of extrema (if exists) occur at critical points

Example # 28 : Find the possible points of Maxima/Minima for $f(x) = |x^2 - 2x|$ $(x \in R)$

Solution.
$$f(x) = \begin{cases} x^2 - 2x & x \geq 2 \\ 2x - x^2 & 0 < x < 2 \\ x^2 - 2x & x \leq 0 \end{cases}$$
$$f'(x) = \begin{cases} 2(x-1) & x > 2 \\ 2(1-x) & 0 < x < 2 \\ 2(x-1) & x < 0 \end{cases}$$
$f'(x) = 0$ at $x = 1$ and $f'(x)$ does not exist at $x = 0, 2$. Thus these are critical points.

Example # 29 : Let $f(x) = \begin{cases} x^3 + x^2 - 10x & x < 0 \\ 3\sin x & x \geq 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

Solution : f(x) is continuous at x = 0.
$$f'(x) = \begin{cases} 3x^2 + 2x - 10 & x < 0 \\ 3\cos x & x > 0 \end{cases}$$
$f'(0^+) = 3$ and $f'(0^-) = -10$ thus f(x) is non-differentiable at $x = 0$ $\Rightarrow$ $x = 0$ is a critical point.
Also derivative changes sign from negative to positive, so x = 0 is a point of local minima.

Example # 30 : Find the critical points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$ if (i) $x \in [0, 3]$ (ii) $x \in [-3, 3]$ (iii) $x \in [-1, 2]$.

Solution : $f'(x) = 12(x^2 - x - 2)$
$= 12(x - 2)(x + 1)$
$f'(x) = 0 \quad \Rightarrow \quad x = -1$ or 2
(i) if $x \in [0, 3]$, $x = 2$ is critical point.
(ii) if $x \in [-3, 3]$, then we have two critical points $x = -1, 2$.
(iii) If $x \in [-1, 2]$, then no critical point as both $x = -1$ and $x = 2$ become boundary points.

Note : Critical points are always interior points of an interval.

Global extrema for continuous functions :

(i) Function defined on closed interval

Let $f(x)$, $x \in [a, b]$ be a continuous function
Step - I : Find critical points. Let it be $c_1, c_2, \ldots, c_n$
Step - II : Find $f(a), f(c_1), \ldots, f(c_n), f(b)$
Let $M = \max \cdot \{f(a), f(c_1), \ldots, f(c_n), f(b)\}$
$m = \min \cdot \{f(a), f(c_1), \ldots f(c_n), f(b)\}$
Step - III : M is global maximum.
m is global minimum.

(ii) Function defined on open interval.
Let $f(x)$, $x \in (a, b)$ be continuous function.
Step - I Find critical points . Let it be $c_1, c_2, \ldots c_n$
Step - II Find $f(c_1), f(c_2), \ldots, f(c_n)$
Let $M = \max \cdot \{f(c_1), \ldots f(c_n)\}$
$m = \min \cdot \{f(c_1), \ldots, f(c_n)\}$
Step - III $\lim_{x \to a^+} f(x) = \ell_1$ (say), $\lim_{x \to b^-} f(x) = \ell_2$ (say).
Let $\ell = \min. \{\ell_1, \ell_2\}$, $L = \max. \{\ell_1, \ell_2\}$
Step - IV
(i) If $m \le \ell$ then m is global minimum
(ii) If $m > \ell$ then $f(x)$ has no global minimum
(iii) If $M \ge L$ then M is global maximum
(iv) If $M < L$, then $f(x)$ has no global maximum

Example # 31 : Find the greatest and least values of $f(x) = x^3 - 12x \quad x \in [-1, 3]$

Solution : The possible points of maxima/minima are critical points and the boundary points.
for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$
$x = 2$ is the only critical point.
Examining the value of $f(x)$ at points $x = -1, 2, 3$. We can find greatest and least values.

x	f(x)
–1	11
2	–16
3	–9

$\therefore$ Minimum $f(x) = -16$ & Maximum $f(x) = 11$.

Self Practice Problems :

(22) Let $f(x) = x^3 + x^2 - x - 4$

(i) Find the possible points of Maxima/Minima of f(x) for $x \in R$.

(ii) Find the number of critical points of f(x) for $x \in [1, 3]$.

(iii) Discuss absolute (global) maxima/minima value of f(x) for $x \in [-2, 2]$

(iv) Prove that for $x \in (1, 3)$, the function does not has a Global maximum.

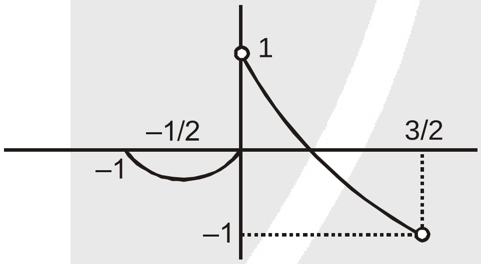
Ans. (i) $x = -1, \frac{1}{3}$ (ii) zero

(iii) f(–2) = –6 is global maximum, f(2) = 6 is global maximum

Example # 32 : Let $f(x) = \begin{cases} x^2 + x & ; & -1 \le x < 0 \\ \lambda & ; & x = 0 \\ \log_{1/2}\left(x + \frac{1}{2}\right) & ; & 0 < x < \frac{3}{2} \end{cases}$

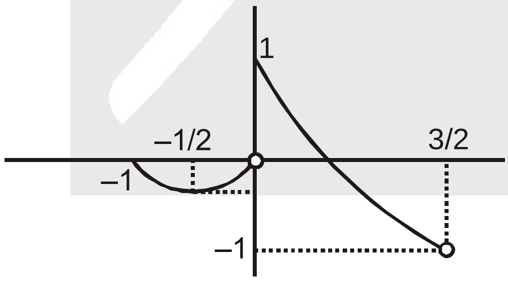
Discuss global maxima, minima for $\lambda = 0$ and $\lambda = 1$. For what values of λ does f(x) has global maxima

Solution : Graph of y = f(x) for $\lambda = 0$



No global maxima, minima

Graph of y = f(x) for $\lambda = 1$



Global maxima is 1, which occurs at x = 0

Global minima does not exists

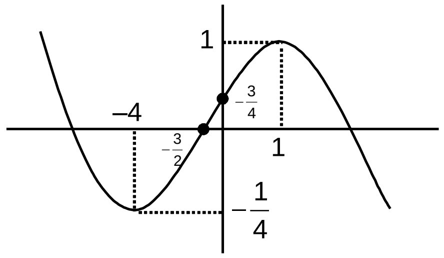
$\lim\limits_{x \to 0^-} f(x) = 0$, $\lim\limits_{x \to 0^+} f(x) = 1$, $f(0) = \lambda$

For global maxima to exists

$f(0) \ge 1 \quad \Rightarrow \quad \lambda \ge 1$.

Example # 33 : Find extrema of f(x) = $\frac{x^2+4}{2x+3}$. Draw graph of g(x) = $\frac{1}{f(x)}$ and comment on its local and global extrema.

Solution : $f'(x) = \frac{2(x^2+3x-4)}{(2x+3)^2} = \frac{2(x+4)(x-1)}{(2x+3)^2} = 0$



local minima occurs at x = –4

local maxima occurs at x = 1

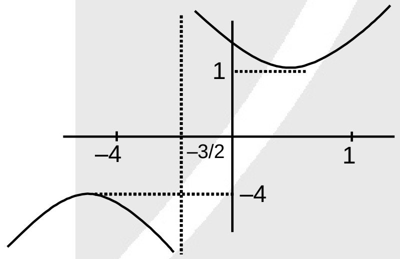
$g(x) = \frac{1}{f(x)} = \left(\frac{2x+3}{x^2+4}\right)$

$g'(x) = \frac{-2(x+4)(x-1)}{(x^2+4)^2}$

local maxima at x = –4

local minima at x = 1

global maxima & minima do not exists



Self Practice Problems :

(23) Let f(x) = x + $\frac{1}{x}$. Find local maximum and local minimum value of f(x). Can you explain this discrepancy of locally minimum value being greater than locally maximum value.

(24) If $f(x) = \begin{cases} (x+\lambda)^2 & x < 0 \\ \cos x & x \geq 0 \end{cases}$, find possible values of λ such that f(x) has local maxima at x = 0.

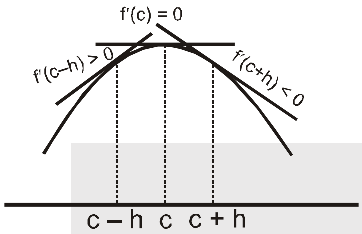
Answers : (23) Local maxima at x = –1, f(–1) = – 2 ; Local minima at x = 1, f(1) = 2.

(24) $\lambda \in [-1, 1)$

Maxima, Minima by higher order derivatives :

Second derivative test :

Let f(x) have derivatives up to second order
Step - I. Find $f'(x)$
Step - II. Solve $f'(x) = 0$. Let $x = c$ be a solution
Step - III. Find $f''(c)$
Step - IV.
(i) If $f''(c) = 0$ then further investigation is required
(ii) If $f''(c) > 0$ then $x = c$ is a point of minima.
(iii) If $f''(c) < 0$ then $x = c$ is a point of maxima.



For maxima $f'(x)$ changes from positive to negative (as shown in figure).
$\Rightarrow$ $f'(x)$ is decreasing hence $f''(c) < 0$

Example # 34 : Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

Solution : $f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0 \Rightarrow \cos 2x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\dfrac{\pi}{6}\right) < 0 \Rightarrow$ Maxima at $x = \dfrac{\pi}{6}$

$f''\left(\dfrac{5\pi}{6}\right) > 0 \Rightarrow$ Minima at $x = \dfrac{5\pi}{6}$

Self Practice Problems :

(25) Let $f(x) = \sin x\,(1 + \cos x)$; $x \in (0, 2\pi)$. Find the number of critical points of f(x). Also identify which of these critical points are points of Maxima/Minima.

Ans. Three

$x = \dfrac{\pi}{3}$ is point of maxima.

$x = \pi$ is not a point of extrema.

$x = \dfrac{5\pi}{3}$ is point of minima.

n^{th} Derivative test :

Let f(x) have derivatives up to nth order
If $f'(c) = f''(c) = ... = f^{n-1}(c) = 0$ and
$f^n(c) \neq 0$ then we have following possibilities
(i) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima
(ii) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.
(iii) n is odd, $f^{(n)}(c) < 0 \Rightarrow f(x)$ is decreasing about $x = c$
(iv) n is odd, $f^{(n)} > 0 \Rightarrow f(x)$ is increasing about $x = c$.

Example # 35 : Find points of local maxima or minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^2 (x - 1)(x - 3)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

Now, $f''(1) < 0 \quad \Rightarrow$ Maxima at $x = 1$

$f''(3) > 0 \quad \Rightarrow$ Minima at $x = 3$

and, $f''(0) = 0 \quad \Rightarrow$ II^{nd} derivative test fails

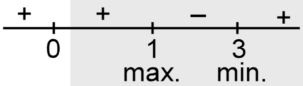
so, $f'''(x) = 30(2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at $x = 0$.

Note : It was very convenient to check maxima/minima at first step by examining the sign change of $f'(x)$ no sign change of $f'(x)$ at $x = 0$

$f'(x) = 5x^2 (x - 1)(x - 3)$



Application of Maxima, Minima :

For a given problem, an objective function can be constructed in terms of one parameter and then extremum value can be evaluated by equating the differential to zero. As discussed in n^{th} derivative test maxima/minima can be identified.

Useful Formulae of Mensuration to Remember :

Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

volume of cube = ℓ^3 , Total Surface area of cube = $6\ \ell^2$

volume of cuboid = ℓbh, Total Surface area of cube = $2(\ell b + bh + h\ell)$

3-D Figures	Volume	Total Surface area	Curved/lateral Surface area
Cone	$\frac{1}{3}\pi r^2 h$	$\pi r\ell + \pi r^2$	Curved Surface area = $\pi r\ell$
Cylinder	$\pi r^2 h$	$2\pi rh + 2\pi r^2$	Curved Surface area =$2\pi rh$
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$	
Prism	(area of base) × (height)	lateral surface area + 2 (area of base)	lateral Surface area = (perimeter of base) × (height)
Right Pyramid	$\frac{1}{3}$ ×(area of base) × (height)	Curved surface area + (area of base)	Curved Surface area = $\frac{1}{2}$ ×(perimeter of base) ×(slant height)

(Note that lateral surfaces of a prism are all rectangle).
(Note that slant surfaces of a pyramid are triangles).

Example # 36: If the equation $x^3 + px + q = 0$ has three real roots, then show that $4p^3 + 27q^2 < 0$.

Solution: $f(x) = x^3 + px + q$, $f'(x) = 3x^2 + p$

$\therefore$ f(x) must have one maximum > 0 and one minimum < 0. $f'(x) = 0$

$\Rightarrow$ $x = \pm\sqrt{\dfrac{-p}{3}}$, $p < 0$

f is maximum at $x = -\sqrt{\dfrac{-p}{3}}$ and minimum at $x = \sqrt{\dfrac{-p}{3}}$

$$f\left(-\sqrt{\frac{-p}{3}}\right) \ f\left(\sqrt{\frac{-p}{3}}\right) < 0$$

$$\left(q - \frac{2p}{3}\sqrt{\frac{-p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{-p}{3}}\right) < 0$$

$q^2 + \dfrac{4p^3}{27} < 0$, $4p^3 + 27q^2 < 0$.

Example # 37 : Find two positive numbers x and y such that x + y = 60 and xy^3 is maximum.

Solution : $x + y = 60$

$\Rightarrow$ $x = 60 - y$ $\Rightarrow$ $xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)\,y^3$; $y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2(60 - y) - y^3 = 0$

$f'(y) = y^2(180 - 4y) = 0$

$\Rightarrow$ $y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at y = 45.

So x = 15 and y = 45.

Example # 38 : Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

Solution : Let sides of rectangle be x and y (as shown in figure).

$\Rightarrow$ A = xy.

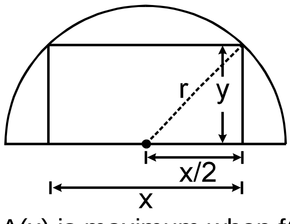
Here x and y are not independent variables and are related by Pythogorus theorem with r.

$\dfrac{x^2}{4} + y^2 = r^2$ $\Rightarrow$ $y = \sqrt{r^2 - \dfrac{x^2}{4}}$

$\Rightarrow$ $A(x) = x\sqrt{r^2 - \dfrac{x^2}{4}}$

$\Rightarrow$ $A(x) = \sqrt{x^2r^2 - \dfrac{x^4}{4}}$

Let $f(x) = r^2x^2 - \dfrac{x^4}{4}$; $x \in (0, r)$



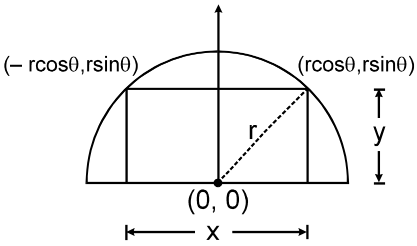
A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0$ $\Rightarrow$ $x = r\sqrt{2}$

also $f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \dfrac{r}{\sqrt{2}}$.

Aliter Let us choose coordinate system with origin as centre of circle (as shown in figure).



A = xy

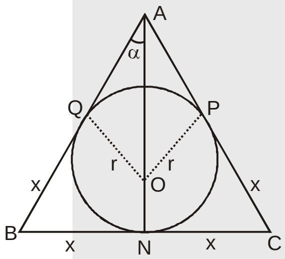
$\Rightarrow$ A = 2 (rcosθ) (rsinθ) $\Rightarrow$ A = r^2 sin2θ, $\theta \in \left(0, \frac{\pi}{2}\right)$

Clearly A is maximum when $\theta = \frac{\pi}{4}$ $\Rightarrow$ $x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Example # 39. Show that the least perimeter of an isosceles triangle circumscribed about a circle of radius 'r' is $6\sqrt{3}$ r .

Solution : AQ = r cot α = AP

AO = r cosecα



$\frac{x}{AO + ON}$ = tanα

x = (r cosecα + r) tanα

x = r(secα + tanα)

Perimeter = p = 4x + 2AQ

p = 4r(secα + tanα) + 2rcotα

p = r(4secα + 4tanα + 2cotα)

$\frac{dp}{d\alpha}$ = r[4secα tanα + 4sec^2 α – 2cosec^{2}α]

for max or min $\frac{dp}{d\alpha} = 0 \Rightarrow$ 2sin^3 α + 3sin^{2}α – 1 = 0

$\Rightarrow$ (sinα + 1) (2sin^{2}α + sinα – 1) = 0

(sinα + 1)^2 (2sinα – 1) = 0 $\Rightarrow$ sinα = 1/2 $\Rightarrow$ α = 30° = π/6

$p_{least} = r\left[\frac{4.2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + 2\sqrt{3}\right] = r\left[\frac{8+4+6}{\sqrt{3}}\right] = \frac{(6\sqrt{3}\sqrt{3})}{\sqrt{3}}$ r = $6\sqrt{3}$ r

Example # 40 : Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

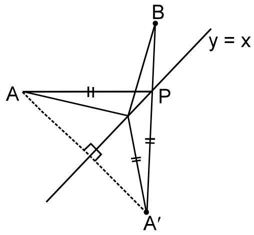
Solution. Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB)

Let A′ be the mirror image of A in the line y = x (see figure).

F(P) = PA + PB

F(P) = PA′ + PB

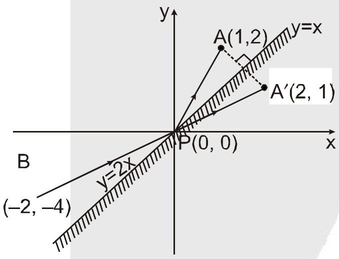
But for ΔPA′B



PA′ + PB $\geq$ A′B and equality hold when P, A′ and B becomes collinear. Thus for minimum path length point P is that special point for which PA and PB become incident and reflected rays with respect to the mirror y = x.

Equation of line joining A′ and B is y = 2x intersection of this line with y = x is the point P.

Hence P $\equiv$ (0, 0).



Note : Above concept is very useful because such problems become very lengthy by making perimeter as a function of position of P and then minimizing it.

Self Practice Problems :

(26) Find the two positive numbers x and y whose sum is 35 and the product $x^2 y^5$ maximum.

(27) A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the slops to form a box. What should be the side of the square to be cut off such that volume of the box is maximum possible.

(28) Prove that a right circular cylinder of given surface area and maximum volume is such that the height is equal to the diameter of the base.

(29) A normal is drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Find the maximum distance of this normal from the centre.

(30) A line is drawn passing through point P(1, 2) to cut positive coordinate axes at A and B. Find minimum area of ΔPAB.

(31) Two towns A and B are situated on the same side of a straight road at distances a and b respectively perpendiculars drawn from A and B meet the road at point C and D respectively. The distance between C and D is c. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find position of P.

Ans. (26) x = 25, y = 10. (27) 3 cm (29) 1 unit

(30) 4 units (31) P is at distance of $\frac{ac}{a+b}$ from C.

Use of monotonicity for proving inequalities

Comparison of two functions f(x) and g(x) can be done by analysing the monotonic behaviour of h(x) = f(x) – g(x)

Example # 41 : For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin^2 x < x^2 < \tan^2 x$

Solution : Let $f(x) = x^2 - \sin^2 x$

$= (x + \sin x)\ (x - \sin x) > 0$

for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin^2 x < x^2$ .....(i)

Let $g(x) = x^2 - \tan^2 x$

$= (x + \tan x)\ (x + \tan x) < 0$

for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow x^2 < \tan^2 x$ .....(ii)

From (i) & (ii) $\sin^2 x < x^2 < \tan^2 x$

Example # 42 : For $x \in (0, 1)$ prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ hence or otherwise find $\lim\limits_{x \to 0} \left[\frac{\tan^{-1} x}{x}\right]$

Solution : Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$ $f'(x) = 1 - x^2 - \frac{1}{1+x^2}$ $f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1) \Rightarrow$ f(x) is M.D.

$\Rightarrow f(x) < f(0) \Rightarrow x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow x - \frac{x^3}{3} < \tan^{-1}x$ ....(i)

Similarly $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$,$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$ $g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1) \Rightarrow$ g(x) is M.I.

$\Rightarrow g(x) > g(0)$ $x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x$ ....(ii)

from (i) and (ii), we get $x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$ Hence Proved

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1} x}{x} < 1 - \frac{x^2}{6}$, for x > 0

Hence by sandwich theorem we can prove that $\lim\limits_{x \to 0} \frac{\tan^{-1} x}{x} = 1$ but it must also be noted that as $x \to 0$, value of $\frac{\tan^{-1} x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1} x}{x} < 1$

$\Rightarrow \lim\limits_{x \to 0} \left[\frac{\tan^{-1} x}{x}\right] = 0$

NOTE : In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality occur at end point of the interval or will be easily predicted by hit and trial.

Example # 43 : For $x \in \left(0, \frac{\pi}{2}\right)$, prove that $\sin x > x - \frac{x^3}{6}$

Solution : Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

we cannot decide at this point whether $f'(x)$ is positive or negative, hence let us check for monotonic nature of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is M.I. for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) > f'(0) \quad \Rightarrow f'(x) > 0$

$\Rightarrow f(x)$ is M.I. $\quad \Rightarrow f(x) > f(0)$

$\Rightarrow \sin x - x + \frac{x^3}{6} > 0 \quad \Rightarrow \sin x > x - \frac{x^3}{6}$. Hence proved

Example # 44 : Examine which is greater : sin x tan x or x^2. Hence evaluate $\lim_{x \to 0} \left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Solution : Let $f(x) = \sin x \tan x - x^2$

$f'(x) = \cos x \,.\, \tan x + \sin x \,.\, \sec^2 x - 2x$

$\Rightarrow f'(x) = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow f''(x) = \cos x + \cos x \sec^2 x + 2\sec^2 x \sin x \tan x - 2$

$\Rightarrow f''(x) = (\cos x + \sec x - 2) + 2 \sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$ and $2 \sec^2 x \tan x \,.\, \sin x > 0$ because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f''(x) > 0 \quad \Rightarrow f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

$\Rightarrow f'(x) > 0 \quad \Rightarrow f(x)$ is M.I. $\quad \Rightarrow f(x) > 0$

$\Rightarrow \sin x \tan x - x^2 > 0$

Hence $\sin x \tan x > x^2 \Rightarrow \frac{\sin x \tan x}{x^2} > 1 \Rightarrow \lim_{x \to 0} \left[\frac{\sin x \tan x}{x^2}\right] = 1.$

Example # 45 : Prove that f(x) = $\left(1+\frac{1}{x}\right)^x$ is monotonically increasing in its domain. Hence or otherwise draw graph of f(x) and find its range

Solution : f(x) = $\left(1+\frac{1}{x}\right)^x$, for Domain of f(x), 1 + $\frac{1}{x}$ > 0

$\Rightarrow \quad \frac{x+1}{x} > 0 \quad \Rightarrow \quad (-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}\right]$

$\Rightarrow \quad f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of f′(x) depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and

So lets assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2} + \frac{1}{(x+1)^2} \quad \Rightarrow \quad g'(x) = \frac{-1}{x(x+1)^2}$

(i) for $x \in (0, \infty)$, $g'(x) < 0$ $\Rightarrow$ g(x) is M.D. for $x \in (0, \infty)$
$g(x) > \lim\limits_{x\to\infty} g(x)$
$g(x) > 0$. and since $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

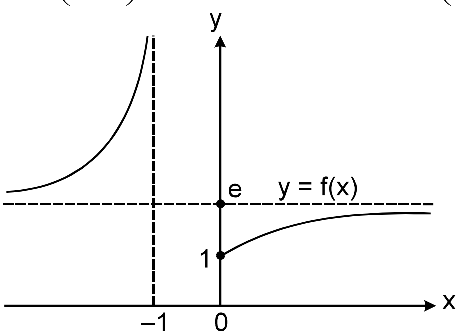
(ii) for $x \in (-\infty, -1)$, $g'(x) > 0$ $\Rightarrow$ g(x) is M.I. for $x \in (-\infty, -1)$
$\Rightarrow$ $g(x) > \lim\limits_{x\to-\infty} g(x)$ $\Rightarrow$ $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

Hence from (i) and (ii) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$
$\Rightarrow$ f(x) is M.I. in its Domain

For drawing the graph of f(x), its important to find the value of f(x) at boundary points
i.e. $\pm\infty$, 0, –1

$\lim\limits_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$

$\lim\limits_{x\to0^+}\left(1+\frac{1}{x}\right)^x = 1$ and $\lim\limits_{x\to-1}\left(1+\frac{1}{x}\right)^x = \infty$



so the graph of f(x) is

Range is $y \in (1, \infty) - \{e\}$

Example # 46 : Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

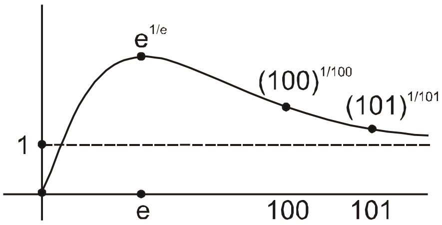
Solution : Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x)

$$f'(x) = x^{1/x} \cdot \left(\frac{1-\ell nx}{x^2}\right)$$

$f'(x) > 0 \quad \Rightarrow \quad x \in (0,e)$

and $f'(x) < 0 \quad \Rightarrow \quad x \in (e,\infty)$

Hence f(x) is M.D. for $x \geq e$



and since 100 < 101

$\Rightarrow \quad f(100) > f(101)$

$\Rightarrow \quad (100)^{1/100} > (101)^{1/101}$

Self Practice Problems :

(32) Prove the following inequalities

(i) $x > \tan^{-1}(x)$ for $x \in (0, \infty)$

(ii) $e^x > x + 1$ for $x \in (0, \infty)$

(iii) $\frac{x}{1+x} \leq \ell n\,(1 + x) \leq x$ for $x \in (0, \infty)$

Rolle's Theorem :

If a function f defined on [a, b] is

(i) continuous on [a, b]

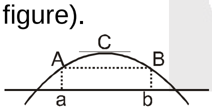
(ii) derivable on (a, b) and

(iii) f(a) = f(b),

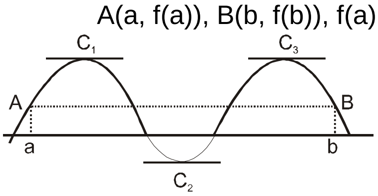
then there exists at least one real number c between a and b (a < c < b) such that $f'(c) = 0$

Geometrical Explanation of Rolle's Theorem :

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn (as shown in figure).



A(a, f(a)), B(b, f(b)), f(a) = f(b), C(c, f(c)), $f'(c) = 0$.



$C_1\ (c_1, f(c_1)),\ f'(c_1) = 0$

$C_2\ (c_2, f(c_2)),\ f'(c_2) = 0$

$C_3\ (c_3, f(c_3)),\ f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of f(x), there exists at least one point where the tangent is parallel to x-axis.

Algebraic Interpretation of Rolle's Theorem :

Between two zeros a and b of f(x) (i.e. between two roots a and b of f(x) = 0) there exists at least one zero of f′(x)

Example # 47 : If 2a + 3b + 6c = 0 then prove that the equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Solution : Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

$f(0) = 0$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c = 2a + 3b + 6c = 0$

If f(0) = f(1) then f′(x) = 0 for some value of $x \in (0, 1)$

$\Rightarrow$ $ax^2 + bx + c = 0$ for at least one $x \in (0, 1)$

Self Practice Problems :

(33) If f(x) satisfies condition in Rolle's theorem then show that between two consecutive zeros of f′(x) there lies at most one zero of f(x).

(34) Show that for any real numbers λ, the polynomial $P(x) = x^7 + x^3 + \lambda$, has exactly one real root.

Lagrange's Mean Value Theorem (LMVT) :

If a function f defined on [a, b] is

(i) continuous on [a, b] and

(ii) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

Proof : Let us consider a function $g(x) = f(x) + \lambda x$, $x \in [a, b]$

where λ is a constant to b determined such that g(a) = g(b).

$\therefore$ $\lambda = -\frac{f(b) - f(a)}{b - a}$

Now the function g(x), being the sum of two continuous and derivable functions it self

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) g(a) = g(b).

Therefore, by Rolle's theorem there exists a real number $c \in (a, b)$ such that $g'(c) = 0$

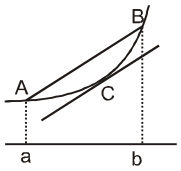
But $g'(x) = f'(x) + \lambda$

$\therefore$ $0 = g'(c) = f'(c) + \lambda$

$f'(c) = -\lambda = \frac{f(b) - f(a)}{b - a}$

Geometrical Interpretation of LMVT :

The theorem simply states that between two points A and B of the graph of f(x) there exists at least one point where tangent is parallel to chord AB.



C(c, f(c)), $f'(c)$ = slope of AB.

Alternative Statement : If in the statement of LMVT, b is replaced by a + h, then number c between a and b may be written as $a + \theta h$, where $0 < \theta < 1$. Thus

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h) \quad \text{or} \quad f(a+h) = f(a) + hf'(a+\theta h),\ 0<\theta<1$$

Example # 48 : Verify LMVT for $f(x) = -x^2 + 4x - 5$ and $x \in [-1, 1]$

Solution : $f(1) = -2$; $f(-1) = -10$

$$\Rightarrow \quad f'(c) = \frac{f(1)-f(-1)}{1-(-1)} \quad \Rightarrow \quad -2c + 4 = 4 \quad \Rightarrow \quad c = 0$$

Example # 49 : Using Lagrange's mean value theorem, prove that if b > a > 0,

then $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

Solution : Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$f'(c) = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$ for a < c < b and $f'(x) = \frac{1}{1+x^2}$,

Now $f'(x)$ is a monotonically decreasing function

Hence if a < c < b

$\Rightarrow \quad f'(b) < f'(c) < f'(a)$

$\Rightarrow \quad \frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$ Hence proved

Example # 50 : Let f : R → R be a twice differentiable function such that $f\left(\frac{\pi}{4}\right) = 0$, $f\left(\frac{5\pi}{4}\right) = 0$ & f(3) = 4 then show that there exists a $c \in (0, 2\pi)$ such that $f''(c) + \sin c - \cos c < 0$.

Solution : Consider g(x) = f(x) – sinx + cosx

$\Rightarrow \quad g'(x) = f'(x) - \cos x - \sin x$

$\Rightarrow \quad g''(x) = f''(x) + \sin x - \cos x$

By LMVT

$$\frac{g(3) - g\left(\frac{\pi}{4}\right)}{3 - \frac{\pi}{4}} = g'(c_1), \quad \frac{\pi}{4} < c_1 < 3 \qquad \text{and} \qquad \frac{g\left(\frac{5\pi}{4}\right) - g(3)}{\frac{5\pi}{4} - 3} = g'(c_2), \qquad 3 < c_2 < \frac{\pi}{4}$$

$g'(c_1) > 0$, $g'(c_2) < 0$

By LMVT

$$\frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = g''(c), \qquad c_1 < c < c_2 \qquad \Rightarrow g''(c) < 0 \qquad \Rightarrow f''(c) + \sin c - \cos c < 0$$

for some $c \in (c_1, c_2)$, $c \in (0, 2\pi)$

Self Practice Problems :

(35) Using LMVT, prove that if two functions have equal derivatives at all points of (a, b), then they differ by a constant

(36) If a function f is

(i) continuous on [a, b],

(ii) derivable on (a, b) and

(iii) $f'(x) > 0$, $x \in (a, b)$, then show that f(x) is strictly increasing on [a, b].