High Level Problems (HLP)

1. A particle moving on a curve has the position at time t given by x = f'(t) sin t + f''(t) cos t, y = f'(t) cos t - f''(t) sin t, where f is a thrice differentiable function. Then prove that the velocity of the particle at time t is f'(t) + f'''(t).

2. Find the interval in which $f(x) = x\sqrt{4ax - x^2}$ (a < 0) is decreasing

3. f : [0, 4] → R is a differentiable function. Then prove that for some a, b ∈ (0, 4), $f^2(4) - f^2(0) = 8f'(a) \cdot f(b)$

4. If all the extreme value of function $f(x) = a^2x^3 - \frac{a}{2}x^2 - 2x - b$ are positive and the minimum is at the point $x_0 = \frac{1}{3}$ then show that when a = –2 ⇒ $b < \frac{-11}{27}$ and when a = 3 ⇒ $b < -\frac{1}{2}$

5. If $f(x) = \begin{cases} 3 + |x - k|, & x \le k \\ a^2 - 2 + \dfrac{\sin(x-k)}{x-k}, & x > k \end{cases}$ has minimum at x = k, then show that |a| > 2

6. The equation $x^3 - 3x + [a] = 0$, where [.] denotes the greatest integer function, will have three real and distinct roots then find the set of all posible values of a.

7. Let $f(x) = \sin\frac{\{x\}}{a} + \cos\frac{\{x\}}{a}$ where a > 0 and { . } denotes the fractional part function. Then find the set of values of 'a' for which f can attain its maximum values.

8. Find the values of the parameter 'k' for which the equation $x^4 + 4x^3 - 8x^2 + k = 0$ has all roots real.

Comprehension (Q. No. 9 to 11)

A function f(x) having the following properties;

(i) f(x) is continuous except at x = 3

(ii) f(x) is differentiable except at x = – 2 and x = 3

(iii) f(0) = 0, $\lim_{x \to 3} f(x) \to -\infty$, $\lim_{x \to -\infty} f(x) = 3$, $\lim_{x \to \infty} f(x) = 0$

(iv) $f'(x) > 0 \ \forall x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \le 0 \ \forall x \in (-2, 3)$

(v) $f''(x) > 0 \ \forall x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0 \ \forall x \in (0, 3) \cup (3, \infty)$

then answer the following questions

9. Find the Maximum possible number of solutions of f(x) = | x |

10. Show that graph of function y = f (– | x |) is continuous but not differentiable at two points, if f ′(0) = 0

11. Show that $f(x) + 3x = 0$ has five solutions if $f'(0) > -3$ and $f(-2) > 6$

12. Let $F(x) = (f(x))^2 + (f'(x))^2$, $F(0) = 7$, where $f(x)$ is thrice differentiable function such that $|f(x)| \le 1 \ \forall \ x \in [-1, 1]$, then prove the followings.
(i) there is atleast one point in each of the intervals $(-1, 0)$ and $(0, 1)$ where $|f'(x)| \le 2$
(ii) there is atleast one point in each of the intervals $(-1, 0)$ and $(0, 1)$ where $F(x) \le 5$
(iii) there exits atleast one maxima of $F(x)$ in $(-1, 1)$
(iv) for some $c \in (-1, 1)$, $F(c) \ge 7$, $F'(c) = 0$ and $F''(c) \le 0$

13. A figure is bounded by the curves, $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$. At what point (a, b), a tangent should be drawn to the curve $y = x^2 + 1$ for it to cut off a trapezium of the greatest area from the figure.

14. If $y = \frac{ax+b}{(x-1)(x-4)}$ has a turning value at $(2, -1)$ find a and b, show that the turning value is a maximum.

15. With the usual meaning for a, b, c and s, if Δ be the area of a triangle, prove that the error in Δ resulting from a small error in the measurement of c, is given by

$$d\Delta = \frac{\Delta}{4}\left\{\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right\} dc$$

16. Find the possible values of 'a' such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution

17. If $(m - 1) a_1^2 - 2m a_2 < 0$, then prove that $x^m + a_1 x^{m-1} + a_2 x^{m-2} + ... + a_{m-1} x + a_0 = 0$ has at least one non real root $(a_1, a_2, ..., a_m \in R)$

18. If $f'(x) > 0$, $f''(x) > 0 \ \forall \ x \in (0, 1)$ and $f(0) = 0$, $f(1) = 1$, then prove that $f(x) f^{-1}(x) < x^2 \ \forall \ x \in (0, 1)$

19. Find the interval of increasing and decreasing for the function $g(x) = 2f\left(\frac{x^2}{2}\right) + f\left(\frac{27}{2} - x^2\right)$, where $f''(x) < 0$ for all $x \in R$.

20. Using calculus prove that $H.M \le G.M. \le A.M$ for positive real numbers.

21. Prove the following inequalities
(i) $1 + x^2 > (x \sin x + \cos x)$ for $x \in [0, \infty)$.
(ii) $\sin x - \sin 2x \le 2x$ for all $x \in \left[0, \frac{\pi}{3}\right]$
(iii) $\frac{x^2}{2} + 2x + 3 \ge (3 - x)e^x$ for all $x \ge 0$
(iv) $0 < x \sin x - \frac{\sin^2 x}{2} < \frac{1}{2}(\pi - 1)$ for $0 < x < \frac{\pi}{2}$

22. Find the interval to which b may belong so that the function f(x) = $\left(1-\frac{\sqrt{21-4b-b^2}}{b+1}\right)$ $x^3 + 5x + \sqrt{6}$ is increasing at every point of its domain.

23. If 0 < x < 1 prove that $y = x\,\ell n\,x - \frac{x^2}{2} + \frac{1}{2}$ is a function such that $\frac{d^2y}{dx^2} > 0$. Deduce that $x\,\ell n\,x > \frac{x^2}{2} - \frac{1}{2}$.

24. Find positive real numbers 'a' and 'b' such that $f(x) = ax - bx^3$ has four extrema on [–1 , 1] at each of which | f(x) | = 1

25. For any acute angled ΔABC , find the maximum value of $\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}$

26. Suppose p,q,r,s are fixed real numbers such that a quadrilateral can be formed with sides p,q,r,s in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle .

27. Find the minimum value of $f(x) = 8^x + 8^{-x} - 4(4^x + 4^{-x})$, $\forall x \in R$

28. Using calculus , prove that $\log_2 3 > \log_3 5 > \log_4 7$.

29. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length ℓ of the median drawn to its lateral side .

30. A tangent to the curve $y = 1 - x^2$ is drawn so that the abscissa x_0 of the point of tangency belongs to the interval (0, 1]. The tangent at x_0 meets the x–axis and y–axis at A & B respectively. Then find the minimum area of the triangle OAB, where O is the origin

31. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and keeping the cut edges of the remaining piece together. Then find the maximum volume attainable for the cone

32. Suppose velocity of waves of wave length λ in the Atlantic ocean is k $\sqrt{\left\{\left(\frac{\lambda}{a}\right)+\left(\frac{a}{\lambda}\right)\right\}}$, where k and a are constants. Show that minimum velocity attained by the waves is independent of the constant a.

33. Find the minimum distance of origin from the curve $ax^2 + 2bxy + ay^2 = c$ where a > b > c > 0

34. Prove that $e^x + \sqrt{1+e^{2x}} \geq (1 + x) + \sqrt{2+2x+x^2}$ $\forall x \in R$

35. Find which of the two is larger $\ell n(1 + x)$ or $\frac{\tan^{-1}x}{1+x}$.

36. Let f′ (sinx) < 0 and f′′ (sin x) > 0, $\forall$ x $\in$ $\left(0, \frac{\pi}{2}\right)$ and g(x) = f(sin x) + f(cos x), then find the intervals of monotonicity of g(x).

37. If f(x) = $(2013)x^{2012} - (2012)x^{2011} - 2014x + 1007$, then show that for x $\in$ $[0, 1007^{1/2011}]$, f(x) = 0 has at least one real root.

39. A function f is differentiable in the interval 0 ≤ x ≤ 5 such that f(0) = 4 & f(5) = – 1. If g(x) = $\frac{f(x)}{x+1}$, then prove that there exists some c $\in$ (0, 5) such that g′(c) = $-\frac{5}{6}$.

39. Let f(x) and g(x) be differentiable functions having no common zeros so that f(x) g′(x) ≠ f′(x) g(x). Prove that between any two zeros of f(x), there exist atleast one zero of g(x).

40. If ϕ(x) is a differentiable function x $\in$ R and a $\in$ R^+ such that $\phi(0) = \phi(2a)$, $\phi(a) = \phi(3a)$ and $\phi(0) \neq \phi(a)$ then show that there is at least one root of equation $\phi'(x + a) = \phi'(x)$ in (0, 2a)

41. Find the set of values of the parameter 'a' for which the function ;
f(x) = 8ax – a sin 6x – 7x – sin 5x increases & has no critical points for all x $\in$ R, is

42. Let h be a twice differentiable positive function on an open interval J. Let
g(x) = ℓn (h(x)) $\forall$ x $\in$ J
Suppose $(h'(x))^2 > h''(x)\, h(x)$ for each x $\in$ J. Then prove that g is concave downward on J.

43. If the complete set of value(s) of 'a' for which the function f (x) = $\frac{ax^3}{3}$ + (a + 2) x^2 + (a – 1) x + 2 possess a negative point of inflection is $(-\infty, \alpha) \cup (\beta, \infty)$, then $|\alpha| + |\beta|$ is :

44. If two curves y = 2sin$\frac{5\pi}{6}$ x and y = $\alpha x^2 - 3\alpha x + 2\alpha + 1$ touch each other at some point then the value of $\frac{\sqrt{3}\alpha}{5\pi}$ is $\left(0 \leq x \leq \frac{18}{5}\right)$

45. The maximum distance of the point (k, 0) from the curve $2x^2 + y^2 - 2x = 0$ is equal to

46_. Let f(x) = $px^3 + qx^2 + qx + p$; where 3p + 2q < 0, 7p + 3q > 0 then prove that
(i) Equation $3px^2 + 2qx + q = 0$ has at least one root lie between (–1, 1/2)
(ii) Equation $3px^2 + 2qx + q = 0$ has at least one root lie between (1/3, 3)
(iii) Equation $3px^4 + 4qx^3 + 6qx^2 + 12px = 0$ has two root lie in (–1, 1)
(iv) Equation $3px^4 + 4qx^3 + 6qx^2 + 12px = 0$ has one root less then –1 and one root greater than 1

47_. Let f, g : [0, 1] → R are differentiable functions with f(0) ≠ g(0). f and g has exactly one point of local minimum in (0, 1) and have same minimum value but different point of minimum. h(x) = f(x) – g(x). f({x}) and g({x}) are differentiable function $\forall$ x $\in$ R {where {.} denotes G.I.F.}. Prove that
(i) (f({x}))′ = 0 has atleast 3 solution in (0, 2)
(ii) (h({x}))′ = 0 has atleast 3 solution in (0, 2)

Answers

2. $[4a, 3a]$

6. $a \in [-1, 2)$

7. $\left(0, \frac{4}{\pi}\right)$

8. $k \in [0,3]$

9. 3

13. $\left(\frac{1}{2}, \frac{5}{4}\right)$

14. $a = 1, b = 0$

16. $a \in \left(-\frac{13}{4}, 3\right)$

18. If $f'(x) > 0, f''(x) > 0 \ \forall\ x \in (0, 1)$ and $f(0) = 0, f(1) = 1$, then prove that $f(x) f^{-1}(x) < x^2 \ \forall\ x \in (0, 1)$

19. g(x) is increasing if $x \in (-\infty, 3] \cup [0, 3]$
g(x) is decreasing if $x \in [-3, 0] \cup [3, \infty)$

22. $[-7, -1) \cup [2, 3]$

24. $a = 3$, $b = 4$

25. $\frac{9\sqrt{3}}{2\pi}$

27. -10

29. $\cos A = 0.8$

30. $\frac{4\sqrt{3}}{9}$

31. $2\pi/3$

33. $\sqrt{\frac{c}{a+b}}$

35. $\ell n(1 + x)$

36. Increasing when $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, decreasing when $x \in \left(0, \frac{\pi}{4}\right)$.

41. $a \in (6, \infty)$

43. 2

44. 1/2

45. $\begin{cases} \sqrt{2k^2 - 2k + 1} & k \in [0, 1) \\ 1 - k & k < 0 \\ k & k \geq 1 \end{cases}$