HINTS & SOLUTIONS

TOPIC : APPLICATION OF DERIVATIVES

EXERCISE # 1

PART–1

Section (A)

A-1. **(i)** $\frac{dy}{dx} = 6x + 4 \quad \Rightarrow \left.\frac{dy}{dx}\right]_{(0,5)} = 4$

Equation of tangent is $\frac{y-5}{x-0} = 4 \Rightarrow y = 4x + 5$

(ii) $\frac{dy}{dx}$ at $(1,1) = -\left(\frac{2x+3y}{3x+2y}\right)_{(1,1)} = -1$

$\Rightarrow$ equation of tangent is $(y–1) = –(x–1)$ and equation of normal is $(y–1) = (x–1)$

(iii) At $\quad t = \frac{1}{2}$, the value of $x = \frac{2a}{5}$ and $y = \frac{a}{5}$

Also $\frac{dx}{dt} = \frac{4at}{(1+t^2)^2}$ and $\frac{dy}{dt} = \frac{2at^2(3+t^2)}{(1+t^2)^2}$

$\frac{dy}{dx} = \frac{t}{2}(3+t^2) \quad \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = \frac{13}{16}$

equation of tangent is $\left(y - \frac{a}{5}\right) = \frac{13}{16}\left(x - \frac{2a}{5}\right) \quad \Rightarrow 13x – 16y = 2a$

equation of normal is $\left(y - \frac{a}{5}\right)\frac{13}{16} + x - \frac{2a}{5} = 0 \quad \Rightarrow 16x + 13y = 9a$

(iv) $\frac{dy}{dx}$ at $(0, 0) = \lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^2 \sin 1/h - 0}{h}$

$= \lim\limits_{h\to 0} (h \sin (1/h)) = 0 \quad \Rightarrow$ equation of tangent is $\frac{y-0}{x-0} = 0 \Rightarrow y = 0$

A-2 $y^2 – 2x^2 – 4y + 8 = 0$

$2y\frac{dy}{dx} - 4x - 4\frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{4x}{2y-4}$

$\left(\frac{dy}{dx}\right)_{(h,\ k)} = \frac{4h}{2k-4}$

Equation of tangent is $\quad (y – k) = \frac{4h}{(2k-4)}(x – h).$

It passes through (1, 2)

$(2 – k)(2k – 4) = 4h (1 – h)$ $\quad$ or, $\quad 4k – 2k^2 – 8 + 4k = 4h – 4h^2$

or, $\quad 4k – k^2 – 4 = 2h – 2h^2$ $\quad$ or, $\quad –2h^2 + k^2 + 2h – 4k + 4 = 0$

$\because \quad k^2 – 2h^2 – 4k + 8 = 0$

$\therefore \quad 2h – 4 = 0 \quad$ or $\quad h = 2 \quad \Rightarrow \quad k = 0$ or 4

Equation of tangent at (2,0); $y = \frac{8}{(-4)}(x – 2)$

or $\quad y = –2x + 4 \quad$ or $\quad y + 2x = 4$

and $\quad$ equation of tangent at (2, 4) is $y = 2x$

(ii) Let point Q be $\left(h, \frac{h^2}{4}\right)$ and point P be the point of contact on the curve.

Now, m_{PQ} = slope of the normal at Q.

Differentiating $x^2 = 4y$ w.r.t we get $2x = 4\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{x}{2}$

or Slope of the normals at Q = $\left.\frac{dy}{dy}\right|_{x=h} = -\frac{2}{h}$

or $\frac{\frac{h^2}{4} - 2}{h - 1} = -\frac{2}{h}$ [From (1)]

or $\frac{h^3}{4} - 2h = -2h + 2$ or $h^3 = 8$ or $h = 2$

Hence, the coordinates of point Q are (2, 1) and so the equation of the required normal becomes $x + y = 3$

A-3. **(i)** Slope of normal equal to –1.

$\Rightarrow \frac{dy}{dx} = \frac{x^2}{6y} = 1$ $x^2 = 6y$ and $9y^2 = x^3$ $\Rightarrow x = 0, 4 \Rightarrow$ point is $\left(4, \frac{8}{3}\right)$

(ii)✍ Differentiating equation of curve w.r.t. x, $2y\frac{dy}{dx} = (2 - x)^2 + 2x(2 - x)(-1)$ $\left.\frac{dy}{dx}\right|_{(1,\ 1)} = \frac{1+(-2)}{2} = -\frac{1}{2}$

Equation of tangent is $(y - 1) = -\frac{1}{2}(x - 1)$ or $2y + x = 3$.

Solving the equations of tangent and curve:

$y^2 = (-2y + 3)(2 - 3 + 2y)^2$ or $y^2 = (3 - 2y)(2y - 1)^2$

or $y^2 = (3 - 2y)(4y^2 + 1 - 4y)$ or $y^2 = 12y^2 + 3 - 12y - 8y^3 - 2y + 8y^2$

or $8y^3 - 19y^2 + 14y - 3 = 0$ or $(y - 1)(8y^2 - 11y + 3) = 0$

or $(y - 1)(8y - 3)(y - 1) = 0$ or $y = 1, 3/8$

P(9/4, 3/8)

(iii) Differentiating, $25x^4 - 30x^2 + 1 + 2y' = 0$

At P(0, – 3), $y' = -\frac{1}{2}$

The normal at P is $y + 3 = 2x$

Eliminating y with the given equation $x(x^2 - 1)^2 = 0 \rightarrow x = 0, 1, -1$

The line is tangent at (1, – 1) and (– 1, – 5).

A-4.✍ **(i)** Slope of $y = x^3$ at (h, k) is $3h^2$

Slope of $y = x^3$ at (h, k) is $3h^2$

$\Rightarrow$ equation of tangent is $= \frac{y-k}{x-h} = 3h^2$

$\Rightarrow y = 3h^2x - 3h^3 + k \Rightarrow y = 3h^2x - 2h^3 \Rightarrow (-2h^3)^2 = 1(3h^2)^2 + 112$

$\Rightarrow 4h^6 - 9h^4 - 112 = 0$

$(h^2 - 4)(4h^4 + 7h^2 + 28) = 0 \Rightarrow h = 2, -2$

Tangent are $y = 12x - 16$ or $y = 12x + 16$

(ii) $\frac{dy}{dx}$ for $y = \frac{1}{x^2}$ is $\frac{-2}{x^3}$

$\left.\frac{dy}{dx}\right|_{(h,x)} = \frac{-2}{h^3}$

equation of normal is $\frac{y-k}{x-h} = \frac{h^3}{2}$

$\Rightarrow \quad 2y - \frac{2}{h^2} = h^3x - h^4$

it passes from $\left(0,\frac{1}{2}\right)$ $\quad \Rightarrow 1-\frac{2}{h^2} = -h^4$ $\quad \Rightarrow h^6 + h^2 - 2 = 0$

$\Rightarrow (h^2 - 1)(h^4 + h^2 + 2) = 0 \quad \Rightarrow h = \pm 1$

$\Rightarrow$ common normals are $x - 2y + 1 = 0$ or $2y + x - 1 = 0$

A-5. **(i)** $xy + ax + by = 0$

As (1,1) lies on curve, so

$1 + a + b = 0$ & $\theta = \tan^{-1} 2$ or $\frac{dy}{dx} = 2$

Differentiating equation of curve w.r.t. x, $x\frac{dy}{dx} + y + a + b\frac{dy}{dx} = 0$,

put x = 1, y = 1 $\quad \Rightarrow \quad \frac{dy}{dx} = -\frac{(1+a)}{1+b} = 2$

or $\quad 1 + a + 2 + 2b = 0 \quad$ or $\quad 3 + a + 2b = 0 \quad b = -2$ & $a = 1$

(ii) (–2, 0) lies on both curves $\Rightarrow -8a + 4b - 6 + 5 = 0$ .....(i)

($\frac{dy}{dx}$ at (–2, 0) for second curve) = ($\frac{dy}{dx}$ at (–2, 0) for first curve) = 0

$\Rightarrow 12a - 4b + 3 = 0$ ...(ii)

$\Rightarrow$ Solving (i) and (ii) we get $a = \frac{1}{2}$, $b = \frac{3}{4}$

Section (B)

B-1. For C_1, $\left.\frac{dy}{dx}\right|_{(1,0)} = \left(\frac{2^x}{x} + \ell nx.2^x.\ell n2\right)_{(1,0)} = 2$

For C_2, $\left.\frac{dy}{dx}\right|_{(1,0)} = \left(x^{2x}.\ell nx \times 2 + 2x(x)^{2x-1}\right)_{(1,0)} = 2$ $\Rightarrow \theta = 0 \Rightarrow \cos\theta = 1$

B-2. For C_1, $\left.\frac{dy}{dx}\right|_{x=1} = \left(\frac{1}{x}\right)_{x=1} = 1$

For C_1, $\left.\frac{dy}{dx}\right|_{x=e} = \left(\frac{1}{x}\right)_{x=e} = \frac{1}{e}$

For C_2, $\left.\frac{dy}{dx}\right|_{x=1} = \left(\frac{2\ln x}{x}\right)_{x=1} = 0$

For C_2, $\left.\frac{dy}{dx}\right|_{x=e} = \left(\frac{2\ln x}{x}\right)_{x=e} = \frac{2}{e}$

angle between curves at (1,0) is $\pi/4$ and angle between curves at (e,1) is $\tan^{-1}\left(\frac{e}{e^2+2}\right)$

B-3. Given curves are $y^2 = 4x + 4$ and $y^2 = 36(9 - x)$ ......(i)

On solving, we get the point (8, 6) and (8,–6)

On differentiating equation (i), we get

$$2y\frac{dy}{dx} = 4 \text{ and } 2y\frac{dy}{dx} = -36$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y} \text{ and } \frac{dy}{dx} = \frac{-18}{y}$$

At point (8, 6), $m_1 = \frac{dy}{dx} = \frac{1}{3}$ and $m_2 = \frac{dy}{dx} = -3$

$m_1m_2 = -1$

B-4. $ax^2 + by^2 = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{-ax}{by}$

$Ax^2 + By^2 = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{-Ax}{By}$

Product of slopes = – 1

$\Rightarrow \quad aAx^2 + bBy^2 = 0$

Eliminating $x^2, y^2 \Rightarrow \begin{vmatrix} a & b & 1 \\ A & B & 1 \\ aA & bB & 0 \end{vmatrix} = 0$

$\Rightarrow \quad (AbB - aAB) - (abB - abA) = 0$

$\quad AB(b - a) - ab(B - A) = 0$

$\Rightarrow \quad ab(A - B) = AB(a - b)$

B-5. Let C_1 is $y = x - 2$ and C_2 is $y = x^2 + 3x + 2$

Slope of common normal is – 1 (if possible)

Now, for C_2, $\frac{dy}{dx} = 2x + 3 = 1 \quad \Rightarrow x = -1$

$\Rightarrow$ Point on C_2 where normal has slope equal to –1 is (–1, 0)

$\Rightarrow$ Shortest distance between C_1 and C_2 is distance of (–1, 0) from $y = x - 2$ which is $\frac{3}{\sqrt{2}}$

B-6. Equation of normal to $y^2 = 4x$ is $y = -tx + 2t + t^3$ at point $(t^2, 2t)$. If it is common normal to $(x - 6)^2 + y^2 = 1$ then (6, 0) satisfies the above equation of normal

$\Rightarrow t^3 - 4t = 0 \Rightarrow t = 0, 2, -2$

$\Rightarrow$ feet of these common normal are (0,0), (4,4), (4,–4)

$\Rightarrow$ Distance of these feet from (6, 0) are 6, $\sqrt{20}, \sqrt{20}$ respectively

$\Rightarrow$ shortest distance between $y^2 = 4x$ and $(x - 6)^2 + y^2 = 1$ is $\sqrt{20} - 1$

Section (C)

C-1. **(i)** Let P be perimeter

$P = 2x + 2y$

$$\frac{dP}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} - 6 + 4 = -2$$

(ii) Let A be area $A = xy$

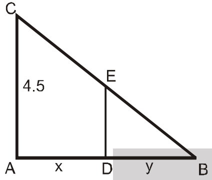
$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = -18 + 20 = 2$$

C-2. We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x} \cdot \frac{dy}{dx}$

$y = x - x^2 \Rightarrow \frac{dy}{dx} = 1 - 2x$

$\frac{d(y^2)}{d(x^2)} = \frac{y}{x}\ (1-2x) = \frac{(x-x^2)(1-2x)}{x} = 2x^2 - 3x + 1$

C-3. Let AC be pole, DE be man and B be farther end of shadow as shown in figure
From triangles ABC and DBE



$\frac{4.5}{x+y} = \frac{1.5}{y}$

$3y = 1.5\,x$

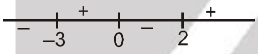
$\frac{dy}{dt} = 2,\ \frac{d}{dt}\ \ (x + y) = \frac{dx}{dt} + \frac{dy}{dt}$

C-4. $V = a^3 \Rightarrow \Delta V = \frac{dV}{da}\ \Delta a = 3a^2\Delta a = 3a^2 \times \frac{2a}{100} = \frac{6 \times 5^3}{100} = 7.5\ m^3$

Section (D)

D-1. $f'(x) = \frac{2+x-2\sqrt{1+x}}{2\ \left(x+1\right)^{3/2}} = \frac{\left(\sqrt{x+1}\ \ -1\right)^2}{2(x+1)^{3/2}} \geq 0,\ x > -1 \Rightarrow f(x)$ is increasing.

D-2. **(i)** Let $f(x) = \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 + 5$



Signs of f′ (x)

$f'(x) = x(x + 3)\ (x - 2)$

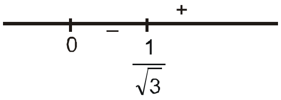
M.D. in $(-\infty, -3]$

M.I. in $[-3, 0]$

M.D. in $[0, 2]$

M.I. in $[2, \infty)$

(ii) Let $f(x) = (\log_3 x)^2 + \log_3 x$



signs of f′(x)

$= \frac{(\ell nx)^2}{(\ell n3)^2} + \frac{\ell nx}{\ell n3}\quad f'(x) = \frac{2(\ell nx) + \ell n3}{x(\ell n3)^2}$

M.D. in $\left(0,\ \frac{1}{\sqrt{3}}\right]$

M.I. in $\left[\frac{1}{\sqrt{3}},\ \infty\right)$

D-3. g(x) is monotonically increasing

$\Rightarrow$ $g'(x) \geq 0$ & f(x) is M.D. $\Rightarrow f'(x) \leq 0$

$$\frac{d}{dx}\ (fog)\ (x) = \frac{d}{dx}\left(\ (f(g(x)\ \right) = f'((g\ \ (x)\ )\ \ g'(x)\ \ \leq 0$$

as $f'(x) \leq 0$ & $g'(x) \geq 0$

$\Rightarrow$ (fog) (x) is monotonically decreasing $\Rightarrow$ $f'(x) \leq 0$

Also $x + 1 > x - 1$

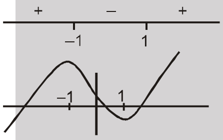
$\Rightarrow$ $f(x + 1) < f(x - 1)$ as f(x) is M.D.

$\Rightarrow$ $g(f(x + 1) < g(f(x - 1))$ as g(x) is M.I.

D-4. $f'(x) = \begin{cases} a & ; \quad x < 0\ , \\ 2x & ; \quad x > 0. \end{cases}$

$f'(x) > 0 \quad \Rightarrow\ a > 0$

D-5. **(i)** $f'(x) = 3x^2 - 3$
$= 3(x - 1)\ (x + 1)$



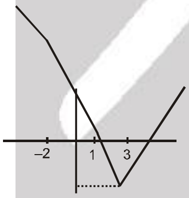
at x = 1 point of minima
$\therefore$ Neither increasing nor decreasing
at x = 2 increasing

(ii) at x = – 2 decreasing

at x = 0 decreasing

at x = 3 neither increasing nor decreasing

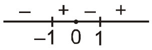
at x = 5 increasing



(iii) $f'(x) = \dfrac{1}{3x^{2/3}} \Rightarrow f'(x) > 0\ \forall\ x \in R - \{0\}$

$\Rightarrow$ strictly increasing at x = 0

(iv) $f'(x) = \dfrac{2(x^2+1)(x-1)(x+1)}{x^3} \Rightarrow$ strictly increasing at x = 2, neither I nor D at x = 1



(v) $f'(x) = \begin{cases} 3x^2+4x+5 & , \quad x < 0 \\ 3\cos x & , \quad x > 0 \end{cases} \quad \Rightarrow \quad f'(0^-) = +5$

and $f'(0^+) = 3 \Rightarrow$ strictly increasing at x = 0

D-6. Let $f(x) = \frac{\sin x}{x} \Rightarrow f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0 \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) is decreasing in $\left(0, \frac{\pi}{2}\right) \Rightarrow f\left(\frac{1}{10}\right) > f\left(\frac{1}{9}\right) \Rightarrow \left(\frac{\sin\left(\frac{1}{10}\right)}{\frac{1}{10}}\right) > \left(\frac{\sin\left(\frac{1}{9}\right)}{\frac{1}{9}}\right)$

D-7. Let $h(x) = f(x) - g(x)$

$h(0) = f(0) - g(0) \Rightarrow \quad h(0) = 0$

$h'(x) = f'(x) - g'(x) \le 0 \quad$ for $x \ge 0$

$\Rightarrow$ h(x) is decreasing for $x \ge 0$

$x \ge 0$

$h(x) \le h(0)$

$h(x) \le 0$

$f(x) - g(x) \le 0$

$f(x) \le g(x)$

D-8. $f'(x) = \begin{cases} -1 \ , & 0 < x < 1 \\ 2x & x > 1 \end{cases}$

f′(x) changes sign from negative to positive.

$f(1^-) = 2, f(1^+) = 1 + \ell nb$ and $f(1) = 1 + \ell nb$.

$f(1^-) \ge f(1) \Rightarrow 2 \ge 1 + \ell nb$

$\Rightarrow \ell nb \le 1 \qquad 0 < b \le e$

D-9. **(i)** $f'(x) = 6(x - 1)(x - 6)$

Local maxima at x = 1

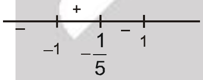
Local minima at x = 6

signs of f′(x)

(ii) $f'(x) = -(x - 1)^2 (x + 1)(5x + 1)$

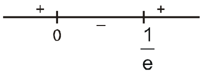
Local minima at x = – 1

Local maxima at $x = -\frac{1}{5}$



signs of f′(x)

Neither local minima nor local maxima at x = 1.



signs of f′(x)

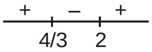
(iii) $f'(x) = \ell nx + 1$

Local minima at $x = \frac{1}{e}$

No local maxima

D-10.✍ **(i)** $g(t) = (t - 1)(t - 2)^2$

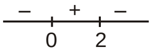
$g'(t) = (3t - 4)(t - 2)$



local maxima at x = $\log_2 \frac{4}{3}$ and local minima at x = 1

(ii) $f'(x) = xe^{-x}(2 - x)$

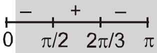
$\Rightarrow$ local min at 0, local max at 2



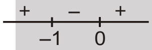
(iii) $f'(x) = -3\sin 2x\,(2\cos x + 1)(\cos x + 2)$

local max at x = 0, $\frac{2\pi}{3}$

local min at x = $\frac{\pi}{2}$, π

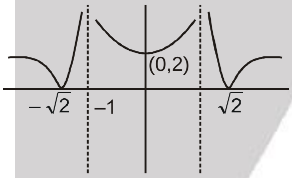


(iv) $f'(x) = \frac{2(1+x^{1/3})}{x^{1/3}}$



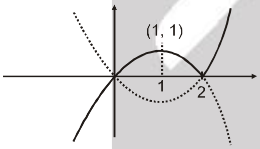
local maxima at –1 and local minma at 0

(v)



local minima at x = ±, $\sqrt{2}$ 0

D-11. $f(x) = \begin{cases} -x^2 + 2x & : \quad x < 2 \\ x^2 - 2x & : \quad x \geq 2 \end{cases}$



Graph of y = f(x)

Section (E)

E-1. **(i)** $f'(x) = 3x^3$

$f'(x) = 0 \Rightarrow x = 0$

$x = -2,\ f(-2) = -8$

$x = 0,\ f(0) = 0$

$x = 2,\ f(2) = 8$

Minimum = – 8, maximum = 8

(ii) $f'(x) = \cos x - \sin x$

$f'(x) = 0 \Rightarrow x = \frac{\pi}{4}$

$x = 0, \qquad f(0) = 1$

$x = \frac{\pi}{4}, \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$x = \pi, \qquad f(\pi) = -1$

Minimum = –1, Maximum = $\sqrt{2}$

(iii) $f'(x) = 4 - x$

$f'(x) = 0 \Rightarrow x = 4$

$x = -2, \qquad f(-2) = -10$

$x = 4, \qquad f(4) = 8$

$x = \frac{9}{2}, \qquad f\left(\frac{9}{2}\right) = \frac{63}{8}$

Minimum = –10, Maximum = 8

(iv) $f'(x) = \cos x - \sin 2x$

$f'(x) = 0 \Rightarrow \cos x = 0, \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{2}, x = \frac{\pi}{6}$

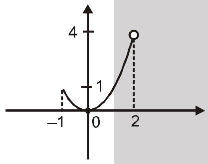
$x = 0, f(0) = \frac{1}{2}$

$x = \frac{\pi}{6}, \ f\left(\frac{\pi}{6}\right) = \frac{3}{4}$

$x = \frac{\pi}{2}, \ f\left(\frac{\pi}{2}\right) = \frac{1}{2}$

Minimum = $\frac{1}{2}$, Maximum = $\frac{3}{4}$

E-2.



Graph of f(x)

x = 0 is local minima and global maximum is not defined

E-3. Let No. of children of john & anglina = y

$\therefore x + (x + 1) + y = 24$

$y = 23 - 2x$

Number of fights

$F = x(x + 1) + x(23 - 2x) + (x + 1)(23 - 2x)$

$F = -3x^2 + 45x + 23$

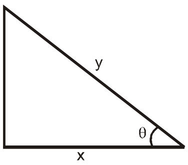
$\frac{df}{dx} = 0 \Rightarrow -6x + 45 = 0 \Rightarrow x = 7.5$

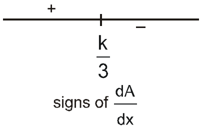
But 'x' wil be integral.

check x = 6 or x = 7

F = 191

E-4. x + y = k constant x + y = k





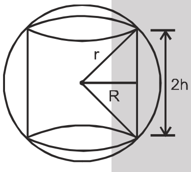
Area of triangle be A.

$A = \frac{1}{2} x \sqrt{y^2 - x^2}$ $A = \frac{1}{2}$ x $\sqrt{k}\sqrt{k-2x}$ $\frac{dA}{dx} = \frac{\sqrt{k}}{2}\frac{(k-3x)}{\sqrt{k-2x}}$ Area is maximum when $x = \frac{k}{3}$.

$\Rightarrow$ $y = \frac{2k}{3}$ $\Rightarrow \cos\theta = \frac{\frac{k}{3}}{\frac{2k}{3}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.

E-5. $R^2 + r^2 = h^2$

$R^2 = h^2 - r^2$



Figure

volume of cylinder,

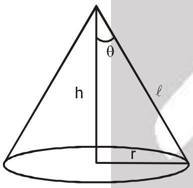
$V = \pi R^2 (2h) = \pi (2h) (\sqrt{r^2 - h^2})^2$

$\frac{dV}{dh} = 2\pi (r^2 - h^2) + 2\pi h(-2h) = 0$

$\Rightarrow r^2 = 3h^2 \Rightarrow h = \frac{r}{\sqrt{3}}$

$\frac{d^2V}{dh^2} < 0$ at $h = \frac{r}{\sqrt{3}} \Rightarrow V_{max} = 2\pi \frac{r}{\sqrt{3}}\left(r^2 - \frac{r^2}{3}\right) = \frac{4\pi r^3}{3\sqrt{3}}$

E-6. $h = \ell\cos\theta$



Figure

$r = \ell\sin\theta$

$V = \frac{1}{3} \pi r^2 h$

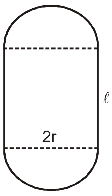
$V = \frac{1}{3} \pi \ell^3 \sin^2\theta \cos\theta$

$\frac{dV}{d\theta} = \frac{1}{3} \pi\ell^3 (2\sin\theta \cos^2\theta - \sin^3\theta)$ $\frac{dV}{d\theta} = \frac{1}{3} \pi\ell^3 \sin\theta (2 - 3\sin^2\theta) = 0$ at

$\sin\theta = \sqrt{\frac{2}{3}}$

$\Rightarrow$ $\tan\theta = \sqrt{2}$

E-7. $2\ell + 2\pi r = 440$



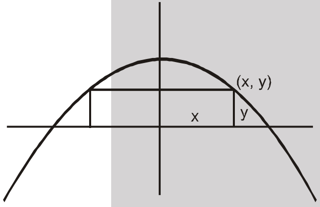
Figure

$A = \ell\ 2r = -2\pi r^2 + 440r$

$\frac{dA}{dr} = -4\pi r + 440 = 0$

at $r = \frac{110}{\pi}$

E-8.✍ Let A be area of rectangle.



Figure

$A = (2x)(y) = 2x(12 - x^2)$

$\frac{dA}{dx} = 6(2 + x)(2 - x)$

At x = 2, A has largest value. Largest A = 32

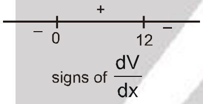
E-9. Let x, y be dimensions of rectangle.

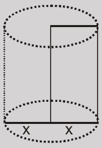
$\Rightarrow$ $2x + 2y = 36$.

Let V be volume sweeped

$V = \pi x^2 y$

$V = \pi x^2(18 - x)$





Figure

$\frac{dV}{dx} = \pi x.3.(12 - x)$

At x = 12, V has maximum value. $\Rightarrow$ y = 6

E-10. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{C - R_1}$

$R = R_1(C - R_1)/C$

$\frac{dR}{dR_1} = \frac{C - 2R_1}{C}$

$\frac{dR}{dR_1} = 0$ at $R_1 = \frac{C}{2}$

$R_2 = \frac{C}{2}$

So $R_1 = R_2$

Section (F)

F-1. f(x) ↓, g(x) ↑

let $h(x) = f(x) - g(x)$

$h(1) = 2, h(2) = -6$

$\Rightarrow$ h(x) = 0 has atleast one root in (1, 2).

but $h'(x) = f'(x) - g'(x) < 0$

$\Rightarrow$ h(x) is always decreasing so only one root

F-2. f(x) is continuous on [a, b], f(x) is differentiable on (a, b), f(a) = p = f(b).

Conditions in Rolle's theorem are satisfied.

$$f'(x) = \left(\frac{x(a+b)}{x^2+ab}\right)\left(1-\frac{ab}{x^2}\right)\frac{1}{(a+b)}$$

$f'(x) = 0 \quad \Rightarrow \quad x = \sqrt{ab} \in (a\ ,\ b)$

i.e $f'(c) = 0, \quad c = \sqrt{ab}$

conclusion of Rolle's theorem also valid.

F-3. Let $f'(x) = 3x^2 + px - 1 \Rightarrow f(x) = x^3 + \frac{px^2}{2} - x + c$

$f(-1) = \frac{p}{2} + c = f(1)$

$\Rightarrow$ f(x) satisfies conditions in Rolle's theorem $\Rightarrow$ $f'(c) = 0$ for atleast one $c \in (-1,1)$

$\Rightarrow$ $3x^2 + px - 1 = 0$ has atleast one root in (–1, 1).

F-4.✍ $f(x) = 0 \Rightarrow \sin\frac{\pi}{x} = 0 \Rightarrow \frac{\pi}{x} = n\pi$

$\Rightarrow x = \frac{1}{n}, n \in N$

$x = \ldots\ldots., \frac{1}{n}, \frac{1}{n-1}, \ldots\ldots., \frac{1}{3}, \frac{1}{2}\ 1.$

Consider interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ $f\left(\frac{1}{n+1}\right) = 0 = f\left(\frac{1}{n}\right)$

By Rolle's theorem f′(x) vanishes at least once in $\left(\frac{1}{n+1}, \frac{1}{n}\right)$

Infinite number of such intervals are there. Hence f′(x) vanishes at infinite number of points in (0, 1)

F-5.✍ Let $h(x) = f(x)\, g'(x)$ $h(a) = 0 = h(b)$

By Rolle's theorem on [a, b] $h'(x) = 0$, for at least one $c \in (a, b)$. $\Rightarrow f'(c)\, g'(c) + f(c)\, g''(c) = 0$

F-6.✍ $f'(c) = \dfrac{f\left(\frac{\pi}{5}\right) - f(0)}{\frac{\pi}{5} - 0}$; $f'(c) = \sec^2 c$ which is strictly increasing $f'(c) = \dfrac{f\left(\frac{\pi}{5}\right) - f(0)}{\frac{\pi}{5} - 0}$; $f'(c) = \sec^2 c$

$$f'(0) < f'(c) < f'\left(\frac{\pi}{5}\right) \quad \sec^2 0 < \frac{\tan\frac{\pi}{5}}{\frac{\pi}{5}} < \sec^2\frac{\pi}{5} < \sec^2\frac{\pi}{4} \quad \frac{\pi}{5} < \tan\frac{\pi}{5} < \frac{2\pi}{5}$$

F-7. Let $\phi(x) = f(x) - 2g(x)$; $x \in [0, 23]$ $\Rightarrow$ $\phi'(x) = f'(x) - 2g'(x)$

Also $\phi(0) = f(0) - 2g(0) = 2 - 0 = 2$

$\phi(23) = f(23) - 2g(23) = 22 - 20 = 2$

Since f(x) and g(x) are differentiable in [0, 23] hence $\phi(x)$ is also continous in [0, 23] and differentiable in [0, 23], so all the conditions of Rolle's theorem are satisfied. Hence there exist a number c, 0 < c < 23 for which $\phi'(x) = 0$

F-8. f(a) = f(b) = 0 and we know $\sin^3x$, xe^x, $\frac{x}{1+x^2}$ are continuous and differentiable for $x \in R$ therefore f(x) is also continuous and differentiable in [a, b] hence by Rolle's theorem there exist a real number $c \in [a, b]$ such that $f'(c) = 0$.

F-9. f(0) = 0 and f(6) = 2
so $f(0) \neq f(6)$
f(x) is discontinuous at x = 4 and nondifferentiable at x = 1, 4 but f'(3) = 0

F-10. Let $g(x) = \frac{f(x)}{x}$, $x \in [a, b]$. By Rolle's theorem $g'(x_0) = 0 \Rightarrow \frac{f'(x_0)x_0 - 1.f(x_0)}{x_0^2} = 0 \Rightarrow f'(x_0) = \frac{f(x_0)}{x_0}$

PART - II

Section (A)

A-1. $f'(0) = \lim\limits_{x\to 0} \frac{\frac{\sin x^2}{x} - 0}{x - 0} = 1$ (slope of tangent)
slope of normal is – 1
Equation of normal is y – 0 = – (x – 0)

A-2. $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{(1,\,1)} = -\frac{1}{2}$
Slope of normal = 2
Equation of normal is 2x – y =1

A-3. $\left.\frac{dy}{dx}\right]_{(3,0)}$ is 4 $\Rightarrow$ $\theta = \tan^{-1} 4$

A-4. $\frac{dy}{dx} = 5x^4$

Equation of tangent at (h,k) is $\frac{y-k}{x-h} = 5\,h^4$ Which passes through (2, 2)

$\Rightarrow (2 - k) = 5h^4\,(2 - h) \Rightarrow -h^5 = 10h^4 - 5h^5 \Rightarrow 4h^5 = 10h^4 \Rightarrow h = 0,\ \frac{5}{2}$

$\Rightarrow$ Equation of tangents are y – 2 = 0 and $16(y - 2) = 5^5\,(x-2)$

A-5. Equation of tangent is
$y - 4/h = -\,4/h^2\,(x - h)$
It passes through (0, 1) $\Rightarrow$ $1 - 4/h^2 = 4h/h^2$
$\Rightarrow$ h – 4 = 4 $\Rightarrow$ h = 8 $\Rightarrow$ tangent is y – 1/2 = – 1/16 (x – 8).

A-6. $y - e^{xy} + x = 0$

Differentiating w.r.t. to y

$1 - e^{xy}\left(\frac{dx}{dy}.\quad y + x\right) + \frac{dx}{dy} = 0$

$\frac{dx}{dy} = 0$

$1 - xe^{xy} = 0$

$xe^{xy} = 1 \qquad x = 1\ ,\ y = 0$

Point is (1, 0)

A-7. $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\ \ (-\sin\theta)}{a\ \ (1 + \cos\theta)}$

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{-\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$

$\tan\alpha = -\frac{1}{\sqrt{3}} \quad\Rightarrow\quad \alpha = \pi - \frac{\pi}{6} \quad\Rightarrow\quad \alpha = \frac{5\pi}{6}$

A-8.✍ $\left.\frac{dy}{dx}\right]_{(3t, 4/t)} = \left.\frac{-12}{x^2}\right]_{(3t,4/t)} = \frac{-4}{3t^2}$

equation of normal $\frac{y - 4/t}{x - 3t} = \frac{3t^2}{4}$ which passes through $(3t_1, 4/t_1)$

$\Rightarrow \frac{4}{t_1} - \frac{4}{t} = \frac{3t^2}{4}(3t_1 - 3t) \Rightarrow \frac{4}{tt_1} = \frac{-9t^2}{4} \Rightarrow t_1 = \frac{-16}{9t^3}$

A-9.✍ $\frac{dy}{dx}$ for $y = x^2 + \frac{1}{x}$ is $2x - \frac{1}{x^2}$

$\frac{dy}{dx}$ at (h, k) is $2h - \frac{1}{h^2}$

equation of tangent at (h, k) is $\frac{y - (h^2 + 1/h)}{x - h} = 2h - \frac{1}{h^2}$

$y = \left(2h - \frac{1}{h^2}\right)x - 2h^2 + \frac{1}{h} + h^2 + \frac{1}{h}$

$y = \left(2h - \frac{1}{h^2}\right)x + \frac{2}{h} - h^2 \Rightarrow \frac{2}{h} - h^2 = \frac{1}{2h - 1/h^2} \Rightarrow \frac{2 - h^3}{h} = \frac{h^2}{2h^3 - 1} \Rightarrow h^6 - 2h^3 + 1 = 0$

$\Rightarrow (h^3 - 1)^2 = 0 \Rightarrow (h - 1)^2\,(h^2 + h + 1)^2 = 0 \Rightarrow h = 1 \Rightarrow$ equation of tangent is $y = x + 1$

A-10_. $1 = 1 + b + c \quad\Rightarrow c = -b$

$\frac{dy}{dx} = 2x + b = 2 + b$ at (1, 1)

equation of tangent is $y - 1 = (2 + b)(x - 1) \qquad\Rightarrow$ x intercept $= \frac{b+1}{b+2}$

y intercept $= -(b + 1) \quad\Rightarrow \left|\frac{1}{2}\times\left(\frac{b+1}{b+2}\right)\times(b+1)\right| = 2$

$b^2 + 2b + 1 = 4b + 8$ on $-4b - 8$

$b^2 - 2b - 7 = 0$ on $b^2 + 6b + 9 = 0 \Rightarrow$ integral $b = -3$

Section (B)

B-1. For C_1, $\left.\frac{dy}{dx}\right|_{(0,1)} = a^x \ell na\Big|_{(0,1)} = \ell na$

For C_2, $\left.\frac{dy}{dx}\right|_{(0,1)} = b^x \ell nb\Big|_{(0,1)} = \ell nb$ $\Rightarrow$ angle between curve is $\left|\frac{\log a - \log b}{1+\log a \log b}\right| = \left|\frac{\log(a/b)}{1+\log a \log b}\right|$

B-2. Both curves are confocal, where focus of both curves is $(\sqrt{24},0)$ and $(-\sqrt{24},0)$

$\Rightarrow$ angle between curves $x^2 + 4y^2 = 32$ and $x^2 - y^2 = 12$ is $\frac{\pi}{2}$

B-3. $\left.\frac{dy}{dx}\right]_{C_1} = \frac{x^2-y^2}{2xy}$, $\left.\frac{dy}{dx}\right]_{C_2} = \frac{-2xy}{x^2-y^2} \Rightarrow \left.\frac{dy}{dx}\right]_{C_1} \times \left.\frac{dy}{dx}\right]_{C_2} = -1$

B-4. For C_1, $\frac{dy}{dx} = \frac{-4x}{a^2y}$ and For C_2, $\frac{dy}{dx} = \frac{16}{3y^2}$ $\Rightarrow$ $\frac{-4x}{a^2y} \times \frac{16}{3y^2} = -1 \Rightarrow \left(\frac{4}{3a^2}\right) \times \left(\frac{16x}{y^3}\right) = -1 \Rightarrow a^2 = 4/3$

B-5. Equation of normal to the curve $y^2 = 8x$ and $y^2 = 4(x-3)$ are $y = mx - 4m - 2m^3$ and $y = m(x-3) - 2m - m^3$ respectively.
$\Rightarrow -4m - 2m^3 = -3m - 2m - m^3 \Rightarrow m^3 - m = 0 \Rightarrow m = 0, 1, -1$
$\Rightarrow$ feet of common normal with slope equal to -1 on the curves $y^2 = 8x$ and $y^2 = 4(x-3)$ are (4,2) and (2,4) respectively
Now, distance between points (4, 2) and (2, 4) is $2\sqrt{2}$

B-6. Equation of normal at $\frac{x^2}{32} + \frac{y^2}{18} = 1$ at (h, k) is $\frac{32x}{h} - \frac{18y}{k} = 14$ which passes through $\left(\frac{7}{4}, 0\right)$

$\Rightarrow \frac{56}{h} - 0 = 14$ $\quad\Rightarrow h = 4$ $\quad\Rightarrow k = 3$

Now, distance between (4, 3) and $\left(\frac{7}{4}, 0\right)$ is $\sqrt{\left(\frac{9}{4}\right)^2 + 9} = \frac{15}{4}$

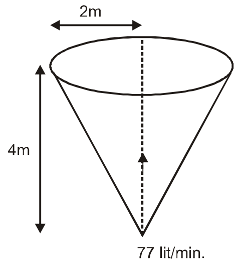
$\Rightarrow$ shortest distance between curves $\frac{x^2}{32} + \frac{y^2}{18} = 1$ and $\left(x - \frac{7}{4}\right)^2 + y^2 = 1$ is $\frac{11}{4}$

Section (C)

C-1. $V = \frac{1}{3}\pi r^2 h$ $\quad \left(\because \frac{r}{h} = \frac{2}{4} = \frac{1}{2}\right)$

$V = \frac{1}{3}\pi \frac{h^3}{4} = \frac{\pi}{12}h^3$

$77 \times 10^3 = \frac{22}{7} \times \frac{1}{4} \times 70 \times 70 \times \frac{dh}{dt}$ $\quad$ ($\because$ 1 litre = 10^3 c.c.)



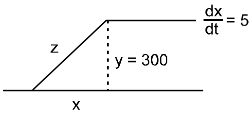
$\therefore \frac{dh}{dt} = 20$ cm/min.

C-2 $x^3 = 12y$

$3x^2 \frac{dx}{dt} = 12 \frac{dy}{dt} \quad \Rightarrow \quad \frac{dx}{dt} > \frac{dy}{dt}$

$\Rightarrow \quad 12 \frac{dy}{dt} \cdot \frac{1}{3x^2} > \frac{dy}{dt}$

$\Rightarrow \quad x^2 < 4 \quad \Rightarrow \quad x \in (-2, 2)$



Figure

C-3. From figure $z^2 = x^2 + y^2$

$z \frac{dz}{dt} = x \frac{dx}{dt}$

If z = 500 then x = 400 $\Rightarrow \quad 500 \frac{dz}{dt} = 400(5) \quad \Rightarrow \quad \frac{dz}{dt} = 4$

C-4. Let $y = \tan x \Delta y = \tan (x + \Delta x) - \tan x \quad \Rightarrow \quad \frac{dy}{dx} \Delta x = \tan (x + \Delta x) - \tan x$

$\Rightarrow (\sec^2 x)\Delta x = \tan (x + \Delta x) - \tan x$

put $x = 45°, \Delta x = 1° \Rightarrow \frac{2\pi}{180} = \tan 46° - 1 \Rightarrow \quad \tan 46° = 1 + \frac{\pi}{90}$

C-5. $V = 4\frac{\pi}{3}(10 + r)^3 \ , \ 0 \le r \le 15$

$\because \quad \frac{dV}{dt} = -50.$

$4\pi (10 + r)^2 \frac{dr}{dt} = -50 \Rightarrow \frac{dr}{dt} = \frac{-1}{18 \ \pi}$ (where r = 5)

Section (D)

D-1. $f'(x) = 3(a + 2)x^2 - 6ax + 9a \le 0 \ \ \forall \ x \in R$

$\Rightarrow \quad a + 2 < 0 \quad \text{and} \quad D \le 0$

$\Rightarrow \quad a < -2 \quad \text{and} \quad a \in (-\infty, -3] \cup [0, \infty)$

$\Rightarrow \quad a \in (-\infty, -3]$

D-2. $f'(x) = 3x^2 + 2ax + b + 5 \sin 2x \ge 0 \ x \in R$

$\because \quad \sin 2x \ge -1$

$\Rightarrow \quad f'(x) \ge 3x^2 + 2ax + b - 5 \quad x \in R \Rightarrow 3x^2 + 2ax + b - 5 \ge 0 \quad x \in R$

$\Rightarrow \quad 4a^2 - 4.\ 3.\ (b - 5) \le 0$

$\Rightarrow \quad a^2 - 3b + 15 \le 0$

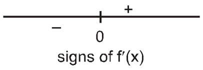
D-3. $f(x) = \begin{cases} \frac{1-x}{x^2} & x < 1, \ x \ne 0 \\ \frac{x-1}{x^2}, & x \ge 1, \end{cases}$

The given function is not differentiable at x = 1

$f'(x) = \begin{cases} \frac{1}{x^2} - \frac{2}{x^3}, & x < 1, \ x \ne 0 \\ \frac{2}{x^3} - \frac{1}{x^2}, & x > 1 \end{cases}$ Now $f'(x) < 0 \Rightarrow \begin{cases} \frac{x-2}{x^3} < 0 & \text{given} \quad x < 1 \\ \frac{2-x}{x^3} < 0 & \text{when} \quad x > 1 \end{cases}$

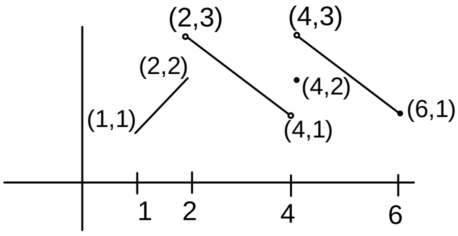
f(x) decreasing $\forall \ x \in (0, 1) \cup (2, \infty)$ and f(x) increases $\forall \ x \in (-\infty, 0) \cup (1, 2)$
here f(x) is decreasing at all points in $x \in (0, 1) \cup (2, \infty)$ so will also be decreasing at x = 3 at x = 1 minima and at x = 2 maxima

D-4. $f'(x) = (2^2 + 4^2x^2 + 6^2x^4 + ... + 100^2x^{98})\,x$



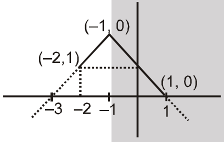
Minimum at x = 0

D-6.



Section (E)

E-1. $f(x) = 2 - |x + 1|$



Figure

From figure it is clear that greatest, least values are respectively 2, 0

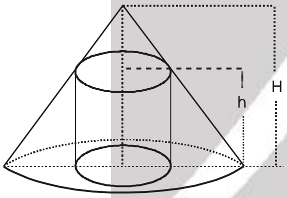
E-2. Since coefficient of x^2 is (+ve)

$$\Rightarrow m(b) = -\frac{D}{4a} \quad m(b) = -\frac{(4b^2 - 4(1+b^2))}{4(1+b^2)} \quad \Rightarrow \quad m(b) = \frac{1}{1+b^2}$$

$$\Rightarrow \; b^2 \geq 0 \;\; \Rightarrow 1 + b^2 \;\; \geq 1 \Rightarrow \quad 0 < \frac{1}{1+b^2} \leq \;\; 1 \;\; \Rightarrow \quad m(b) \in \;\; (0, 1]$$

E-3. $\frac{H}{R} = \frac{H-h}{r}$

$S = 2\pi rh$



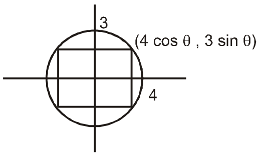
Figure

$$= 2\pi H\left(r - \frac{r^2}{R}\right) \quad \frac{dS}{dr} = 2\pi H\left(1 - \frac{2r}{R}\right)$$

Maximum at $r = \frac{R}{2}$

E-4. $x = 4\cos\theta,\ y = 3\sin\theta$

Let A be area. $A = 4\,(4\cos\theta)\,(3\sin\theta) = 24\sin 2\theta$

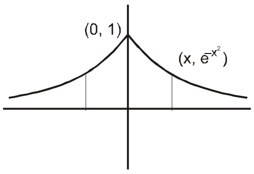


Figure

A is maximum when $2\theta = \frac{\pi}{2} \Rightarrow$ Dimensions are. $\frac{2\ .\ 4}{\sqrt{2}},\ \frac{2.3}{\sqrt{2}}$

E-5. Let A be area A = $(2x)(e^{-x^2})$, x > 0

$$\frac{dA}{dx} = -2\left(x+\frac{1}{\sqrt{2}}\right)\left(x-\frac{1}{\sqrt{2}}\right)e^{-x^2}$$



Figure

At x = $\frac{1}{\sqrt{2}}$, A is maximum. Largest area is $2\frac{1}{\sqrt{2}}\ e^{-1/2}$

E-6. $f(1^-) \le f(1)$ and $f(1^+) \le f(1) - 2 + \log_2(b^2 - 2) \le 5$

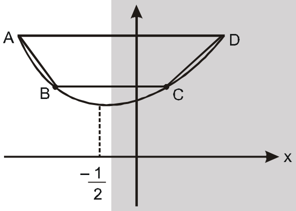
$0 < b^2 - 2 \le 128\ 2 < b^2 \le 130$

E-7. A(– 2, 3) ⇒ 3 = 4a – 2b + c

B(– 1, 1) ⇒ 1 = a – b + c

D(2, 7) ⇒ 7 = 4a + 2b + c

⇒ $y = x^2 + x + 1$



Figure

C $(h, h^2 + h + 1)$, $-1 < h < 2$

Area = $\frac{3}{2}\ (-h^2 + h + 6)$

Maximum at h = $\frac{1}{2}$ $\Rightarrow C\left(\frac{1}{2}, \frac{7}{4}\right)$

Section (F)

F-1. $f(x) = x^3 - 6x^2 + ax + b$

f(x) satisfies condition in Rolle's theorem on [1, 3]

f(1) = f(3) ⇒ 1 – 6 + a + b = 27 – 54 + 3a + b

2a = 22

a = 11 and b ∈ R.

F-2. f′ (x) = 0 ⇒ x = – 2, 3

x = – 2 ∈(–3, 0)

∴ c = – 2

F-3. For x∈(0, 2)

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

(Here c∈(0, x))

⇒ f(x) = 2.f '(x)

f(x) ≤ 1

F-4_.✍ Let f(x) = $\frac{ax^5}{5}+\frac{bx^3}{3}$ + cx then f (1) = 0, f(–1) = 0, f(0) = 0

$\Rightarrow$ there exist atleast one root of equation $ax^4 + bx^2 + c = 0$ in (–1, 0) and there exist atleast one root of equation f'(x) = 0 in (0, 1)

$\Rightarrow$ there exist atleast one root of equation $ax^4 + bx^2 + c = 0$ in (–1, 0) and there exist atleast one root of equation f'(x) = 0 in (0, 1)

But $\frac{-b}{a}$ equal to negative so equation $ax^4 + bx^2 + c = 0$ has two real and two non-real roots. Hence there exist exactly one root of equation $ax^4 + bx^2 + c = 0$ in (–1, 0) and there exist exactly one root of equation f'(x) = 0 in (0, 1)

F-5. $\frac{x^3-2x^2-5x+6}{x-1}=\frac{(x-1)(x^2-x-6)}{(x-1)}$ = $x^2 - x - 6 \Rightarrow$ f(x) = $\begin{cases} x^2-x-6 & ; \quad x \neq 1 \\ -6 & ; \quad x = 1 \end{cases}$

$\lim\limits_{x\to1^+}$ f(x) = –6 ; $\lim\limits_{x\to1^-}$ f(x) = – 6 $\Rightarrow$ LHL = RHL = f(1)

$\therefore$ f(x) is continuous

LHD at x = 1 is 1

RHD at x = 1 is 1

$\therefore$ f(x) is differentiable at x = 1

f(–2) = 0 ; f(3) = 0 all the conditions of Rolle's are holding

f'(x) = 2x – 1 = 0

$\Rightarrow$ x = $\frac{1}{2}$ $\therefore$ $\frac{1}{2} \in [-2, 3]$

PART - III

1. **(A)** $y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

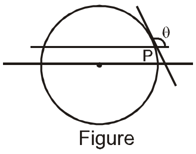
$y = e^{\frac{-x}{2a}} \Rightarrow \frac{dy}{dx} = \frac{-1}{2a}e^{\frac{-x}{2a}} = \frac{-1}{2a}y$

Product of slopes = $\left(\frac{2a}{y}\right)\left(\frac{-y}{2a}\right) = -1$

(B) $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

y = [| sin x | + | cos x|] $\Rightarrow$ y = 1

P(2, 1)



Figure

$x^2 + y^2 = 5$

$\frac{dy}{dx} = \frac{-x}{y}$ $\tan\theta = \left|\frac{-2-0}{1+0}\right|$ $\text{cosec}^2\theta = \frac{5}{4}$

(C) Let $y^2 = 4a\,(x + a)$ ....(1)
and $y^2 = 4b\,(x + b)$ ....(2)
intersect each other at (h, k) then $h = -\,(b + a)$
$(h, k)\ h = -\,(b + a)$

Now $\left.\frac{dy}{dx}\right]_{(h,k)}$ for curve (1) is $\frac{4a}{2k}$ and $\left.\frac{dy}{dx}\right]_{(h,k)}$ for curve (2) is $\frac{4b}{2k}$

$\Rightarrow \frac{4a}{2k} \times \frac{4b}{2k} = -1 \quad \Rightarrow \frac{4ab}{k^2} = -1 \Rightarrow \frac{4ab}{4a(h+a)} = -1 \Rightarrow \frac{b}{h+a} = -1 \Rightarrow \frac{b}{-b} = -1$

Which is always true $\Rightarrow \frac{a}{b}$ can take any value from interval, R– {0}

(D) Inverse curves touches each other at line y = x
$\Rightarrow$ y = x is tangent to both curves $\Rightarrow$ equation $x = x^2 + 3x + c$ has both equal roots
$\Rightarrow$ c = 1 and x = –1 = h and y = –1 = k $\Rightarrow |h + k + c| = 1$

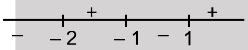
2. **(A)** $f'(x) = 2x - \frac{2}{x^3} = \frac{2(x^2-1)(x^2+1)}{x^3}$ No point of local maxima

(B) Given expression = $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$= \left(\frac{\pi}{2}\right)^3 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right) . \frac{\pi}{2} = \frac{3\pi}{2}(\sin^{-1}x)^2 - \frac{3\pi^2}{4}\sin^{-1}x + \frac{\pi^3}{8}$

This is quadratic in $\sin^{-1}x$. Therefore it will give maximum value when $\sin^{-1}x = -\frac{\pi}{2} \Rightarrow x = -1$

(C) $f'(x) = 12\,(x + 2)(x + 1)\,(x - 1)$



sings of f'(x)

$\Rightarrow \quad a = -\,2,\ b = -\,1$

(D) $\frac{a^3+b^3}{48} = \frac{a^3+(8-a)^3}{48} = \frac{8}{48}(3a^2 - 24a + 64)$

Minimum $\frac{a^3+b^3}{48}$ is $\frac{8}{48}\,\frac{(4.3.64-24^2)}{4.3} = \frac{8}{3}$

3. **(A)** f(x) is continuous and differentiable f(0) = f(π)
Hence condition in Rolle's theorem and LMVT are satisfied.

(B) $f(1^-) = -1,\ f(1) = 0,\ f(1^+) = 1$
f(x) is not continuous at x = 1, belonging to $\left[\frac{1}{2},\frac{3}{2}\right]$
Hence, atleast one condition in LMVT and Rolle's theorem is not satisfied

(C) $f'(x) = \frac{2}{5}\,(x-1)^{-3/5},\ x \neq 1$
At x = 1, f(x) is not differentiable.
Hence at least one condition in LMVT and Rolle's theorem is not satisfied.

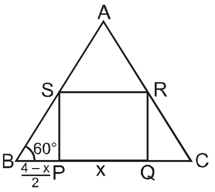
(D) At x = 0

$$\text{L.H.D.} = \lim_{x\to 0^-} t\,\frac{x\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right) - 0}{x-0} = \frac{0-1}{0+1} = -1$$

R.H.D. = 1
At x = 0, f(x) is not differentiable
Hence at least one condition in LMVT and Rolle's theorem is not satisfied.

4. **(A)** Let PQ = x

Then BP = $\frac{4-x}{2}$



Figure

$\therefore$ PS = $\frac{4-x}{2}$ tan60º = $\frac{\sqrt{3}\ (4-x)}{2}$

$\therefore$ area A of rectangle = $\frac{\sqrt{3}}{2}$ (4 – x) x

$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(4-2x) = 0 \quad \Rightarrow \quad x = 2 \ \frac{d^2A}{dx^2} = -\sqrt{3} < 0$

$\therefore$ A is maximum, when x = 2.

$\therefore$ Maximum area = $\frac{\sqrt{3}}{2}\ 2.2 = 2\sqrt{3}$.

Square of maximum area = 12

(B) Dimensions be x, 2x, h

72 = x. 2x. h

$36 = x^2h$ ....(1)

$S = 4x^2 + 6xh$

$S = 4x^2 + 6\frac{36}{x}$

$\frac{dS}{dx} = 8x - \frac{216}{x^2} = \frac{8(x^3-3^3)}{x^2}$

For least S, x = 3 and least S is 108.

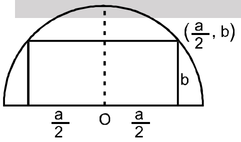
(C) $f(x) = x^3y = x^3(60-x)$

$f'(x) = 4x^2(45-x)$

f(x) is maximum at x = 45

(D) $x^2 + y^2 = 5$

$\frac{a}{2} = \sqrt{5}\ \cos\theta,\ b = \sqrt{5}\ \sin\theta$



Figure

Let f(θ) be perimeter $f(\theta) = 2a + 2b = 2\sqrt{5}\ (2\cos\theta + \sin\theta)$

$f'(\theta) = 2\sqrt{5}\ (-2\sin\theta + \cos\theta)$

$f''(\theta) = 2\sqrt{5}\ (-2\cos\theta - \sin\theta)$

$f'(\theta) = 0 \Rightarrow \tan\theta\ \frac{1}{2} =$ and $f''(\theta) < 0 \Rightarrow$ f(θ) is greatest

a = 4, b = 1

$a^3 + b^3 = 65$

EXERCISE # 2

1. $\frac{y}{b} = 1 - \frac{x}{a}$ $\frac{y}{b} = e^{-x/a} \Rightarrow e^{-x/a} = 1 - \frac{x}{a}$ put t = $-\frac{x}{a}$ $e^t = 1 + t$

Draw graph of $y = e^t$, $y = 1 + t$

From graph it is clear that t = 0 is the only Solution

$\Rightarrow$ x = 0 $\Rightarrow$ y = b (0, b)

2.✍ $x^3 + y^3 = 8xy$, $y^2 = 4x$ $\Rightarrow$ $x^3 + 8x^{3/2} = 8x\ 2x^{1/2}$

$x^3 - 8x^{3/2} = 0$ $\Rightarrow$ $x^6 - 64x^3 = 0$

$\Rightarrow$ $x^3 = 0$ or $x^3 = 64$

x = 4, y = 4.Point of intersection (4, 4) $3x^2 + 3y^2\, y' = 8y + 8xy'$

$y' = -1$

slop of normal is 1

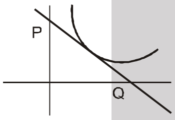
Equation of normal is y – 4 = 1 (x – 4) $\Rightarrow$ x = y

3. Here $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating w.r.t. to x $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

$\left.\frac{dy}{dx}\right|_{(h,\ \ k)} = -\left(\frac{k}{h}\right)^{1/3}$

Equation $(y - k) = -\left(\frac{k}{h}\right)^{1/3}(x - h)$



$P(0, k^{1/3}(h^{2/3} + k^{2/3}))$ or $P(0, a^{2/3}k^{1/3})$
And, Q $(h^{1/3}a^{2/3}, 0)$
PQ = $\sqrt{h^{2/3}a^{4/3} + a^{4/3}k^{2/3}}$ = |a| = constant.

4.✍ The tangent at $(x_1, \sin x_1)$ is $y - \sin x_1 = \cos x_1 (x - x_1)$

It passes through the origin $\sin x_1 = x_1 \cos x_1 = x_1\sqrt{1 - \sin^2 x_1}$

$y_1^2 = \sin^2 x_1 = x_1^2(1 - y_1^2)$ $\Rightarrow$ $(x_1 y_1)$ lies on the curve
$y^2 = x^2(1 - y^2)$ $\Rightarrow$ $x^2 - y^2 = x^2y^2$

5.✍ $y = e^{\{x\}} = e^{x-a}$ in $x \in [a, a + 1)$ $\frac{dy}{dx} = e^{x-a} = e^{\{x\}}$

equation of tangent $(Y - y) = \frac{dy}{dx}(X - x)$

passing through (–1/2, 0) $(0 - y) = e^{\{x\}}(-1/2 - x)$

$\Rightarrow -1 = -\frac{1}{2} - x \Rightarrow x = \frac{1}{2}$

$\therefore$ point $\left(\frac{1}{2},\ e^{1/2}\right)$ Number of tangent = 1

6.✍ Let y = mx + c be tangent touching both branches.

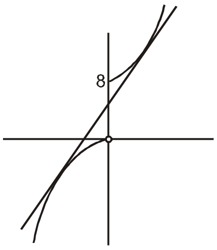
$f(x) = -x^2$, y = mx + c , x < 0

$x^2 + mx + c = 0$, m > 0 (∵ x < 0) (negative roots)

$D = 0 \Rightarrow m^2 = 4c$

$f(x) = x^2 + 8$, y = mx + c, x > 0

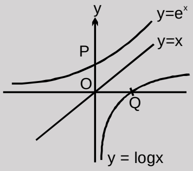
$x^2 - mx + 8 - c = 0$, m > 0 (positive roots)



Figure

$D = 0 \Rightarrow m^2 = 32 - 4c \Rightarrow c = 4, m^2 = 16 \Rightarrow c = 4, m = 4$

7.✍ $f(x) = e^x$ & $g(x) = \ell nx$ are image of each other in line mirror y = x hence minimum distance between these will be equal to distance between parallel tangents of f(x) & g(x) which are parallel y = x.

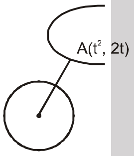


⇒ $e^x = 1$ & = 1

⇒ x = 0 & x = 1

P ≡ (0, 1) ; Q = (1, 0); PQ = $\sqrt{2}$

8.



shortest distance always lie along the common normal

Equation of normal at $(t^2, 2t)$ to the parabola is $y + xt = 2t + t^3$ ..... (i)

above equation passes through the center of the circle c(0, 12)

∴ $12 = 2t + t^3$

$t^3 + 2t - 12 = 0$

t = 2

∴ point is (4, 4)

9. $f(x) = a^{\{a^{|x|} \operatorname{sgn} x\}}$; $g(x) = a^{[a^{|x|} \operatorname{sgn} x]}$

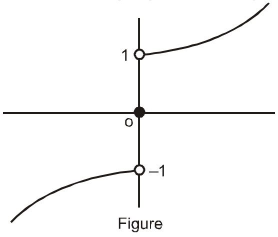
for a > 1, a ≠ 1 and x ∈ R

$\ell n\, a\, h(x) = \ell n\, f(x) + \ell n g(x)$

⇒ $(\ell n\, a)\, h(x) = \left\{ a^{|x|} \operatorname{sgn} x \right\} \ell na + [\, a^{|x|} \operatorname{sgn} x]\ \ell n\, a \Rightarrow h(x) = \left\{ a^{|x|} \operatorname{sgn} x \right\} + [\, a^{|x|} \operatorname{sgn} x]$

⇒ $h(x) = a^{|x|} \operatorname{sgn} x$

Now $h(-x) = a^{|-x|} \operatorname{sgn}(-x) = -h(x)$ ⇒ h(x) is an odd function Also graph of h(x) is



Figure

It is clear from the graph that h(x) is an increasing function

10.✍ x > 1 ⇒ f(x) ≥ f(1)

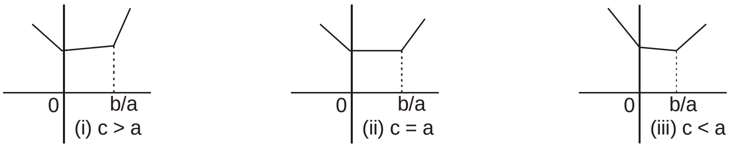
$x > 1 \quad \Rightarrow \quad g(x) \le g(1) \Rightarrow \quad f(g(x)) \le f(g(1))$

$\Rightarrow \quad h(x) \le 1$ .... (i)

Range of $h(x)$ is subset of $[1, 10]$ $\quad\Rightarrow\quad h(x) \ge 1$ .... (ii)

By (i), (ii) we have $h(x) = 1$ $\quad\Rightarrow\quad h(2) = 1$

11.✍ $f(x) = \begin{cases} b-(a+c)x \ , & x < 0 \\ b+(c-a)x \ , & 0 \le x < b/a \\ (a+c)x-b \ , & x \ge b/a \end{cases}$



(i) c > a (ii) c = a (iii) c < a

12. $f'(x) = \dfrac{e^x + e^{-x}}{2} > 0$

$\therefore$ f(x) increasing hence g(x) is also increasing function

13. $f'(x) = 3x^2 - 3p2x + 3p^2 - 3 = 3((x-p)^2 - 1) = 3(x - (p+1))\,(x - (p-1))$

$\Rightarrow p - 1 > -2$ and $p + 1 < 4 \Rightarrow p > -1$ and $p < 3 \Rightarrow -1 < p < 3$

14.✍ $f'(x) = \dfrac{x^{1/x}}{x^2}$ $(1 - \ell nx)$ $f'(x) \le 0$, when $x \ge e$

$\therefore$ f(x) is decreasing function, when $x \ge e$

$\pi > e \quad\Rightarrow\quad f(\pi) < f(e)$

$\pi^{1/\pi} < e^{1/e} \quad\Rightarrow\quad e^{\pi} > \pi^{e}$

$\therefore$ Statement-1 is True, Statement-2 is False

15^. $\dfrac{x^2 + x + 2}{x^2 + 5x + 6} < 0 \Rightarrow x \in (-3, -2)$

For maximum or minimum of the function, put $f'(x) = 0 \Rightarrow a^2 - 3x^2 = 0 \Rightarrow x = -\dfrac{a}{\sqrt{3}}, \dfrac{a}{\sqrt{3}}$

If $a > 0$, then point of minima is $x = -\dfrac{a}{\sqrt{3}} \Rightarrow -3 < -\dfrac{a}{\sqrt{3}} < -2$ or $2\sqrt{3} < a < 3\sqrt{3}$

if $a < 0$, then point of minima is $x = \dfrac{a}{\sqrt{3}} \Rightarrow -3 < \dfrac{a}{\sqrt{3}} < -2 \Rightarrow -3\sqrt{3} < a < -2\sqrt{3}$

16.✍ $f'(x) = \sin x \cos x\,(3 \sin x + 2\lambda)$

$f'(x) = 0$

$\Rightarrow \quad \sin x = 0 \quad$ or $\quad \cos x = 0 \quad$ or $\quad \sin x = \dfrac{-2\lambda}{3}$

$\Rightarrow \quad x = 0 \quad$ or $\quad \sin x = \dfrac{-2\lambda}{3}$ (as cos x = 0 is not possible).

If $\lambda = 0$ then $f'(x) \ge 0$

$\Rightarrow$ no extrema,

hence $\lambda \ne 0$

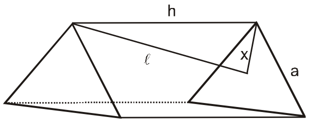
$\Rightarrow \quad -1 < \dfrac{-2\lambda}{3} < 0 \quad$ or $\quad 0 < \dfrac{-2\lambda}{3} < 1$

$\Rightarrow \quad 0 < \lambda < \dfrac{3}{2} \quad$ or $\quad -\dfrac{3}{2} < \lambda < 0$

17. $\ell^2 = h^2 + x^2$

Area of base (triangle) is $\frac{\sqrt{3}}{4}a^2$

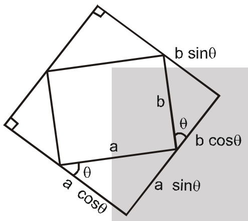
3x = $\frac{\sqrt{3}}{2}$ a Volume V = $\frac{\sqrt{3}}{4}$ ha^2 = h $\frac{\sqrt{3}}{4}$. 4. 3. x^2 = 3$\sqrt{3}$ h (ℓ^2 – h^2) $\frac{dV}{dh}$ = 3$\sqrt{3}$ (ℓ^2 – 3h^2)



Figure

V is maximum when h = $\frac{\ell}{\sqrt{3}}$.

18.

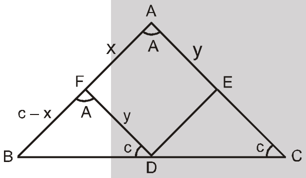


Figure

Area = ab + $\left(\frac{1}{2}a^2\sin\theta\quad\cos\theta+\frac{1}{2}b^2\sin\theta\cos\theta\right)$. 2 = ab + $\frac{(a^2+b^2)}{2}$ sin 2θ

Maximum area is ab + $\frac{(a^2+b^2)}{2}$

19.✍



Figure

From similar triangles ΔABC, ΔFBD $\frac{c-x}{y}$ = $\frac{c}{b}$

Area of AFDE = xy sin A = $\frac{b}{c}$ (c – x) sin A

It is maximum when x = $\frac{c}{2}$

∴ Maximum area = $\frac{bc}{4}$ sinA

statement -1 is true
statement-2 is obvious.

20.✍ Let f(x) = $\sqrt{x}$ If x ∈ 7

$$\frac{f(x+1)-f(x)}{x+1-x}=f'(c)$$

$\sqrt{x+1}-\sqrt{x}$ = $\frac{1}{2\sqrt{c}}$, N^2 < x < c < x + 1 ; c > N^2

$$\sqrt{x+1}-\sqrt{x}=\frac{1}{2\sqrt{c}}<\frac{1}{2N}$$

21.✍ f(x) = $\frac{\log x}{x}$ is differentiable and continous for every x > 0. Now for RMV to be applicable f(a) = f(b)

$\Rightarrow \frac{\ell na}{a} = \frac{\ell nb}{b} \Rightarrow a^b = b^a \Rightarrow a = 2, b = 4$

hence $a^2 + b^2 = 20$.

22. Let $g(x) = f(x) - x^2 \Rightarrow g(x)$ has atleast 3 real roots which are x = 1,2,3

By langrange mean value theorem (LMVT)

$\Rightarrow g'(x)$ has atleast 2 real roots in $x \in (1, 3) \Rightarrow g''(x)$ has atleast 1 real roots in $x \in (1, 3)$

$\Rightarrow f''(x) - 2 = 0$ for atleast 1 real roots in $x \in (1, 3) \Rightarrow f''(x) = 2$ for atleast one $x \in (1, 3)$

PART-II

1. Parametric form of curve is $x = 3t^2$, $y = 2t^3$ $\frac{dy}{dx} = t$

Let $P\left(3t_1^2,\ 2t_1^3\right)$, Q $\left(3t_2^2,\ 2t_2^3\right)$

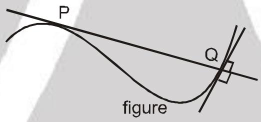
Conditions are

(i) $\left(\frac{dy}{dx}\bigg|_P\right)\left(\frac{dy}{dx}\bigg|_Q\right) = -1$

(ii) $\frac{dy}{dx}\bigg|_P$ = Slope of line segment PQ

$t_1 t_2 = -1$ ...(i);

$t_1 = \frac{2}{3} \quad \frac{t_2^2 + t_1t_2 + t_1^2}{t_2 + t_1}$ ...(ii)



$\Rightarrow \quad 3(-1 + t_1^2) = 2\left(t_2^2 - 1 + t_1^2\right)$

$t_1^2 = \frac{2}{t_1^2} + 1$

$t_1^2 = 2, -1$

$t_1^2 = 2 \Rightarrow t_1 = \pm\sqrt{2} \Rightarrow t_2 = \mp\frac{1}{\sqrt{2}}$

If $t_1 = \sqrt{2}$, $t_2 = -\frac{1}{\sqrt{2}} \Rightarrow P(6, 4\sqrt{2}), Q\left(\frac{3}{2}, -\frac{1}{\sqrt{2}}\right) \Rightarrow$ Required line is $y = \sqrt{2}\,x - 2\sqrt{2}$

If $t_1 = -\sqrt{2}$, $t_2 = \frac{1}{\sqrt{2}} \Rightarrow P(6, -4\sqrt{2}), Q\left(\frac{3}{2}, \frac{1}{\sqrt{2}}\right) \Rightarrow$ Required line is $y = -\sqrt{2}\,x + 2\sqrt{2}$

2. Any point on $y = x^2 + 4x + 8$ is $P(h, h^2 + 4h + 8)$. $\frac{dy}{dx}\bigg|_p = 2h + 4$

Tangent at P is $(2h + 4)\,x - y = h^2 - 8$.

It is sufficient to have only one solution for equations $y = x^2 + 8x + 4$,

$y = (2h + 4)\,x + 8 - h^2$

$\Rightarrow \quad x^2 + (4 - 2h)\,x + h^2 - 4 = 0 \quad \Rightarrow \quad D = 0$

$(4 - 2h)^2 - 4(h^2 - 4) = 0 \quad \Rightarrow \quad h = 2$

$8x - y + 4 = 0$

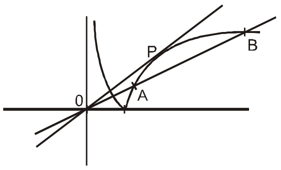
coordinates of point of contact are (2, 20) and (0, 4)

3. $|\ell nx| = px$

It is sufficient to find values of p for which $y = |\ell nx|$ and $y = px$ has three points in common.

If y = px is passing through points O, A, B then we obtain three roots. Let us consider line y = px. When it is passing through points O, P. (tangent)

Let $\quad P(\alpha, p\alpha)\ (\alpha > 1) \Rightarrow p\alpha = |\ell n\alpha|$



Figure

$p\alpha = \ell n\alpha \qquad .....(1)$

$$\frac{dy}{dx} = \frac{1}{x}$$

$p = \frac{1}{\alpha} \qquad .....(2)$

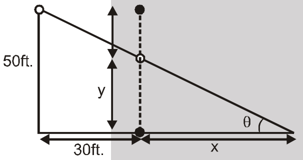
$(1)\ ,\ (2) \Rightarrow \qquad \alpha = e$

Slope of tangent (OP) is $\frac{1}{e}$.

For three roots condition is $0 < p < \frac{1}{e}$.

4. $y = 50 - 16t^2$ So, $\frac{dy}{dt} = -32t$

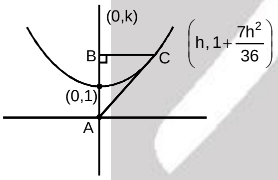
$\tan\theta = \frac{y}{x} = \frac{50}{30+x} \Rightarrow y = \left(\frac{50}{30+x}\right) . x$



Figure

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = \frac{1500}{(30+x)^2}\cdot\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -16\,\frac{(375)^2}{1500} = -1500 \text{ ft/sec.} = 100\lambda \text{ ft/sec} \Rightarrow \lambda = -15 \Rightarrow |\lambda| = 15$$

5. at t = 0 ; x = 0, y = 1



$\frac{dy}{dt}$ = 2cm/sec

$$A = \frac{1}{2}\times h\times\left(1+\frac{7h^2}{36}\right)$$

$$\frac{dA}{dt} = \left(\frac{1}{2}\left(1+\frac{7h^2}{36}\right)+\frac{h}{2}\left(\frac{14h}{36}\right)\right)\ \frac{dh}{dt}\ ;\ \frac{dA}{dt} = \left(\frac{1}{2}\times 8+3\times\frac{14\times 6}{36}\right)\times\ \frac{6}{(7/2)}$$

(At, t = 7/2 sec, change in y -co-ordinate = 7 hence, pt. C has

y-co-ordinate = 8 and x- co-ordinate = 6 at t = 7/2 sec.) $= (4 + 7) \times \frac{6}{7} \times 2 = \frac{132}{7}$ cm²/sec

6. $$f'(x) = \frac{2x\left(e^{x^2}+e^{-x^2}\right)\ \left(e^{x^2}+e^{-x^2}\right)-\left(e^{x^2}-e^{-x^2}\right)\ \left(2xe^{x^2}-2xe^{-x^2}\right)}{\left(e^{x^2}+e^{-x^2}\right)^2}$$

$$= \frac{2x\ \left(\left(e^{x^2}+e^{-x^2}\right)^2-\left(e^{x^2}-e^{-x^2}\right)^2\right)}{\left(e^{x^2}+e^{-x^2}\right)^2} = \frac{8x}{\left(e^{x^2}+e^{-x^2}\right)^2} \geq 0\ x \in [0, \infty) \Rightarrow \text{least value of } \alpha \text{ is } 2.$$

7. Let $f(x) = 2\sin x + \tan x - 3x$

$$f'(x) = 2\cos x + \sec^2 x - 3 = \frac{(\cos x - 1)^2\ \ (2\cos x + 1)}{\cos^2 x} > 0$$

f(x) is M.I.

$x > 0$

$f(x) > f(0)$

$2\sin x + \tan x > 3x$

$$3x < 2\sin x + \tan x \Rightarrow \frac{3x}{2\sin x + \tan x} < 1 \text{ for } x \in \left(0\ ,\ \frac{\pi}{2}\right)$$

and $$\underset{x\to 0^+}{\text{Lt}}\ \frac{3x}{3\sin x + \tan x} = 1 \Rightarrow \underset{x\to 0^+}{\text{Lt}}\ \left[\frac{3x}{3\sin x + \tan x}\right] = 0$$

and $$\lim_{x\to 0^+} \frac{\tan^3 x - \sin^3 x}{x^5} = \lim_{x\to 0^+} \frac{\tan^3 x(1-\cos^3 x)}{x^5} = \lim_{x\to 0^+} \frac{\tan^3 x}{x^3}\ \frac{(1-\cos x)}{x^2}(\cos^2 x + \cos x + 1) = \frac{3}{2}$$

Hence $$\lim_{x\to 0^+} \left(2 + \left[\frac{3x}{2\sin x + \tan}\right]\right)^{\frac{\tan^3 x - \sin^3 x}{x^5}} = (2+0)^{3/2} = 2\sqrt{2} = 2.828$$

8.✍ $f(x) = 2e^x - ae^{-x} + (2a-3)x - 3$

$$f'(x) = 2e^x + ae^{-x} + (2a-3) \geq 0\ \forall\ x \in R \Rightarrow a \geq \frac{e^x(3-2e^x)}{(1+e^x)}$$

Let $y = \frac{e^x(3-2e^x)}{(1+e^x)}$ Let $e^x = t$

$$y = t\left(\frac{3-2t}{1+2t}\right) t \in (0, \infty)\ \ \frac{dy}{dt} = -\frac{(2t+3)(2t-1)}{(1+2t)^2}$$

hence maximum value of y will be at $t = \frac{1}{2}$ $y_{\text{max.}} = \frac{1}{2}$ hence minimum value of a is $\frac{1}{2}$

9. $f'(x) = 4\left[\cos^2 x - 2(a+1)\cos x + a^2 + 2a - 4\right]$

Let $\phi(t) = t^2 - 2(a+1)t + a^2 + 2a - 4,\ \ -1 \leq t \leq 1$

It is sufficient to find values of t when $\phi(t) = 0$ has no root in [–1, 1]

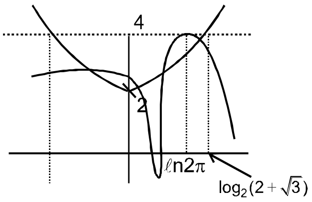
D = 20

Case-I : $\phi(-1) > 0$, $a + 1 < -1$, $D = 20 > 0$

$$\Rightarrow a \in \left(-\infty,\ \ -2-\sqrt{5}\right)\ \cup\ \left(-2,\ \ +\sqrt{5},\ \ \infty\right) \text{and } a < -2 \Rightarrow a \in\ \left(-\infty,\ \ -2-\sqrt{5}\right)$$

Case - II : $\phi(1) > 0$, $a + 1 > 1$, $D = 20 > 0$ $\Rightarrow a \in \left(-\infty,\ \ -\sqrt{5}\right)\ \cup \left(\sqrt{5},\ \ \infty\right)$ and $a > 0$ $\Rightarrow a \in \left(\sqrt{5}, \infty\right)$

10. Using graph of expressions on both the sides, we get only two roots.



11. ✍ $-1 \le p \le 1$

Consider $f(x) = 4x^3 - 3x - p = 0$

$f\left(\frac{1}{2}\right) \cdot \le 0$

$f(1) \ge 0$

$\therefore$ f(x) has at least one root between $\left[\frac{1}{2},\ 1\right]$

Also $f'(x) = 12x^3 - 3 > 0\ \forall \left[\frac{1}{2},\ 1\right] \Rightarrow$ f is increasing on $\left[\frac{1}{2},\ 1\right]$

$\Rightarrow$ f(x) has only one real root between $\left[\frac{1}{2},\ 1\right]$ To find root put $x = \cos\theta$

$\Rightarrow \cos 3\theta = p \quad \Rightarrow \quad \theta = \frac{1}{3}\cos^{-1} p$

$\therefore$ Root is $\cos\left(\frac{1}{3}\cos^{-1}\ p\right)$

12. Let $y = 2^{x^2} - 1 + \dfrac{200}{2^{x^2}+1}$

let $2^{x^2} = t \Rightarrow y = t - 1 + \dfrac{200}{t+1}$ $t \in [1,\infty) \Rightarrow \dfrac{dy}{dt} = \dfrac{(t+1)^2 - 200}{(t+1)^2} = \dfrac{(t+1+10\sqrt{2})(t+1-10\sqrt{2})}{(t+1)^2}$

$\Rightarrow$ hence function has minimum value at $t = -1 + 10\sqrt{2}$ $y_{\text{min.}} = 20\sqrt{2} - 2 = 26.28$

13. $f(x) = (x-1)^{2013} + (x-2)^{2013} + \ldots\ldots\ldots.. + (x-2013)^{2013}]$

$f'(x) = 2013\,[(x-1)^{2012} + (x-2)^{2012} + \ldots\ldots\ldots.. + (x-2013)^{2012}]$

$f'(x) > 0 \Rightarrow f(x) \uparrow$

$f(x) \in (-\infty, \infty) \Rightarrow$ f(x) can have only one real root

Due to symmetric nature, real root is $\dfrac{1+2013}{2} = \dfrac{2014}{2} = 1007$

14. ✍ $f(x) = \dfrac{a}{3}x^3 + (a+2)x^2 + (3a-10)$

$f'(x) = g(x) = ax^2 + 2(a+2)x + (a-1) = a(x-\alpha)(x-\beta)\ (\beta > \alpha)$

Case - I : If $a = 0 \Rightarrow f'(x) = 4x - 10 \Rightarrow x = \dfrac{10}{4}$ is point of minimum (which is not negative)

Case- II : If a > 0 then β will be point of minima which is negative hence both root of g(x) = 0 must be negative. $\left.\begin{matrix} g(0) > 0 \\ D > 0 \\ -b/2a < 0 \end{matrix}\right\} \Rightarrow a \in \left(\dfrac{10}{3}, \dfrac{7+\sqrt{57}}{2}\right)$

Case-III : If a < 0 then α will be point of minima which is negative hence β can be negative, zero or positive but g(0) < 0 is hence β can be negative only hence again g(x) = 0 must have both negative roots $\left.\begin{matrix} g(0) < 0 \\ D > 0 \\ -b/2a < 0 \end{matrix}\right\} \Rightarrow a \in \phi$ hence exhaustive set of values of a is $\left(\dfrac{10}{3}, \dfrac{7+\sqrt{57}}{2}\right)$

15. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$

$\lim\limits_{x \to 0} \left(1 + \frac{f(x)}{x^3}\right) = 1 \Rightarrow \quad a_0 = a_1 = a_2 = a_3 = 0$

$\lim\limits_{x \to 0} \left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e^2$

$\lim\limits_{x \to 0} e^{(a_4 + a_5x + a_6x^2)} = e^2 \Rightarrow a_4 = 2$

$f(x) = 2x^4 + a_5x^5 + a_6x^6$

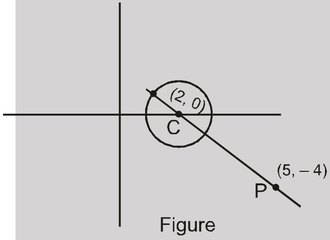
$f'(x) = x^3 (8 + 5a_5x + 6a_6x^2)$ $f'(1) = 0, \ f'(2) = 0$

$a_5 = -\frac{12}{5}, \ a_6 = \frac{2}{3} \quad f(x) = 2x^4 - \frac{12}{5}x^5 + \frac{2}{3}x^6$

16. Let $y = \sqrt{-3 + 4x - x^2}$

$x^2 + y^2 - 4x + 3 = 0$

$(x - 2)^2 + y^2 = 1$, center C = (2, 0)



Figure

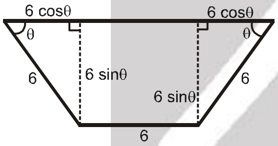
Consider point P(5, – 4)

$CP = \sqrt{9 + 16} = 5$

Maximum value of $\left(\sqrt{-3 + 4x - x^2} + 4\right)^2 + (x - 5)^2$ is $(5 + 1)^2 = 36$.

17. Area = 36 (1 + cosθ) sinθ = 36.4. $\cos^3(\theta/2)\sin(\theta/2)$

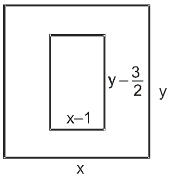
Maximum occurs when $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{3}} \Rightarrow \theta = \frac{\pi}{3}$



Figure

18. xy = 18

Area of printed space = $(x - 1)\left(y - \frac{3}{2}\right) = 18 + \frac{3}{2} - \left(\frac{3x}{2} + \frac{18}{x}\right)$



Figure

Maximum when $\frac{3x}{2} = \frac{18}{x} \Rightarrow x = 2\sqrt{3} \ \ y = 3\sqrt{3}$

19. Fuel charges per hour = $k\nu^2 \Rightarrow \quad 48 = k.\ 16^2$

$\Rightarrow$ Fuel charges per hour = $\frac{3}{16}\ \nu^2$

Charges per hour = $\frac{3}{16}\ \nu^2 + 300$

Expenses of journey = $(\frac{3}{16}\nu^2 + 300)\ \frac{s}{\nu}$

where ν = speed s = distance

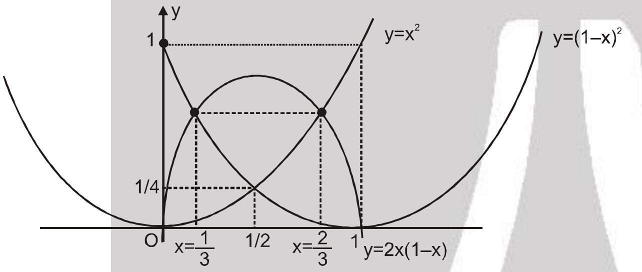
Maximum occures when $\frac{3\nu}{16} = \frac{300}{\nu}$

($\because$ $ax + \frac{b}{x}$, a, b, > 0, x > 0, has minimum when $ax = \frac{b}{x}$)

$\nu^2 = 16.100$

$\nu = 40$

20.



From figure we can see Rolle'stheorem is applicable for $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and f'(c)= 0 = 2 – 4c

$\Rightarrow$ c = $\frac{1}{2}$ a + b + c = $\frac{1}{3}+\frac{2}{3}+\frac{1}{2}=\frac{3}{2}$

21. There exist atleast one $c \in (\alpha, \alpha + d)$ such that $f'(c) = \frac{f(\alpha+d)-f(\alpha)}{d} \Rightarrow f'(c) \in \left[\frac{-6}{d},\frac{6}{d}\right]$.

If d is very large then f'(c) will approach to zero. Also $(f'(\alpha))^2 \in [77, 83]$ Because f'(x) is continuous function so $(f'(x))^2$ can take all values from (0, 77] $\Rightarrow$ $(f'(x))^2$ can take 76 integral values in (0, 77)

PART - III

1. $2y^3 = ax^2 + x^3 \Rightarrow 6y^2\ \frac{dy}{dx} = 2ax + 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{(a,\ a)} = \frac{5a^2}{6a^2} = \frac{5}{6}$

Tangent at (a, a) is $5x - 6y = -a \Rightarrow \alpha = \frac{-a}{5}, \beta = \frac{a}{6}$

$\alpha^2 + \beta^2 = 61 \quad \Rightarrow \quad \frac{a^2}{25}+\frac{a^2}{36} = 61$

$a^2 = 25.36$

$a = \pm 30$

2. $x = 2 \quad \Rightarrow \quad t^2 + 3t - 10 = 0 \quad \Rightarrow \quad t = 2, -5$

$y = -1 \quad \Rightarrow \quad t^2 - t - 2 = 0 \quad \Rightarrow \quad t = 2, -1$

$\Rightarrow \quad t = 2$ (common value)

$\frac{dy}{dx} = \frac{4t-2}{2t+3} \Rightarrow .\ \left.\frac{dy}{dx}\right|_{t=2} = \frac{6}{7} = \frac{-1}{\frac{6}{7}} = -\frac{7}{6} = \frac{-1}{\frac{6}{7}}\sqrt{1+\frac{36}{49}} = -\frac{\sqrt{85}}{6}$

3. Let $f(x) = x + \sin x \Rightarrow f'(x) = 1 + \cos x$

As $f'(x) \ge 0 \ \forall \ x \in R$, f(x) is increasing

Let $g(x) = \sec x$

$g'(x) = \sec x \tan x$

$g'(x)$ changes sign.

g(x) is neither increasing nor decreasing.

4.✍ $f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} = 1 - \frac{1}{1+x^2} + 1 - \frac{1}{\sqrt{x^2+1}} = \frac{x^2}{1+x^2} + \left(1 - \frac{1}{\sqrt{x^2+1}}\right) \ge 0$

5.✍ $g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$ and $g'(x) = f'(x/2) - f'(1-x)$

Now g(x) is increasing if $g'(x) \ge 0$

$f'\left(\frac{x}{2}\right) \ge f'(1-x)$

[$\because$ $f''(x) < 0$ i.e. $f'(x)$ is decreasing]

$\Rightarrow \quad \frac{x}{2} \le 1 - x \Rightarrow x \le 2 - 2x \Rightarrow 3x \le 2 \Rightarrow x \le 2/3 \Rightarrow 0 \le x \le \frac{2}{3} \Rightarrow$ g(x) increases in $0 \le x \le 2/3$

and $g'(x) \le 0$ for decreasing $\Rightarrow f'\left(\frac{x}{2}\right) \le f'(1-x) \Rightarrow \frac{x}{2} \ge 1 - x \Rightarrow x \ge 2/3 \Rightarrow 2/3 \le x \le 1$

6. $f'(x) = \frac{m}{n} x^{\left(\frac{m-n}{n}\right)}$

m – n is odd.

$f'(x) < 0 \ \forall \ x \in (-\infty, 0)$

$f'(x) > 0 \ \forall \ x \in (0, \infty)$

7. Let $h(x) = f(x)\,g(x)$

$h'(x) = f'(x)\,g(x) + g'(x)\,f(x)$

As $f'(x) < 0, g(x) \le 0 \Rightarrow f'(x)\,g(x) \ge 0$ and $g'(x) > 0, f(x) \ge 0 \Rightarrow f(x)\,g'(x) \ge 0$

$\Rightarrow h'(x) \ge 0 \Rightarrow$ h(x) is increasing.

Let $x_1, x_2 \in I$

$x_1 < x_2$

$g(x_1) < g(x_2)$

$f(g(x_1)) > f(g(x_2))$

$fog(x_1) > fog(x_2)$

$\Rightarrow$ fog(x) is monotonically decreasing.

8.✍ $\phi'(x) = (3(f(x))^2 - 6(f(x)) + 4)f'(x) + 5 + 3\cos x - 4\sin x$

$5 - \sqrt{9+16} \le 5 + 3\cos x - 4\sin x \le 5 + \sqrt{9+16}$

adding $(3(f(x))^2 - 6(f(x)) + 4)f'(x)$

$(3(f(x))^2 - 6(f(x)) + 4)f'(x) \le \phi'(x) \le (3(f(x))^2 - 6(f(x)) + 4)f'(x) + 10$

$\because \quad 3(f(x))^2 - 6f(x) + 4 = 3(f(x) - 1)^2 + 1 > 0$

$(3(f(x))^2 - 6(f(x)) + 4)f'(x) \ge 0$ when ever f(x) is increasing.

$\Rightarrow \phi'(x) \ge 0 \Rightarrow \phi(x)$ is increasing, when ever f(x) is increasing.

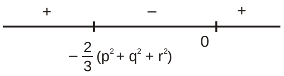
If $f'(x) = -11$ then

$(3(f(x))^2 - 6f(x) + 4)\,f'(x) + 10 = -33\,(f(x) - 1)^2 - 1 < 0$

$\Rightarrow \phi'(x) < 0 \Rightarrow \phi(x)$ is decreasing.

9. $f(x) = \begin{vmatrix} x+p^2 & pq & pr \\ pq & x+q^2 & qr \\ pr & qr & x+r^2 \end{vmatrix} = x^3 + (p^2 + r^2 + q^2)\,x^2$

$f'(x) = 3x^2 + 2x(p^2 + q^2 + r^2) \;=\; x\,\{3x + 2(p^2 + q^2 + r^2)\}$



Here f(x) is increasing if $x < -\frac{2}{3}$ $(p^2 + q^2 + r^2$

13. $f'(x) = (x-1)^{n-1}(x+1)^{n-1}[2(n+1)x^3 + (2n+1)x^2 + 2(n-1)x - 1]$

At x = 1 $2(n+1)x^3 + (2n+1)x^2 + 2(n-1)x - 1 \neq 0$

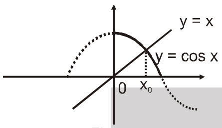
for $n \in N$

$\therefore$ n – 1 must be odd

$\Rightarrow$ n is even

14. $f'(x) = \dfrac{\sec^2 x(\cos x + x)\quad(\cos x - x)}{(1 + x\quad \tan x)^2}$

The only factor in f′(x) which changes sign is cosx – x.



Figure

Let us consider graph of y = cos x and y = x

It is clear from figure that for $x \in (0, x_0)$, cos x – x > 0 and for $x \in \left(x_0, \dfrac{\pi}{2}\right)$

cos x – x < 0, $\Rightarrow$ f′ (x) has maxima at x_0

15. $f'(x) = \dfrac{a}{x} + 2bx + 1$

f′(– 1) = 0

– a – 2b + 1 = 0

a + 2b = 1

f′(2) = 0

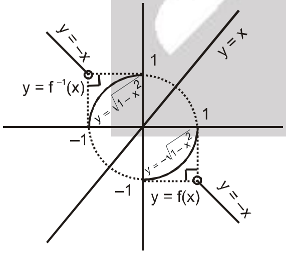
$\dfrac{a}{x} + 4b + 1 = 0 \Rightarrow a + 8b + 2 = 0$

$-6b = 3 \Rightarrow b = \dfrac{-1}{2}, a = 2$

16. From graph $f^{-1}(x) = \begin{cases} -x & ; \quad x < -1 \\ \sqrt{1-x^2} & ; \quad -1 \leq x \leq 0 \end{cases}$

Maximum of f(x) exist at x = 1

Minimum of $f^{-1}(x)$ exist at x = – 1



Figure

17. $f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{2}\ \dfrac{1}{x}$, x > 0 $= \dfrac{-(x-1)^2}{2x(1+x^2)} \leq 0 \ \forall\ x > 0.$

f(x) is decreasing $\forall$ x > 0.

On $\left[\dfrac{1}{\sqrt{3}}, \ \sqrt{3}\right]$, greatest value is $f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} - \dfrac{1}{2}\ \ell n\left(\dfrac{1}{\sqrt{3}}\right)$ and least value is $f(\sqrt{3}) = \dfrac{\pi}{3} - \dfrac{1}{2}\ \ell n.\ \sqrt{3}$

18.✍ $f'(x) = \dfrac{-\ 40.12x(x+3)(x-1)}{(3x^4+8x^3-18x^2+60)^2}$

$f'(x) = 0$

at $x = 0, x = -3, x = 1$

so at $x = 0$, $f(x)$ has local minima.

and at $x = -3$, $x = 1$; $f(x)$ has local maxima

$f(1) = \dfrac{40}{53}$, $f(-3) = \dfrac{-\ 40}{75}$. $f(-3) < 0$, $f(1) > 0$ and $f(x) \neq 0$

$\Rightarrow$ f(x) is undefined at point(s) in (–3, 1). Hence f(x) has no absolute maxima.

19. $f(x) = \dfrac{x-2}{x+3}$; $x \neq 1, -3$ $\Rightarrow f'(x) = \dfrac{5}{(x+3)^2}$

20.✍ $(2x)^2 + 2.2x \,.\, 3y + (3y)^2 + (y-1)(y-3) = 0 \Rightarrow (2x+3y)^2 + (y-1)(y-3) = 0$

$x \in R$

So $D \geq 0$

$144y^2 - 16(10y^2 - 4y + 3) \geq 0$

$16[-y^2 + 4y - 3] \geq 0$

$y^2 - 4y + 3 \leq 0$

$(y-1)(y-3) \leq 0$

$\therefore 1 \leq y \leq 3$

So $y_{max} = 3$ and $y_{min} = 1$

21.✍ $f(0) = 0 \neq f(1)$

there will be no $x \in (0, \infty)$ ($\therefore$ Rolle's theorem is not applicable)

for which f'(x) = 0 i.e, $\cot^{-1} x = \dfrac{x}{1+x^2}$

$$f''(x) = \frac{-1}{1+x^2} - \frac{\left(1+x^2\right)-2x^2}{\left(1+x^2\right)^2} = \frac{-1}{1+x^2} + \frac{x^2-1}{\left(x^2+1\right)^2}$$

$$f''(x) = \frac{-2}{(x^2+1)^2} < 0$$

f'(x) is strictly decreasing $\lim\limits_{x\to\infty} f'(x) = \lim\limits_{x\to\infty}\left(\dfrac{-x}{1+x^2} + \cot^{-1} x\right) = 0$

$$f(0+) = \lim_{x\to 0^+}\left(\cot^{-1} x - \frac{x}{1+x^2}\right) = \frac{\pi}{2} \qquad \frac{f\left(x+\frac{2}{\pi}\right)-f(x)}{2/\pi} = f'(c) \ \ c \in \left(0, \frac{\pi}{2}\right)$$ ($\therefore$ LMVT is applicable)

$\therefore\ f'(c) < \dfrac{\pi}{2}$

$$f\left(x+\frac{2}{\pi}\right) - f(x) < \frac{2}{\pi} \times \frac{\pi}{2}$$

$$f\left(x+\frac{2}{\pi}\right) - f(x) < 1$$

$f'(x) \geq 0$; f(x) is increasing

$f(x) \in [f(0), f(\infty))$

$f(0) = 0$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\cot^{-1} x = \lim_{x\to 0}\frac{\cot^{-1} x}{1/x} = \lim_{x\to\infty}\frac{-1}{1+x^2} \times (-x^2) = 1$$

$f(x) \in [0, 1) \Rightarrow f(x) = \sec x$ will have no solution

22^. $f(4) = f(5) = f(6) = f(7) = 0$

By Rolle's theorem on interval

$[4, 5], [5, 6], [6, 7]$ we have

$f'(x) = 0$ for at least once in each intervals $(4, 5), (5, 6), (6, 7)$.

23. (A) let $f(x) = \tan^{-1}x$

$$f'(x) = \frac{1}{1+x^2}$$

$\therefore$ $|f'(c)| = \dfrac{1}{1+c^2} < 1$

$$\left|\frac{\tan^{-1}x - \tan^{-1}x}{x-y}\right| < 1$$

(B) Let $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x > 0,\ x \in [0, 1]$

$\Rightarrow$ f(x) is increasing.

(C) Suppose $f(x) = ax^3 - 2bx^2 + cx$, then clearly $f(0) = 0$

and $f(1) = a - 2b + c = 0$,

$\because$ $f(0) = f(1)$

$\therefore$ By Rolle's theorem $f'(x) = 3ax^2 - 4bx + c = 0$

for atleast one x in (0, 1) which is positive

(D) $f(x) = 3\tan x + x^3 - 2$

$\Rightarrow f'(x) = 3\sec^2x + 3x^2 > 0$

$\Rightarrow$ f(x) is always increasing

24.✍ Apply Roll's theorem on f(x), $g(x) = e^x f(x)$ and $h(x) = e^{-x} f(x)$

25.✍ (A) Let $x \Rightarrow x + h$ and $y \to x$

$|\tan^{-1}x - \tan^{-1}y| \le |x - y|$

$|\tan^{-1}(x + h) - \tan^{-1}x| \le |h|$

$$\left|\frac{d}{dx}(\tan^{-1}x)\right| \le 1$$

$\left|\dfrac{1}{1+x^2}\right| \le 1$ hence true

(C) $|\sin x - \sin y| \le |x - y|$

$x \to x + h$ $y \to x$

$$\left|\frac{\sin(x+h) - \sin x}{h}\right| \le 1$$

$|\cos x| \le 1$ hence true

Alternative solutions

For x = y, this is true

$\therefore$ Let $x, y \in R$ and $x < y$

consider $f(t) = \tan^{-1}t,\ t \in [x, y]$

Using LMVT, $\dfrac{\tan^{-1}y - \tan^{-1}x}{y-x} = \dfrac{1}{1+c^2}$, $c \in (x, y)$

$\Rightarrow \tan^{-1} y - \tan^{-1}x = \dfrac{y-x}{1+c^2} \le y - x$ ....(i)

similarly $x > y$, $\tan^{-1} x - \tan^{-1}y \le x - y$ ....(ii)

From (i) and (ii) we get $\left|\tan^{-1}x - \tan^{-1}y\right| \le |x - y|$

Similarly considering g(t) = sin t in [x, y] we get $\dfrac{\sin y - \sin x}{y-x} = \cos c$

$\Rightarrow$ $\sin y - \sin x = (\cos c)(y - x) \le y - x$ ....(iii)

and $\sin x - \sin y \le x - y$ ....(iv)

(iii), (iv) $\Rightarrow |\sin x - \sin y| \le |x - y|$

26. ✍ $|f''(c)| \le 2 \Rightarrow$ $|f'(x) - f'(a)| \le 2|x - a|$ $\Rightarrow$ $|f'(x) - f'(a)| < 2$

a is that point on the interval when $f'(a) = \left|\dfrac{f(2)-f(1)}{2-1}\right| = 0$ as $f(1) = f(2)$

27_. **(A)** Apply L. M. V. T. in (a, a + 6) we get $f'(c) = \dfrac{f(a+6)-f(a)}{6}$

Now $f(a + 6) - f(a) \in [-6, 6] \Rightarrow f'(c) \in [-1, 1] \Rightarrow (f'(c))^2 \le 1$

(B) Apply L. M. V. T. in (–d, 0) where d is positive then $f'(c) = \dfrac{f(0)-f(-d)}{d} \Rightarrow f'(c) \in \left[\dfrac{-6}{d}, \dfrac{6}{d}\right]$

If d is approaching to infinity then f'(c) approaches to zero. And $(f'(0))^2 \in [76, 85]$, Also f'(x) is continuous function. So $(f'(x))^2$ can take all real value from (0, 76]

(C) By using option (A) we get there exist a value $x_0 \in (-6, 0)$ such that $(f'(x))^2 \le 1$

also $f(x_0)^2 \le 9 \Rightarrow (f'(x_0))^2 + (f(x_0))^2 \le 10$

But $f(x))^2 + (f'(x)^2$ is continuous function and $(f(0)^2 + (f'(0)^2 = 85$ hence there exists $\alpha \in [x_0, 0]$ such that $(f'(\alpha))^2 + (f(\alpha))^2 = 10 \Rightarrow$ there exist $\alpha \in (-6, 0)$ such that $(f'(\alpha))^2 + (f(\alpha))^2 = 10$.

(D) Similarly by using option (C) we say that there exists $\beta \in (0, 6)$ such that $(f(\beta))^2 + (f'(\beta))^2 = 10$

Assume $H(x) = (f(x))^2 + (f'(x))^2$ where $H(0) = 85$

Let p is largest negative number lies between (–6, 0) such that H(p) = 10

Similarly let q is smallest positive number lies between (0, 6) such that H(q) = 10

Apply Rolle's in H (x) in interval (p, q) we get there exists r in (p, q) such that H'(r) = 0

$\Rightarrow$ $2f'(r)\,(f(r) + f''(r)) = 0$

Because in (p, q), $(f(x))^2 \le 9$ & $H(x) \ge 10$

$\Rightarrow (f'(x))^2 \ge 1 \Rightarrow f'(r) \ne 0 \Rightarrow f(r) + f''(r) = 0 \Rightarrow |f(r)| = |f''(r)|$

(E) There exist one $c \in (-3, 3)$ such that $f'(c) = \dfrac{f(3)-f(-3)}{6} \in [-1,1]$ and $f'(0)^2 \in [76, 85]$

let f'(0) ≥ $\sqrt{76}$ then f'(x) is mix up of increasing as well as decreasing function

$\Rightarrow$ f'''(c) < 0 for some c $\in$ (–3, 3) $\Rightarrow$ f'(c) f'''(c) < 0

PART - IV

1^. Here, $m = \left.\dfrac{dy}{dx}\right|_{x=0}$

$\dfrac{dy}{dx} = 3x^2 + 6x + 4 \quad \Rightarrow \quad m = 4$

and, $k = y(0) \quad \Rightarrow \quad k = -1$

$\ell = |k|\sqrt{1 + \dfrac{1}{m^2}} \quad \Rightarrow \quad \ell = |(-1)|\sqrt{1+\dfrac{1}{16}} = \dfrac{\sqrt{17}}{4}$

2. |yy'| = |y/y'| at (0, 1) $\Rightarrow$ (y') at (0, 1) equal ± 1 $\Rightarrow$ $(pe^{px} + p)_{(0,1)} = \pm 1 \Rightarrow 2p = \pm 1 \Rightarrow p = \pm 1/2$

3. Length of subnormal = $\left|y\dfrac{dy}{dx}\right| = \left|-3\sin\left(\dfrac{-\pi}{4}\right)\dfrac{-3\cos\left(\dfrac{-\pi}{4}\right)}{\left(-\sqrt{2}\sin\left(\dfrac{-\pi}{4}\right)\right)}\right| = \dfrac{3}{\sqrt{2}}\,\dfrac{3}{\sqrt{2}} = \dfrac{9}{2}$

(4 to 6)

Let $g(x) = \dfrac{x+\sin x}{2}$, $x \in [0,\pi]$. g(x) is increasing function of x.

$\therefore$ range of g(x) is $\left[0\ ,\ \dfrac{\pi}{2}\right]$

$\therefore$ $f(x) = \dfrac{x+\sin x}{2}$, $x \in [0, \pi]$

Now let $\pi \le t \le 2\pi$, then $f(t) + f(2\pi - t) = \pi$

i.e $f(t) + \dfrac{2\pi - t + \sin(2\pi - t)}{2} = \pi$

i.e $f(t) + \pi - \dfrac{t}{2} - \dfrac{\sin\ t}{2} = \pi$

i.e $f(t) = \dfrac{t+\sin t}{2}$

$\therefore$ $f(x) = \dfrac{x+\sin x}{2}$ for $\pi \le x \le 2\pi$

Thus $f(x) = \dfrac{x+\sin x}{2}$ for $0 \le x \le 2\pi$

Also f(x) = f(4π – x) for all x $\in$ [2π, 4π] $\Rightarrow$ f(x) is symmetric about x = 2π

$\therefore$ from graph of f(x)

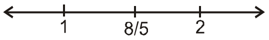
$\therefore$ $\alpha = 2\pi - 0 = 2\pi$

$\therefore$ $\beta = \alpha$

Maximum value is $f(2\pi) = \pi = \dfrac{\beta}{2}$

7. $y = (x-1)^3 (x-2)^2$

$$\frac{dy}{dx} = 3(x-1)^2(x-2)^2 + 2(x-2)(x-1)^3 = (x-1)^2(x-2)[3(x-2)+2(x-1)] = (x-1)^2(x-2)(5x-8)$$



$(x^2 - 2x + 1)(5x^2 - 18x + 16)$

$$\frac{d^2y}{dx^2} = (2x-2)(5x^2-18x+16) + (10x-18)(x^2-2x+1) = 0 = 20x^3 - 42x^2 + 11x - 50 = 0$$

$= 10x^3 - 42x^2 + 57x - 25 = 0$

$(x - 1)(10x^2 - 32x + 25) = 0$

$x = 1$ or $x = \dfrac{32 \pm \sqrt{24}}{20}$

no. of points of inflections = 3

8. $f(x) = x^4 + ax^3 + \dfrac{3x^2}{2} + 1$

$f'(x) = 4x^3 + 3ax^2 + 3x$

$f''(x) = 12x^2 + 6ax + 3$

Now, f(x) will be concave upward along the entire real line iff

$f''(x) \geq 0 \ \forall \ x \in R$

$12x^2 + 6ax + 3 > 0 \quad \Rightarrow \quad D \leq 0$

$36a^2 - 144 \leq 0$

$a^2 - 4 \leq 0 \quad \Rightarrow \quad a \in [-2, 2]$

9. sin x is concave downward in $(0, \pi)$ and sin x is concave upward in $(\pi, 2\pi)$

10. e^x, 2^x, $\tan^{-1}x$ (If $x \in R^-$) is concave upward and ℓnx is concave downward

11. f(x) concave downward ($f''(x) < 0$)

f(x) increasing ($\therefore f'(x) \leq 0$)

Let $g(x) = f^{-1}(x) = x$

$f'(g(x)) = x$

$f'(g(x)).\, g'x = 1$

$$g'(x) = \frac{1}{f'(g(x))} > 0$$

$$g''(x) = -\frac{1}{(f'(g(x))^2} \times f''(x).g'(x)$$

$g''(x) > 0$

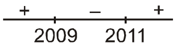
$g(x) = f^{-1}(x)$ concave upward

EXERCISE # 3

PART - I

1. $f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$

$f(x) = \ell n\,(g(x))$



$\Rightarrow \quad g(x) = e^{f(x)}$

$\Rightarrow \quad g'(x) = e^{f(x)} \,.\, f'(x)$

only point of maxima [Applying first derivative test]

2. Clearly $f(x) = e^{x^2} + e^{-x^2}$

$f'(x) = 2x\,(e^{x^2} - e^{-x^2}) \ge 0$ increasing $\Rightarrow f_{max} = f(1) = e + \frac{1}{e}$

$g(x) = x\,e^{x^2} + e^{-x^2} \Rightarrow g'(x) = e^{x^2} + 2x^2 e^{x^2} - 2x\, e^{-x^2} > 0$ increasing

$\Rightarrow g_{max} = g(1) = e + \frac{1}{e}$

$h(x) = x^2\, e^{x^2} + e^{-x^2} \Rightarrow h'(x) = 2x\,e^{x^2} + 2x^3 e^{x^2} - 2x\,e^{-x^2} = 2x\left(e^{x^2} + x^2 e^{x^2} - e^{-x^2}\right) > 0$

$\Rightarrow \quad h_{max} = h(1) = e + \frac{1}{e}$, so $a = b = c$

3. **(A)** $\text{Re}\left(\frac{2i\ \ (x+iy)}{1-(x^2-y^2+2xyi)}\right) = \text{Re}\left(\frac{-2y\ \ +2ix}{1-x^2+y^2-2xyi)}\right) = \text{Re}\left(\frac{-2y\ \ +2ix}{2y\ \ (y-ix)}\right) = \text{Re}\,(-1/y) = \frac{-1}{y}$

$= -1 \le y \le 1 \;=\; \frac{-1}{y} \ge 1$ or $\frac{-1}{y} \le -1$

Alternate

$\text{Re}\left(\frac{2ie^{i\theta}}{1-e^{2i\theta}}\right) = \text{Re}\left(\frac{2i(\cos\theta + i\sin\theta)}{1-(\cos 2\theta + i\sin 2\theta)}\right)$

$= \text{Re}\left(\frac{2i(\cos\theta + i\sin\theta)}{2\sin^2\theta - 2i\sin\theta\cos\theta)}\right) = \text{Re}\left(\frac{i\ \ (\cos\theta + i\sin\theta)}{\sin\theta\ \ (\sin\theta - i\cos\theta)}\right) = \text{Re}\left(\frac{(\cos\theta + i\sin\theta)}{-\sin\theta\ \ (\cos\theta + i\sin\theta)}\right) = \text{Re}\left(\frac{-1}{\sin\theta}\right)$

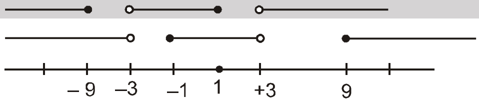
as $-1 \le \sin\theta \le 1$ $(-\infty, 0) \cup (0, \infty)$

(B) $-1 \le \frac{8.3^{x-2}}{1-3^{2x-2}} \le 1 \Rightarrow \quad -1 \le \frac{8t}{9-t^2} \le 1$

$\Rightarrow \quad -1 \le \frac{8t}{9-t^2} \le 1 \Rightarrow 0 \le \frac{9-t^2+8t}{9-t^2} \cap \frac{8t}{9-t^2} - 1 \le 0$

$\Rightarrow \quad 0 \le \frac{t^2-8t-9}{t^2-9} \cap \frac{8t-9+t^2}{9-t^2} \le 0 \Rightarrow 0 \le \frac{(t-9)\ \ (t+1)}{(t-3)\ \ (t+3)} \cap \frac{(t+9)\ \ (t-1)}{(t-3)\ \ (t+3)} \ge 0$

$\Rightarrow \quad t \in (-\infty\,, -9] \cup [-1\,, 1] \cup [9, \infty) \Rightarrow x \in (-\infty\,, 0) \cup [2\,, \infty)$



$\Rightarrow \quad t \in (-\infty\,, -9] \cup [-1\,, 1] \cup [9, \infty) \quad \Rightarrow \quad x \in (-\infty\,, 0) \cup [2\,, \infty)$

(C) $f(\theta) = 2\sec^2\theta \quad \Rightarrow \quad f(\theta) \ge 2 \quad \Rightarrow \quad f(\theta) \in [2, \infty)$

(D) $f(x) = x^{3/2}\,(3x - 10) \quad \Rightarrow \quad f'(x) = x^{3/2}\,3 + \frac{3}{2}\,x^{1/2}\,(3x - 10)$

as $f'(x) \ge 0 \quad \Rightarrow \quad x^{1/2}\left[3x + \frac{3}{2}(3x-10)\right] \ge 0 \Rightarrow 3x + \frac{9x}{2} - 15 \ge 0$

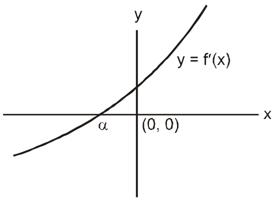
$\Rightarrow \quad \frac{15x}{2} - 15 \ge 0 \quad \Rightarrow \quad x \ge 2 \Rightarrow x \in [2, \infty)$

4. $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f''(x) = 12x^2 - 24x + 24 = 12\,(x^2 - 2x + 2) > 0\ x \in R$

$\therefore$ f′ (x) is S.I. function



Let α is a real root of the eqution $f'(x) = 0$

$\therefore$ f(x) is MD for $x \in (-\infty, \alpha)$ and M.I. for $x \in (\alpha, \infty)$ where $\alpha < 0$

$\therefore$ $f(0) = -1$ and $\alpha < 0 \Rightarrow f(\alpha)$ is also negative

$\therefore$ $f(x) = 0$ has two real & distinct roots.

5. $p' = \lambda(x - 1)(x - 3) = \lambda(x^2 - 4x + 3)$

$p(x) = \lambda(x^3/3 - 2x^2 + 3x) + \mu$

$p(1) = 6$

$6 = \lambda(1/3 - 2 + 3) + \mu$

$6 = \lambda(1/3 + 1) + \mu$

$18 = 4\lambda + 3\mu \quad ...(i)$

$p(3) = 2$

$2 = \lambda(27/3 - 2 \times 9 + 9) + \mu$

$2 = \mu$

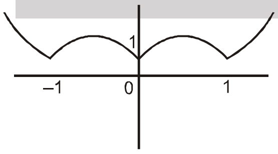
$\mu = 2 \Rightarrow \lambda = 3$

$p'(x) = 3(x - 1)(x - 3)$

$p'(0) = 3(-1)(-3) = 9$

6. $f(x) = |x| + |x^2 - 1|$

$$f(x) = \begin{cases} -x + x^2 - 1 & x < -1 \\ -x - x^2 + 1 & -1 \le x \le 0 \\ x - x^2 + 1 & 0 < x < 1 \\ x + x^2 - 1 & x \ge 1 \end{cases}$$



$$f(x) = \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \le x \le 0 \\ -x^2 + x + 1 & 0 < x < 1 \\ x^2 + x - 1 & x \ge 1 \end{cases}$$

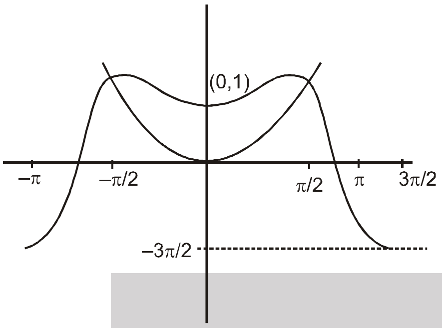
7. $x^2 = x \sin x + \cos x$

$f(x) = x^2$

$g(x) = x \sin x + \cos x$

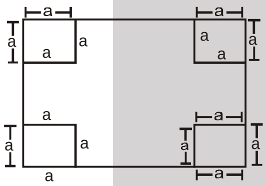
$g'(x) = \sin x + x \cos x - \sin x$

$g'(x) = x \cos x$



Only two solution.

8*.



Let $\ell = 8x$, $b = 15x$

$\therefore$ Volume $= (8x - 2a)(15x - 2a)(a) = 4a^3 - 46a^2x + 120\,ax^2$

$$\frac{dV}{da} = 6a^2 - 46ax + 60x^2$$

$$\left(\frac{dV}{da}\right)_{\text{at } x=5} = 0$$

$\therefore$ $x = 3$ and $\frac{5}{6}$

$$\frac{d^2V}{da^2} = 6a - 23x$$

$$\left(\frac{d^2V}{da^2}\right)_{\text{at } a=5\ \&\ x=3} < 0,$$

So, at x = 3 gives maxima $\left(\frac{d^2V}{da^2}\right)_{\text{at } a=5\ \&\ x=\frac{5}{6}} > 0$

So, at $x = \frac{5}{6}$ gives minima. $\frac{dV}{da} = 0$ when a = 5 given

($\therefore$ $4a^2 = 100$ given for maximum volume)

at a = 5

by $\frac{dV}{da} = 0$

$\Rightarrow$ $6x^2 - 23x + 15 = 0$

$x = 3$ or $5/6$

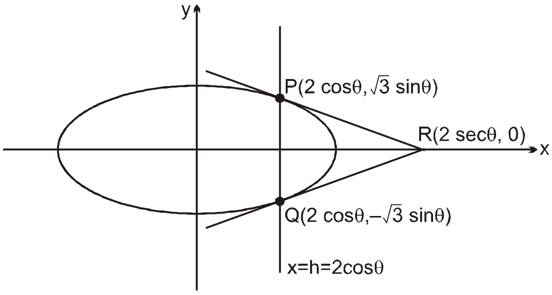
So by x = 3 (for max volume)

$8x = 24$, $15x = 45$

9. Point of intersection of tangents at P and Q is R(2 secθ, 0)

Area of ΔPQR = $\frac{1}{2} \cdot 2\sqrt{3}\ \sin\theta \cdot (2\sec\theta - 2\cos\theta)$

$\Rightarrow \quad \Delta = 2\sqrt{3} \cdot \frac{\sin^3\theta}{\cos\theta}$; where $\cos\theta \in \left[\frac{1}{4}, \frac{1}{2}\right]$



Now $\frac{d\Delta}{d\theta} = \frac{2\sqrt{3}\left[\cos\theta.3\sin^2\theta\cos\theta - \sin^3\theta(-\sin\theta)\right]}{\cos^2\theta} > 0$

As θ increases, Δ increases $\Rightarrow$ when cosθ decreases, Δ increases

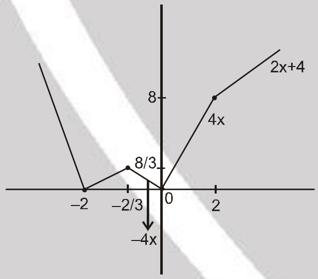
$\therefore \quad \Delta_{\text{min.}}$ occurs at cosθ = 1/2, Therefore $\Delta_2 = 2\sqrt{3}\ .\ \frac{(1-1/4)^{3/2}}{1/2} = 4\sqrt{3}\ .\ \frac{3\sqrt{3}}{8} = \frac{36}{8}$

$\Delta_{\text{max.}}$ occurs at cosθ = 1/4, Therefore $\Delta_1 = 2\sqrt{3}\ .\ \frac{(1-1/16)^{3/2}}{1/4} = 8\sqrt{3}\ .\ \frac{15.\sqrt{15}}{4.4.4} = \frac{2\sqrt{3}.15.\sqrt{3}\sqrt{5}}{16}$

$\Rightarrow \quad \Delta_1 = \frac{45}{8}\sqrt{5}$

Now $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$

10*. f(x) = 2|x| + |x + 2| – | |x + 2| – 2 |x| |



$$= \begin{cases} -2x-4 & x \le -2 \\ 2x+4 & -2 < x \le -2/3 \\ -4x & -2/3 < x \le 0 \\ 4x & 0 < x \le 2 \\ 2x+4 & x > 2 \end{cases}$$

Graph of y = f(x) is minima at x = –2, 0; maxima at x = –2/3

11. $f''(x) - 2f'(x) + f(x) \ge e^x$

$f''(x)\cdot e^{-x} - f'(x)e^{-x} - f'(x)e^{-x} + f(x)e^{-x} \ge 1$

$\frac{d}{dx}(f'(x)e^{-x}) - \frac{d}{dx}(f(x)\cdot e^{-x}) \ge 1$

$\frac{d}{dx}(f'(x)\, e^{-x} - f(x)\, e^{-x}) \ge 1 \ \Rightarrow \ \frac{d^2}{dx^2}(e^{-x}f(x)) \ge 1 \quad \forall\ x \in [0, 1]$

Let $\phi(x) = e^{-x} f(x)$

$\Rightarrow \phi(x)$ is concave upward f(0) = f(1) = 0

$\Rightarrow \phi(0) = 0 = \phi(1) \Rightarrow \quad \phi(x) < 0 \Rightarrow f(x) < 0$

12. $\phi'(x) < 0,\ x \in (0, 1/4)$ and $\phi'(x) > 0,\ x \in (1/4, 1) \Rightarrow e^{-x} f'(x) - e^{-x} f(x) < 0,\ x \in (0, 1/4)$
$f'(x) < f(x),\ 0 < x < 1/4$

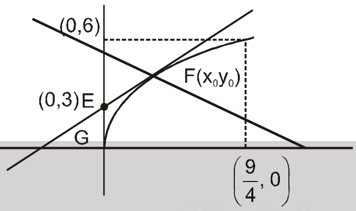
13. tangent at Fyt = $x + 4t^2$

a : x = 0y = 4t (0, 4t)

$(4t^2, 8t)$ satisfies the line

$8t = 4mt^2 + 3$

$4mt^2 - 8t + 3 = 0$



$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 0 & 4t & 1 \\ 4t^2 & 8t & 1 \end{vmatrix}$$

$$= \frac{1}{2}\ (4t^2\ (3 - 4t)) = 2t^2\ (3 - 4t)$$

$$A = 2[3t^2 - 4t^3] \quad \frac{dA}{dt} = 2[6t - 12t^2] = 24\ t(1 - 2t)$$

t = 1/2 maxima

G(0, 4t) $\Rightarrow$ G(0, 2)

$y_1 = 2$

$(x_0, y_0) = (4t^2, 8t) = (1, 4)$

$y_0 = 4$

$$\text{Area} = 2\left(\frac{3}{4} - \frac{1}{2}\right) = 2\left(\frac{3-2}{4}\right) = \frac{1}{2}$$

14*. $f(x) = x^5 - 5x + a = 0$

$x^5 - 5x = -a$

$x(x^4 - 5) = -a$

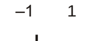
$x(x^2 - \sqrt{5})(x^2 + \sqrt{5}) = -a$

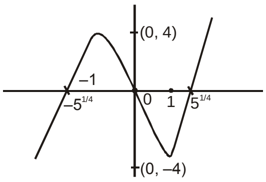
$x(x - 5^{1/4})(x + 5^{1/4})(x^2 + \sqrt{5}) = -a$ ..(1)

$f'(x) = 5x^4 - 5 = 0$

$(x^2 - 1)(x^2 + 1) = 0$

$(x - 1)(x + 1)(x^2 + 1) = 0$





15. $(y - x^5)^2 = x(1 + x^2)^2$

$$2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right) = (1 + x^2)^2 + 2x\,(1 + x^2)\,2x$$

at point (1, 3)

$$\therefore \quad 2(3 - 1)\left(\frac{dy}{dx} - 5\right) = 4 + 8$$

$$\frac{dy}{dx} - 5 = \frac{12}{4} = 3$$

$$\frac{dy}{dx} = 8$$

16.✍ Volume of material $V = \pi r^2 h$

$\Rightarrow \quad V_1 = \pi(r + 2)^2 2 + \pi(r + 2)^2 h - \pi r^2 h \qquad \Rightarrow \quad V_1 = 2\pi(r + 2)^2 + \pi h(4 + 4r)$

$\Rightarrow \quad V_1 = 2\pi(r + 2)^2 + 4\pi h(r + 1) \qquad \Rightarrow \quad V_1 = 2\pi\left((r+2)^2 + \frac{2(r+1)V}{\pi r^2}\right)$

$\Rightarrow \quad \frac{dV_1}{dr} = 2\pi\left(2(r+2) + \frac{2v}{\pi}\left(\frac{-1}{r^2} - \frac{2}{r^3}\right)\right) = 0 \qquad \Rightarrow \quad 24 + \frac{2V}{\pi}\left(\frac{-2-10}{10^3}\right) = 0$

$\Rightarrow \quad \frac{24v}{10^3\pi} = 24 \qquad \Rightarrow \quad v = 10^3\pi \qquad \Rightarrow \quad \frac{v}{250\pi} = 4$

17*.✍ Let h(x) = f(x) – 3g(x)

$$\left.\begin{array}{l} h(-1) = 3 \\ h(0) = 3 \end{array}\right\}$$

$\Rightarrow$ h'(x) = 0 has atleast one root in (–1, 0) and atleast one root in (0, 2)

h(2) = 3

But since h"(x) = 0 has no root in (–1, 0) & (0, 2) therefore h'(x) = 0 has exactly 1 root in (–1, 0) & exactly 1 root in (0, 2)

18. $\lim\limits_{x\to 2} \frac{f(x)\,g(x)}{f'(x)\,g'(x)} = 1$

$\because \quad \lim\limits_{x\to 2} \frac{f(x)\,g(x)}{f'(x)\,g'(x)} \left(\frac{0}{0}\right)$ Indeterminant form as f'(2) = 0, g(2) = 0

$\because \quad$ Using L.H.

$$\lim_{x\to 2} \frac{f'(x)\,g(x) + g'(x)f(x)}{f''(x)\,g'(x) + g''(x)f'(x)} = \frac{f'(2)\,g(2) + g'(2)f(2)}{f''(2)\,g'(2) + g''(2)f'(2)} = \frac{g'(2)f(2)}{f''(2)g'(2)} = 1 \Rightarrow f''(2) = f(2)$$

and f'(2) = 0 & range of f(x) $\in (0, \infty)$ so f"(2) = f(2) = +ve

so f(x) has point of minima at x = 2

and f(2) = f"(2) so f(x) = f"(x) have atleast one solution in $x \in R$

(19 to **21)**

$f(x) = x + \ell nx - x\, \ell nx$

$f'(x) = 1 + \frac{1}{x} - \ell nx - x\left(\frac{1}{x}\right) = \frac{1}{x} - \ell nx$

$f''(x) = -\frac{1}{x^2} - \frac{1}{x} < 0 \ \forall x \in (0, \infty)$

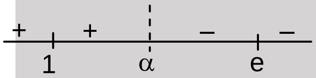
$\therefore$ f'(x) is strictly decreasing function for $x \in (0, \infty)$

$\left.\begin{matrix} \lim\limits_{x\to\infty} f'(x) = -\infty \\ \lim\limits_{x\to 0^+} f'(x) = \infty \end{matrix}\right\} \Rightarrow$ f'(x) = 0 has only one real root in $(0, \infty)$

$f'(1) = 1 > 0$

$f'(e) = \frac{1}{e} - 1 < 0 \ \therefore$ f '(x) = 0 has one root in (1, e)

Let $f'(\alpha) = 0$, where $\alpha \in (1, e)$



$\therefore$ f(x) is increasing in $(0, \alpha)$ and decreasing in (α, ∞)

$f(1) = 1$ and $f(e^2) = e^2 + 2 - 2e^2 = 2 - e^2 < 0$

$\Rightarrow$ f(x) = 0 has one root in $(1, e^2)$

From column 1 : I and II are correct.

From column 2 : ii, iii, and iv are correct.

From column 3 : P, Q, S are correct

22. $f''(x) > 0$ for all $x \in R$, $f(1/2) = 1/2$, $f(1) = 1$

$\Rightarrow f'(x)$ increases

Let $g(x) = f(x) - x$, $x \in [1/2, 1]$

Then $g'(x) = 0$ has atleast one real root in (1/2, 1)

$f'(x) = 1$ has atleast one real root in (1/2, 1)

Hence $f'(x)$ increases $\Rightarrow f'(1) > 1$

23. $f'(x) - 2f(x) > 0$

$\Rightarrow \quad \frac{d}{dx}\left(f(x).e^{-2x}\right) > 0$

$\Rightarrow$ $g(x) = f(x).e^{-2x}$ is an increasing function. for $x > 0$, $g(x) > g(0)$

$\Rightarrow \quad f(x).e^{-2x} > 1$

$\Rightarrow \quad f(x) > e^{2x}$

Now $f'(x) > 2f(x) > 2.e^{2x}$

$\therefore$ f(x) is an increasing function

24. $f(x) = \begin{vmatrix} \cos 2x & \cos 2x & \sin 2x \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = \cos 2x - \cos 2x(-\cos^2 x + \sin^2 x) + \sin 2x(-2\sin x\cos x)$

$f(x) = \cos 4x + \cos 2x$

$\therefore$ $f(x) = 2\cos^2 2x + \cos 2x - 1$

Let $\cos 2x = t$

$\Rightarrow$ $f(x) = 2t^2 + t - 1$ and $t \in [-1, 1]$

f(x) attains its minima at $t = -\frac{1}{4} \in [-1, 1]$

$f(x),\ t = -\frac{1}{4} \in [-1, 1]$

$\therefore$ $f(x)\big|_{\min} = \frac{2}{16} - \frac{1}{4} - 1 = \frac{-9}{8}$

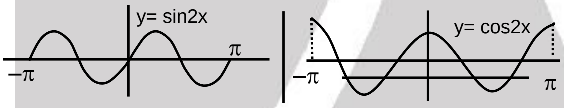
$\therefore$ $f(x)\big|_{\max} = 2 + 1 - 1 = 2$……..(when $\cos 2x = 1$)

$f'(x) = -4\sin 4x - 2\sin 2x$

$f'(x) = 0 \Rightarrow 4\sin 4x + 2\sin 2x = 0$

$\Rightarrow$ $8\sin 2x \cos 2x + 2\sin 2x = 0 \Rightarrow 2\sin 2x(4\cos 2x + 1) = 0$

$\Rightarrow$ $\sin 2x = 0$ or $\cos 2x = -\frac{1}{4}$



Hence option (B), (C)

25. $f^2(0) + (f'(0))^2 = 85$ $\quad f : R \to [-2, 2]$

(A) This is true of every continuous function

(B) $f'(c) = \frac{f(-4) - f(0)}{-4 - 0}$

$|f'(c)| = \left|\frac{f(-4) - f(0)}{4}\right|$

$-2 \le f(-4) \le 2$

$-2 \le f(0) \le 2$

$-4 \le f(-4) - f(0) \le 4$

This $|f'(c)| \le 1$

(C) $\lim_{x\to\infty} f(x) = 1$

Note f(x) should have a bound ∞ which can be concluded by considering

$f(x) = 2\sin\left(\frac{\sqrt{85}\,x}{2}\right)$ $f'(x) = \sqrt{85}\cos\left(\frac{\sqrt{85}\,x}{2}\right)$

$f^2(0) + (f'(0)^2) = 85$

and $\lim_{x\to\infty} f(x)$ does not exist

(D) Consider $H(x) = f^2(x) + (f'(x)^2$

$H(0) = 85$

By (B) choice there exists some x_0 such that $(f'(x_0))^2 \le 1$ for some x_0 in (-4, 0)

hence $H(x_0) = f^2(x_0) + (f'(x_0))^2 \le 4 + 1$

$H(x_0) \le 5$

Hence let $p \in (-4, 0)$ for which $H(p) = 5$

(note that we have considered p as largest such negative number)

similarly let q be smallest positive number $\in (0, 4)$ such that $H(q) = 5$

Hence By Rolle's theorem is (p, q)

$H'(c) = 0$ for some $c \in (-4, 4)$ and since H(x) is greater than 5 as we move from x = p to x = q and $f^2(x) \le 4 \Rightarrow (f'(x))^2 \ge 1$ in (p, q) Thus $H'(c) = 0$ $\Rightarrow f'f + f'f'' = 0$

so $f + f'' = 0$ and $f' \ne 0$

26.

$$f(x) = \begin{cases} x^5+5x^4+10x^3+10x^2+3x+1 & x<0 \\ x^2-x+1 & 0 \le x < 1 \\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3} & 1 \le x < 3 \\ (x-2)\ln(x-2)-x+\frac{10}{3} & x \ge 3 \end{cases}$$

$$f'(x) = \begin{cases} 5(x+1)^4-2 & x<0 \\ 2x-1 & 0 \le x < 1 \\ 2x^2-8x+7 & 1 \le x < 3 \\ \ln(x-2) & x \ge 3 \end{cases}$$

$x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1$ takes value between $-\infty$ to 1

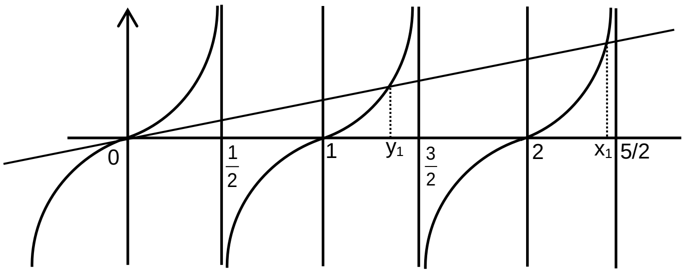
Also $(x - 2) \ln(x - 2) - x + \frac{10}{3}$ takes value between $\frac{1}{3}$ to ∞

So, range of f(x) is R. So option (A) is correct $f''(1^-) = 2$ and $f''(1^+) = -4$

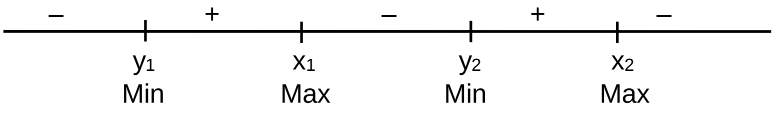
so f'(x) is non-diff at x = 1 so option (B) is correct f'(x) has local maxima at x = 1 so option (C) is correct

27.

$$f'(x) = \frac{2x\cos\pi x\left(\frac{\pi x}{2} - \tan\pi x\right)}{x^4}$$



For f'(x)



PART - II

1. $\lim\limits_{x\to -1^+} f(x) = 1$

$f(-1) = k + 2$

$\lim\limits_{x\to (-1)^-} f(x) = k + 2$

$\because$ f has a local minimum at x = – 1

$\therefore$ $f(-1^+) \ge f(-1) \le f(-1^-)$

$1 \ge k + 2 \le k + 2$

$\Rightarrow$ $k \le -1$

possible value of k is – 1

Hence correct option is (3)

2. $e^x + 2e^{-x} \ge 2\sqrt{2}$ (AM ≥ GM)

$$\frac{1}{e^x + 2e^{-x}} \le \frac{1}{2\sqrt{2}}$$

$\frac{1}{2\sqrt{2}} \ge f(x) > 0$ so statement- 2 is correct

As f(x) is continuous and $\frac{1}{3}$ belongs to range $\left(0, \frac{1}{2\sqrt{2}}\right]$ of f(x),

$\Rightarrow$ $f(c) = \frac{1}{3}$ for some C.

Hence correction option is (4).

3. $y = x + \frac{4}{x^2}$

$y' = 1 - \frac{8}{x^3} = 0 \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2$

$y = 2 + \frac{4}{2^2} = 3$

(2, 3) is point of contact

Thus y = 3 is tangent

Hence correct option is (3)

4. $$f(x) = \begin{cases} \frac{\tan x}{x} & x \ne 0 \\ 1 & x = 0 \end{cases}$$

In right neighbourhood of ‘0’

tan x > x

$\frac{\tan x}{x} > 1$

In left neighbourhood of ‘0’

tan x < x

$\frac{\tan x}{x} > 1$ as (x < 0)

at x = 0, f(x) = 1

$\Rightarrow$ x = 0 is point of minima

so statement 1 is true.

statement 2 obvious

5. $y - x = 1$

$y^2 = x$

$2y\,\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{2y} = 1$

$y = \frac{1}{2}$

$x = \frac{1}{4}$

tangent at $\left(\frac{1}{4}\ ,\ \frac{1}{2}\right)$ $\frac{1}{2}\,y = \frac{1}{2}\left(x + \frac{1}{4}\right)$ $y = x + \frac{1}{4}$ $y - x = \frac{1}{4}$

distance = $\left|\frac{1-\frac{1}{4}}{\sqrt{2}}\right| = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

6. $V = \frac{4}{3}\,\pi r^3$ $\qquad 4500\,\pi = \frac{4\pi r^3}{3}$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$ $\qquad 45 \times 25 \times 3 = r^3$

$r = 15$ m

after 49 min = $(4500 - 49.72)\pi = 972\,\pi$ m^3

$972\,\pi = \pi r^3$

$r^3 = 3 \times 243 = 3 \times 3^5$

$r = 9$

$72\,\pi = 4\pi \times 9 \times 9\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = \left(\frac{2}{9}\right)$

$r^3 = 3 \times 243 = 3 \times 3^5$

$r = 9$

$72\,\pi = 4\pi \times 9 \times 9\left(\frac{dr}{dt}\right) \Rightarrow \qquad \frac{dr}{dt} = \left(\frac{2}{9}\right)$

7. $f\,'(x) = \frac{1}{x} + 2bx + a$

at $x = -1$ $\qquad -1 - 2b + a = 0$

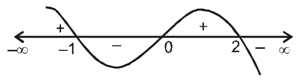
$a - 2b = 1 \qquad ...(i)$

at $x = 2$ $\qquad \frac{1}{2} + 4b + a = 0$

$a + 4b = -\frac{1}{2} \qquad ...(ii)$

On solving (i) and (ii) $\quad a = \frac{1}{2},\ b = -\frac{1}{4}$

$f\,'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x} = \frac{-(x+1)(x-2)}{2x}$



So maxima at $x = -1,\ 2$

8. $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0 \qquad \forall\ x \in R$

$\Rightarrow$ f(x) is strictly increasing function

$\Rightarrow$ f(x) = 0 has only one real root, so two roots are not possible

9. Consider f(x) – 2g(x) = h(x)

Then, h(x) is continuous and differentiable in [0, 1]

Also h(0) = 2 & h(1) = 2

Hence h(x) satisfies conditions of Rolles Theorem in (0, 1)

Thus, There exist a 'c' such that h'(c) = 0 where c $\in$ (0, 1)

$\Rightarrow \qquad f'(c) = 2g'(c)$

10. $f(x) = \alpha\,\ell n|x| + \beta x^2 + x$

(1). $f'(x) = \frac{\alpha}{x} + 2\beta x + 1 = \frac{2\beta x^2 + x + \ell}{x}$

Since x = –1, 2 are extreme points $\Rightarrow$ f '(x) = 0 at these points.

Hence $2\beta - 1 + \alpha = 0$

$8\beta + 2 + \alpha = 0$

$-6\beta - 3 = 0 \Rightarrow \beta = -\frac{1}{2}$ & $\alpha = 2$.

11. $4x + 2\pi r = 2 \qquad$ ....(i)

$x^2 + \pi r^2$ = minimum $\qquad \Rightarrow$ So $\quad f(r) = \left(\frac{1-\pi r}{2}\right)^2 + \pi r^2$

$\frac{df}{dr} = \pi^2\frac{r}{2} - \frac{\pi}{2} + 2\pi r = 0 \quad \Rightarrow r = \frac{1}{\pi + 4}$

using equation (i) $x = \frac{(1-\pi r)}{2} \quad \Rightarrow x = 2r$

12. at $x = \frac{\pi}{6} \qquad \Rightarrow \qquad y = \frac{\pi}{3}$

$f(x) = \tan^{-1}\left(\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|\right) \qquad \because \qquad x \in \left(0, \frac{\pi}{2}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

$f(x) = \frac{\pi}{4} + \frac{x}{2} \qquad f'(x) = \frac{1}{2}$

slope of normal = –2

equation of normal $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$

$y = -2x + \frac{2\pi}{3}$

13. $2r + \ell = 20 \Rightarrow 2r + r\theta = 20 \Rightarrow \theta = \frac{20-2r}{r} \Rightarrow A = \frac{\pi r^2\theta}{360} = \frac{r^2}{2}.\frac{20-2r}{r} = r\,(10-r)$

$A = 10r - r^2 \Rightarrow \frac{dA}{dr} = 10 - 2r = 0 \Rightarrow r = 5$

$\therefore \quad \theta = \frac{10}{5} = 2$

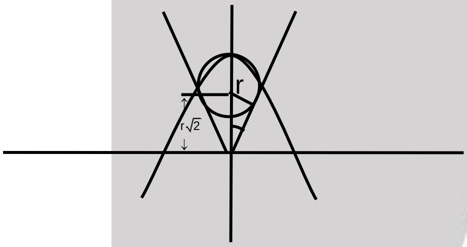
$\therefore$ Maximum area = $\frac{1}{2} \times 25 \times 2 = 25$ sq. m.

14. $y(x-2)(x-3) = x + 6$

Intersection with y-axis; Put $x = 0 \Rightarrow y = 1 \Rightarrow$ Point of Intersection is (0, 1)

Now, $y = \frac{x+6}{x^2-5x+6}$ $y' = \frac{(x^2-5x+6)-(x+6)(2x-5)}{(x^2-5x+6)^2}$ $y' = \frac{6-(-30)}{36} = 1$ at (0, 1)

$\therefore$ Equation of normal is given by $(y-1) = -1\,(x-0)$ $x + y - 1 = 0$

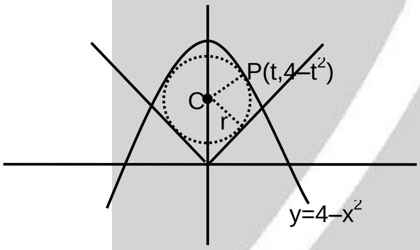


15.

let radius of circle be r, its center lies on y-axis as y-axis bisects the 2 rays of y = |x|

Now $4 - r\sqrt{2} = r \quad \Rightarrow \quad r = \frac{4}{\sqrt{2}+1} = 4(\sqrt{2}-1)$

NOTE : The correct solution should be



due to symmetry center of the circle must be on y-axis let center be (0, k)

Length of perpendicular from (0, k) to y = x, i.e. $r = \left|\frac{k}{\sqrt{2}}\right|$

$\therefore$ Equation of circle : $x^2 + (y-k)^2 = \frac{k^2}{2}$

solving circle and parabola, $4 - y + y^2 - 2ky + \frac{k^2}{2} = 0$

$y^2 - (2k+1)\,y + \left(\frac{k^2}{2} + 4\right) = 0$

Because circle touches the parabola

$\therefore D = 0$

$(2k+1)^2 = 4\left(\frac{k^2}{2} + 4\right) \Rightarrow 4k^2 + 4k + 1 = 2k^2 + 16$

On solving we get $k = \frac{-4+\sqrt{136}}{4}$

Therefore radius = $k/\sqrt{2} \approx 1.3546$

However among the given choices the following method will yield one of the choice.

16. $y^2 = 6x$ and $9x^2 + by^2 = 16$

$2y\frac{dy}{dx} = 6 \Rightarrow \frac{dy}{dx} = \frac{3}{y}$

$18x + 2by\frac{dy}{dx} = 0$

$9x + by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-9x}{by}$

$\frac{3}{y} \times \frac{-9x}{by} = -1$

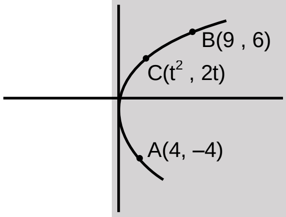
(b) $6x = 27x$

$b = \frac{27}{6} \Rightarrow b = \frac{9}{2}$

17. $f(x)\ x^2 + \frac{1}{x^2}, g(x) = x = \frac{1}{x} \Rightarrow h(x) = \frac{f(x)}{g(x)} = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)} \Rightarrow x - \frac{1}{x} = t$

$h(t) = \frac{t^2 + 2}{t} = t + \frac{2}{t}$ $|t| \geq 2 \Rightarrow$ AM $\geq$ GM . $\frac{t + \frac{2}{t}}{2} \geq \sqrt{t.\frac{2}{t}}$ $t + \frac{2}{t} \geq 2\sqrt{2}$

18.



$\Delta\ ACB = \frac{1}{2}\begin{vmatrix} 4 & -4 & 1 \\ 9 & 6 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$

$\Delta = 30 + 5t - 5t^2$

$\frac{d\Delta}{dt} = 0 \Rightarrow 5 - 10t = 0 \Rightarrow t = \frac{1}{2}$

$\frac{d^2\Delta}{dt^2} = -\ 10 < 0$

$\therefore$ $C\left(\frac{1}{4}, 1\right)$ so $\Delta = 30 + \frac{5}{2} - \frac{5}{4} = 31\frac{1}{4}$

19. $y = 7 + x^{3/2}$

Let the point on curve be $P\left(x_1, 7 + x_1^{3/2}\right)$ and given point be $A\left(\frac{1}{2}, 7\right)$

For nearest point normal at P passes through A

So slope of line AP = Slope of normal at P

$\Rightarrow \frac{x_1^{3/2}}{x_1 - \frac{1}{2}} = -\left.\frac{dx}{dy}\right|_{(x_1,y_1)} = -\frac{2}{3\sqrt{x_1}}$ $\Rightarrow 3x_1^2 = 1 - 2x_1 \Rightarrow 3x_1^2 + 2x_1 - 1 = 0 \Rightarrow (x_1 + 1)(3x_1 - 1) = 0$

$\Rightarrow x_1 = \frac{1}{3}$ ($x_1 = -1$ is not possible as $x_1 > 0$) $\Rightarrow$ Hence point P is $\left(\frac{1}{3}, 7 + \frac{1}{3\sqrt{3}}\right)$

So AP $= \sqrt{\frac{1}{36} + \frac{1}{27}} = \frac{1}{6}\sqrt{\frac{7}{3}}$

20. $$f'(x) = \frac{\sqrt{a^2+x^2} - \frac{x^2}{\sqrt{a^2+x^2}}}{(a^2+x^2)} - \frac{-\sqrt{b^2+(d-x)^2} + \frac{(d-x)^2}{\sqrt{b^2+(d-x)^2}}}{b^2+(d-x)^2} = \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{b^2}{\left(b^2+(d-x)^2\right)^{3/2}}$$

Hence f(x) is increasing.

21. $f''(x) > 0$, $y = f(x)$; $x \in (0, 2)$

$\phi(x) = f(x) + f(2 - x)$

$\phi'(x) = f'(x) - f'(2 - x)$

for $\phi(x)$ to be increasing

$\phi'(x) > 0 \quad \Rightarrow \quad f'(x) > f'(2 - x)$

$\Rightarrow \quad x > 2 - x$ (f'(x) is increasing in (0, 2))

$\Rightarrow \quad x > 1$

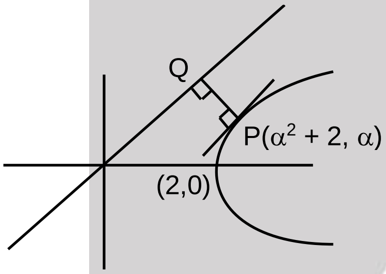
$\Rightarrow \quad x \in (1, 2)$

For $\phi(x)$ to be decreasing

$\phi'(x) < 0 \quad \Rightarrow \quad f'(x) < f'(2 - x)$

$\therefore \quad x \in (0, 1)$

22.



Shortest distance between $y^2 = x - 2$ and $y = x$

$\frac{dy}{dx}$ at point P will be 1. Differentiating the curve

$2yy' = 1 \quad \Rightarrow \quad y' = \frac{1}{2y} = \frac{1}{2\alpha} = 1 \quad \therefore \quad P\left(\frac{9}{4}, \frac{1}{2}\right)$

$\therefore$ minimum distance = PQ = $\left|\frac{\frac{9}{4} - \frac{1}{2}}{\sqrt{2}}\right| = \frac{7}{4\sqrt{2}}$

23. $f(x) = x\sqrt{kx - x^2}$

$$f'(x) = \sqrt{kx - x^2} + \frac{(k-2x)x}{2\sqrt{kx-x^2}} = \frac{2(kx - x^2) + kx - 2x^2}{2\sqrt{kx - x^2}} = \frac{3kx - 4x^2}{2\sqrt{kx - x^2}} = \frac{x(3k - 4x)}{2\sqrt{kx - x^2}}$$

for increasing function

for f'(x) $\geq 0 \quad \forall x \in [0,3]$

$\Rightarrow \quad kx - x^2 \geq 0, \ \forall x \in [0,3]$ and $x(3k - 4x) \geq 0, \ \forall x \in [0,3]$

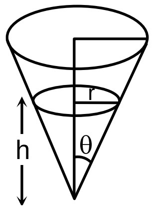
$\Rightarrow \quad x(x - k) \leq 0, \ \forall x \in [0,3]$ and $x(4x - 3k) \leq 0, \forall x \in [0,3]$

$k \geq 3$ and $k \geq 4 \quad \Rightarrow k \geq 4 \ \Rightarrow m = 4$

maximum(f(x) when k = 4 is $3\sqrt{4 \times 3 - 3^2} = 3\sqrt{3}$ = M

(m, M) = $(4, 3\sqrt{3})$

24. $\frac{dv}{dt} = 5m^3 / \min \Rightarrow v = \frac{1}{3}\pi r^2 h$



$\tan\theta = \frac{r}{h} = \frac{1}{2} \Rightarrow 2r = h \quad \Rightarrow v = \frac{1}{3}\pi\frac{h^3}{4} = \frac{\pi h^3}{12} \Rightarrow \frac{dv}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \quad 5 = \frac{\pi}{4}10^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{20}{100\pi} = \frac{1}{5\pi}$ m/min

25. $f(x) = ax^5 + bx^4 + cx^3 \quad \lim_{x\to 0}\left(2 + \frac{ax^5 + bx^4 + cx^3}{x^3}\right) = 4 \Rightarrow 2 + c = 4 \Rightarrow c = 2$

$f'(x) = 5ax^4 + 4bx^3 + 6x^2 = x^2\left(5ax^2 + 4bx + 6\right)$

$f'(1) = 0 \quad \Rightarrow \quad 5a + 4b + 6 = 0$

$f'(-1) = 0 \quad \Rightarrow \quad 5a - 4b + 6 = 0$

$b = 0$

$a = -\frac{6}{5} \quad f(x) = \frac{-6}{5}x^5 + 2x^3 \Rightarrow f'(x) = -6x^4 + 6x^2 = 6x^2(-x^2 + 1) = -6x^2(x+1)(x-1)$

–1 + 1–

1– 1

Minimal at x = –1 ⇒ Maxima at x = 1

26. $f'(x) = x\,(\pi - \cos^{-1}(\sin|x|)) = x\left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(\sin|x|)\right)\right) = x\left(\frac{\pi}{2} + |x|\right)$

$f(x) = \begin{cases} x\left(\frac{\pi}{2} + x\right) & x \ge 0 \\ x\left(\frac{\pi}{2} - x\right) & x < 0 \end{cases} \qquad f'(x) = \begin{cases} \frac{\pi}{2} + 2x & x \ge 0 \\ \frac{\pi}{2} - 2x & x < 0 \end{cases}$

$f'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{-\pi}{2}, 0\right)$

27. $f(3) = f(4) \Rightarrow \alpha = 12 \quad f'(x) = \frac{x^2 - 12}{x(x^2 + 12)}$

$\therefore \quad f'(c) = 0$

$\therefore \quad c = \sqrt{12}$

$\therefore \quad f''(c) = \frac{1}{12}$

28. Lets use LMVT for $x \in [a,c]$ $\frac{f(c) - f(a)}{c - a} = f'(\alpha)$, $\alpha \in (a,c)$

also use LMVT for $x \in [c,b]$ $\frac{f(b) - f(c)}{b - c} = f'(\beta)$, $\beta \in (c,b)$

$\because \quad f''(x) < 0 \Rightarrow f'(x)$ is decreasing $f'(\alpha) > f'(\beta)$ $\frac{f(c) - f(a)}{c - a} > \frac{f(b) - f(c)}{b - c}$ $\frac{f(c) - f(a)}{f(b) - f(c)} > \frac{c - a}{b - c}$

($\because$ f(x) is increasing)

HIGH LEVEL PROBLEMS (HLP)

1. Given x = f'(t) sint + f"(t) cost

y = f'(t) cost - f"(t) sint

from given equation $\frac{dx}{dt} = \{f'(t)+f'''(t)\}\cos t$ & $\frac{dy}{dt} = -\{f'(t)+f'''(t)\}\sin\ t$

Velocity = $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{\{f'(t)+f'''(t)\}^2\cos^2 t+\{f'(t)+f'''(t)\}^2\sin^2 t} = f'(t) + f'''(t)$

2. $f'(x) = \sqrt{4ax-x^2} + \frac{x(4a-2x)}{2\sqrt{4ax-x^2}} = \frac{6ax-2x^2}{\sqrt{4ax-x^2}} < 0, \forall\ x \in (4a, 3a)$

so f(x) is decreasing in [4a, 3a]

3. Here f is a differentiable function then f is continuous function. So by L.M.V. theorem for any a ∈ (0, 4)

$f'(a) = \frac{f(4)-f(0)}{4-0}$ .... (1)

Again from mean value for any b ∈ (0, 4)

$f(b) = \frac{f(4)+f(0)}{2}$ ....(2)

Now multiplying (1) and (2), we get $\frac{f^2(4)-f^2(0)}{8} = f'(a) \cdot f(b) \Rightarrow f^2(4) - f^2(0) = 8f'(a) \cdot f(b)$

4. $f'(x) = 0 \Rightarrow x = \frac{1}{a}, \frac{-2}{3a}$

since, we have a cubic polynomial with coefficient of x^3 +ve , minima will occur after maxima.

Case - 1 : If a > 0

then $\frac{1}{a} = \frac{1}{3} \Rightarrow a = 3$ also $f\left(\frac{1}{3}\right) > 0 \Rightarrow b < -\frac{1}{2}$

Case - 2 : If a < 0

then $-\frac{-2}{3a} = \frac{1}{3} \Rightarrow a = -2$

also $f\left(\frac{1}{3}\right) > 0 \Rightarrow \frac{(-2)^2}{3^2} - \frac{(-2)}{2} \cdot \frac{1}{3^2} - 2\left(\frac{1}{3}\right) - b > 0 \Rightarrow \frac{4}{27}+\frac{1}{9}-\frac{2}{3}-b > 0 \Rightarrow b < -\frac{11}{27}$

5. f(k) = 3

$f(k+h) = a^2 - 2 + \frac{\sin\ h}{h} \Rightarrow \lim_{h \to 0} f(k+h) = a^2 - 1$

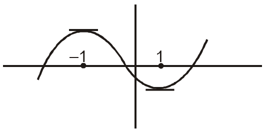
$\lim_{h \to 0} f(k-h) = \lim_{h \to 0} (3 + |k-h-k|) = \lim_{h \to 0} (3 + |-h|) = 3 \Rightarrow a^2 - 1 > 3$

$a^2 > 4 \Rightarrow |a| > 2$

6. $f(x) = x^3 - 3x + k, k = [a]$

$f'(x) = 3(x - 1)(x + 1)$

– 1 is maxima is 1 is minima



Figure

for three roots $f(-1) f(1) < 0 \Rightarrow (k + 2)(k - 2) < 0$

$k \in (-2, 2) \Rightarrow -2 < [a] < 2 \Rightarrow -1 \le a < 2$

7. $f(x) = \sin\frac{\{x\}}{a} + \cos\frac{\{x\}}{a} \quad a > 0$

it attains max. if $\frac{\{x\}}{a} = \frac{\pi}{4}$

$\frac{\{x\}}{a} \in \left[0, \frac{1}{a}\right) \quad \therefore \frac{1}{a} > \frac{\pi}{4}$, for f to have is maxima $\Rightarrow 0 < a < \frac{4}{\pi}$

8. Let $f(x) = x^4 + 4x^3 - 8x^2 + k$

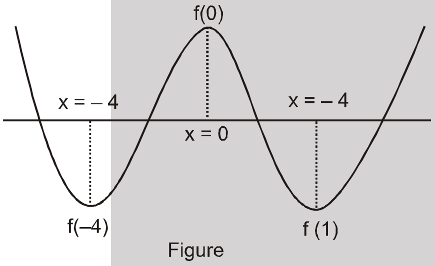
$f'(x) = 4x^3 + 12x^2 - 16x = 4x(x^2 + 3x - 4) = 4x(x + 4)(x - 1) \Rightarrow f'(x) = 0 \Rightarrow x = -4, 0, 1$

$f''(x) = 12x^2 + 24x - 16 = 4(3x^2 + 6x - 4)$

$f''(-4) = 20 > 0$

$f''(0) = -16 < 0$

$f''(1) = 20 > 0 \Rightarrow x = -4$ and $x = 1$ are points of local minima whereas



Figure

$x = 0$ is point of local maxima

for $f(x) = 0$ to have 4 real roots

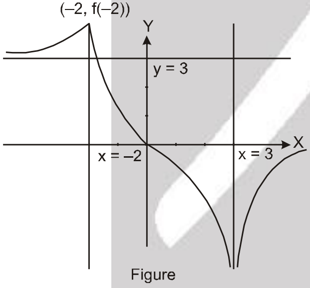
$f(-4) < 0 \qquad \Rightarrow k < 128$

$f(0) > 0 \qquad \Rightarrow k > 0$

$f(1) < 0 \qquad \Rightarrow k < 3 \qquad \Rightarrow \qquad k \in (0, 3)$

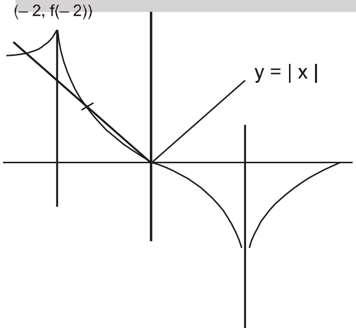
(09 to 11)

Graph of $y = f(x)$



Figure

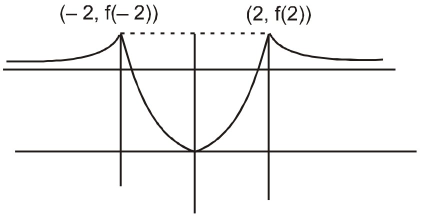
9.



Figure

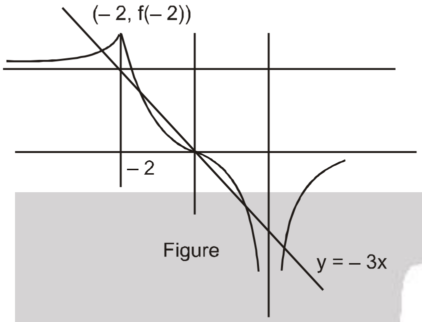
Three points of intersection. Three solutions

10.



Figure

11.



Figure

12. **(i)** $\frac{f(-1)-f(0)}{-1-0} = f'(\alpha)$, where $-1 < \alpha < 0$

$\Rightarrow$ $|f'(\alpha))| = |f(0) - f(1)| \le |f(0)| + |f(-1)| \Rightarrow |f'(\alpha)| \le 1 + 1 = 2$

similarly for $0 < \beta < 1$

(ii) $$\left.\begin{array}{r} |f(x)| \le 1 \Rightarrow (f(x))^2 \le 1 \\ |f'(x)| \le 2 \Rightarrow f'(x))^2 \le 4 \end{array}\right\} \Rightarrow F(x) \le 5$$

(iii) Obvious from (i) and (ii) that there exits atleast one max.

(iv) Also from (i) and (ii) option iv is quite obvious.

13. As (a, b) lies on $y = x^2 + 1$ $\Rightarrow$ $b = a^2 + 1$

$\left.\frac{dy}{dx}\right|_{(a,\ b)} = 2a$

Tangent $y - a^2 - 1 = 2a\,(x - a)$

$x = 0 \quad \Rightarrow \quad y = 1 - a^2$

$x = 1 \quad \Rightarrow \quad y = -a^2 + 2a + 1$

Area = $\frac{1}{2}$ $(1)\,(1 - a^2 - a^2 + 2a + 1) = -a^2 + a + 1$

It is greatest when $a = \frac{1}{2} \Rightarrow b = 1 + \frac{1}{4} = \frac{5}{4}$.

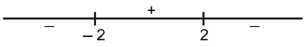
14. $(2, -1) \Rightarrow -1 = \frac{2a+b}{(+1)\quad(-2)}$ $2a + b = 2$

$y' = \frac{a(x-1)(x-4)-(ax+b)(2x-5)}{(x-1)^2(x-4)^2}$

$y' = 0$ at $x = 2$ $\Rightarrow$ $b = 0$ $\Rightarrow$ $a = 1$

$y = \frac{x}{(x-1)\quad(x-4)}$

$y' = \frac{(2+x)(2-x)}{(x-1)^2(x-4)^2}$



signs of y′

At x = 2, y′ changes sign from positive to negative ⇒ x = 2 is point of maxima.

15. Since = $\sqrt{s(s-a)(s-b)(s-c)} = \{s(s-a)(s-b)(s-c)\}^{½}$

Taking logarithm of both sides, we get ln $\Delta = \frac{1}{2}\{\ln s + \ln(s-a) + \ln(s-b) + \ln(s-c)\}$

$$\therefore \quad \frac{1}{\Delta}\frac{d\Delta}{dc} = \frac{1}{2}\left\{\frac{1}{s}.\frac{ds}{dc} + \frac{1}{(s-a)}.\frac{d(s-a)}{dc} + \frac{1}{(s-b)}.\frac{d(s-b)}{dc} + \frac{1}{(s-c)}\frac{d(s-c)}{dc}\right\} \qquad \text{.....................(1)}$$

But s = $\frac{1}{2}(a+b+c)$ $\frac{ds}{dc} = \frac{1}{2} \Rightarrow \frac{d(s-a)}{dc} = \frac{ds}{dc} - \frac{da}{dc} = \frac{1}{2} - 0 = \frac{1}{2}$,

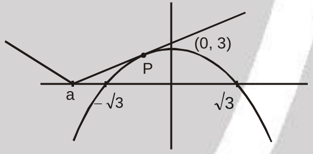
and $\frac{d(s-b)}{dc} = \frac{ds}{dc} - \frac{db}{dc} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\frac{d(s-c)}{dc} = \frac{ds}{dc} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Now from (1), $\frac{1}{\Delta}.\ \frac{d\Delta}{dc} = \frac{1}{2}\left\{\frac{1}{s}.\frac{1}{2} + \frac{1}{(s-a)}.\frac{1}{2} + \frac{1}{(s-b)}.\frac{1}{2} - \frac{1}{(s-c)}.\frac{1}{2}\right\} = \frac{1}{4}\left\{\frac{1}{s} + \frac{1}{(s-a)} + \frac{1}{(s-b)} - \frac{1}{(s-c)}\right\}$

Hence $d\Delta = \frac{\Delta}{4}\left\{\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right\}dc$

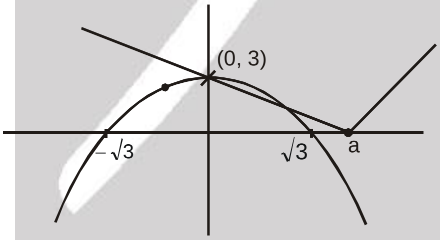
16. $3 - x^2 > | x - a |$

Case (i) a < 0 and y = x – a is tangent of y = 3 – x^2 (see figure)



$-2x = 1 \Rightarrow x = -\frac{1}{2}$ P $\left(-\frac{1}{2}, \frac{11}{4}\right)$

Since y = x – a passes through $\left(-\frac{1}{2}, \frac{11}{4}\right) \Rightarrow a = x - y = -\left(\frac{11}{4} + \frac{1}{2}\right) = -\frac{13}{4}$ (minimum value of a)



Case (ii) a > 0 and y = – x + a passes through (0, 3), then a = 3 (maximum value of a) (see figure)

$\Rightarrow \qquad a \in \left(-\frac{13}{4},\ 3\right)$

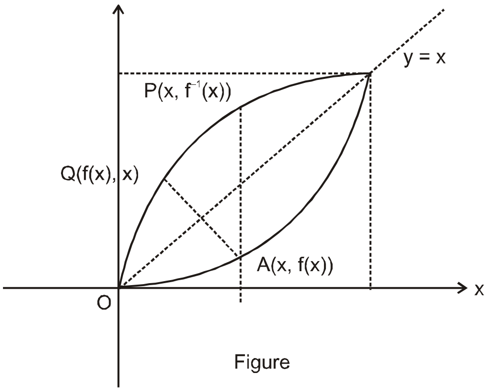
17. Let f(x) = $x^m + a_1\, x^{m-1} + a_2\, x^{m-1} + ... + a_0$ if possible, let f(x) = 0 has 'm' real roots, then by Roll's thearem, f '(x) = 0 must have "(m – 1)" real roots, f "(x) = 0 must have "(m – 2)" real roots and so on, $f^{m-2}(x) = 0$ must have 2 real roots, $\frac{m!}{2}\ x^2 + a_1\,(m-1)!\ x + a_2\,(m-2)! = 0$ must have 2 real roots

or $\frac{m\ \ (m-1)}{2}\ x^2 + a_1\,(m-1) + a_2 = 0$ must have 2 real roots

$D = a_1^2\,(m-1)^2 - 2m\,(m-1)\,a^2 = (m-1)\,[(m-1)\,a_1^2 - 2a_2]$

which is –ve, so our allumption is wrong. Hence proved.

18.



Figure

Slope of OQ > slope of OP $\frac{x}{f(x)} > \frac{f^{-1}(x)}{x} \Rightarrow f(x).\ f^{-1}(x) < x^2$

19. $g'(x) = 2f'\left(\frac{x^2}{2}\right).\ \frac{2x}{2} + f'\left(\frac{27}{2} - x^2\right)\ (-2x) = x\left[f'\left(\frac{x^2}{2}\right) - f'\left(\frac{27}{2} - x^2\right)\right]$

$g'(x) = 0$

$\Rightarrow x = 0$ or $\frac{x^2}{2} = \frac{27}{2} - x^2$

$\Rightarrow x = -3, 0, 3$

for g'(x) $\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & -3 & & 0 & & 3 & \end{array}$

so g(x) is increasing in $x \in (-\infty,\ -3]$ and in [0, 3] and g(x) is decreasing in [–3, 0] and in [3, ∞)

20. Let $f(x) = \ell nx \quad \Rightarrow \quad f''(x) = -\frac{1}{x^2}$

So $\frac{f(x_1) + f(x_2) + ... + f(x_n)}{n} \le f\left(\frac{(x_1 + x_2 + ... + x_n)}{n}\right)$

for $x_1, x_2, ..., x_n \in R^+$

$$\Rightarrow \quad \frac{\ell n\ (x_1) + \ell n\ (x_2) + ... + \ell n\ (x_n)}{n} \le \ell n\ \left(\frac{x_1 + x_2 + ... + x_n}{n}\right)$$

$$\Rightarrow \quad (x_1\ x_2........x_n)^{\frac{1}{n}} \le \frac{x_1 + x_2 + ... + x_n}{n} \quad \Rightarrow \quad G.M.\ \ A.M.$$

Again $\frac{f\left(\frac{1}{x_1}\right) + f\left(\frac{1}{x_2}\right) + ... + f\left(\frac{1}{x_n}\right)}{n} \le f\left(\frac{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}}{n}\right)$

$$\Rightarrow \left(\frac{1}{x_1 x_2 . x_n}\right)^{\frac{1}{n}} \le \frac{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}}{n} \Rightarrow \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}} (x_1\ x_2\ ......\ x_n)^{1/n} \Rightarrow G.M.\ \le A.M.$$

21. **(i)** $1 + x^2 > (x \sin x + \cos x)$

Let $f(x) = 1 + x^2 - x \sin x - \cos x,\ x \in [0, \infty)$

$f'(x) = 2x - \sin x - x \cos x + \sin x = x(2 - \cos x)$

$\Rightarrow$ $f'(x) > 0$ for $x \in (0, \infty)$

$\Rightarrow$ f(x) is an increasing function

$\therefore$ $x > 0$

$\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $1 + x^2 > x \sin x + \cos x$

(ii) $f(x) = \sin x - \sin 2x - 2x$

$f'(x) = \cos x - 2 \cos 2x - 2 = \cos x - 2(2 \cos^2 x - 1) - 2$

$= \cos x - 4 \cos^2 x = \cos x\,(1 - 4 \cos x),\ x \in \left[0, \frac{\pi}{3}\right]$

$\Rightarrow$ $\cos x \geq \frac{1}{2} \Rightarrow \cos x(1 - 4 \cos x) < 0$

$\therefore$ $f'(x) < 0\ \forall\ x \in \left[0, \frac{\pi}{3}\right]$

$f(x) \leq f(0) \Rightarrow$ $\sin x - \sin 2x - 2x \leq 0$

$\Rightarrow$ $\sin x - \sin 2x \leq 2x$

(iii) $f(x) = \frac{x^2}{2} + 2x + 3 - 3e^x + xe^x$

$f'(x) = x + 2 - 3e^x + e^x + xe^x = x + 2 - 2e^x + xe^x$

$f''(x) = 1 - 2e^x + e^x + xe^x = 1 - e^x + xe^x$

$f'''(x) = - e^x + e^x + xe^x = xe^x$

$f'''(x) \geq 0\ \forall\ x \geq 0$

$\Rightarrow$ $f''(x) \geq f''(0) \Rightarrow f''(x) \geq 0$

$\Rightarrow$ $f'(x) \geq f'(0) \Rightarrow f'(x) \geq 0$

$\Rightarrow$ $f(x) \geq f(0) \Rightarrow f(x) > 0$

$\Rightarrow$ $\frac{x^2}{2} + 2x + 3 \geq 3e^x - xe^x$

(iv) $f(x) = x \sin x - \frac{\sin^2 x}{2}$ $f'(x) = x \cos x + \sin x - \sin x \cos x = x \cos x + \sin x (1 - \cos x)$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow f(x) > f(0)$ or $x \sin x - \frac{\sin^2 x}{2} > 0$

and $f(x) < f\left(\frac{\pi}{2}\right)$, $x \sin x - \frac{\sin^2 x}{2} < \frac{\pi}{2} - \frac{1}{2}$

$\Rightarrow x \sin x - \frac{\sin^2 x}{2} < \frac{1}{2}(\pi - 1)$

22. $f(x) = \left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right) x^3 + 5x + \sqrt{6}$

$f'(x) = 3\left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right) x^2 + 5$

f(x) is increasing $\Rightarrow$ $f'(x) \ge 0 \ \forall\ x \in R$

$\Rightarrow\ 1 - \frac{\sqrt{21-4b-b^2}}{b+1} \ge 0$ $\Rightarrow$ $\frac{\sqrt{(b+7)\ (3-b)}}{b+1} \le 1$

Case- I If b + 1 > 0, then $\frac{(b+7)\ (3-b)}{(b+1)^2} \le 1$ and $-7 \le b \le 3$

$\Rightarrow$ $b \le -5$ and $b \ge 2$ $\Rightarrow$ $b \in [2, 3]$

Case-II If $b + 1 < 0 \Rightarrow b \in [-7, -1)$

23. $y = x\,\ell n\,x - \frac{x^2}{2} + \frac{1}{2}$

$y' = 1 + \ell n\,x - x$

$y'' = \frac{1}{x} - 1 = \frac{1-x}{x}$

$y'' > 0 \ \forall\ x \in (0, 1)$

$\Rightarrow$ $y'(x) < y'(1)$

$\Rightarrow$ $y'(x) < 0$

$\therefore$ $y(x) > y(1)$

$\Rightarrow$ $x \log x - \frac{x^2}{2} + \frac{1}{2} > 0$

$\Rightarrow$ $x \log x > \frac{x^2}{2} + \frac{1}{2}$

24. $f'(x) = 0 \ \Rightarrow\ x = \pm\sqrt{\frac{a}{3b}}$

$f\left(-\sqrt{\frac{a}{3b}}\right) = \frac{-2a}{3}\sqrt{\frac{a}{3b}}$

$f\left(\sqrt{\frac{a}{3b}}\right) = \frac{2a}{3}\sqrt{\frac{a}{3b}}$

$f(-1) = b - a$

$f(1) = a - b$

Given that $\left|\frac{2a}{3}\sqrt{\frac{a}{3b}}\right| = \left|-\frac{2a}{3}\sqrt{\frac{a}{3b}}\right| = |\,b - a\,| = |\,a - b\,| = 1$

$\Rightarrow \frac{4a^3}{27b} = 1 \Rightarrow b = \frac{4a^3}{27} \Rightarrow a - b = 1 \Rightarrow a - \frac{4a^3}{27} = 1 \Rightarrow 4a^3 - 27a + 27 = 0$

$a = -3, \frac{3}{2} \Rightarrow a = -3 - 1 : b = \frac{3}{2} - 1 = -\frac{1}{2} = -4$

Also, $b - a = 1 \Rightarrow \frac{4a^3}{27} - a \Rightarrow 4a^3 - 27a - 27 = 0$

$\Rightarrow (a - 3)\ (2a + 3)^2 = 0$

$\Rightarrow a = 3 \Rightarrow b = 4$

Rejecting – ve values, therefore a = 3, b = 4

25. Let $f(x) = \dfrac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0 \;\; \forall \;\; x \in \left(0, \frac{\pi}{2}\right); (\therefore \tan x > x)$$

$$f''(x) = \frac{-x^2 \sin x - 2x\cos x + 2\sin x}{x^3}$$

Let $g(x) = -x^2\sin x - 2x\cos x + 2\sin x$

$g'(x) = -x^2\cos x < 0 \;\forall\; x \in (0, \pi/2)$

for $x > 0$, we have $g(x) < g(0)$ i.e. $g(x) < 0$

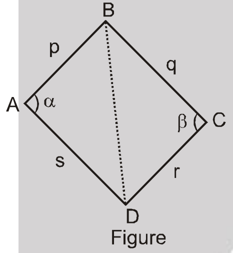
$\therefore f'(x) < 0$ and $f''(x) < 0 \;\forall\; x \in \left(0, \dfrac{\pi}{2}\right)$

$$\Rightarrow \quad f\left(\frac{A+B+C}{3}\right) > \left(\frac{f(A)+f(B)+f(C)}{3}\right)$$

$$\Rightarrow \quad \frac{\sin\left(\frac{A+B+C}{3}\right)}{\frac{A+B+C}{3}} \geq \left(\frac{\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}}{3}\right)$$

$$\Rightarrow \quad \frac{\sin\;\; A}{A} + \frac{\sin\;\; B}{B} + \frac{\sin\;\; C}{C} \leq \frac{9\sqrt{3}}{2\pi}$$

26. Area (ΔABCD) = Area of ΔADB + Area of ΔBDC



Figure

$$A = \frac{1}{2}\; ps \sin\alpha + \frac{1}{2}\; qr \sin\beta$$

$$\frac{dA}{dB} = \frac{1}{2}\; ps(+\cos\alpha) + \frac{1}{2}\; qr\cos\beta\; \frac{d\beta}{d\alpha} = 0 \Rightarrow \frac{d\beta}{d\alpha} = \frac{-ps}{qr}\; \frac{\cos\alpha}{\cos\beta}$$

$BD^2 = p^2 + s^2 - 2ps\cos\alpha = q^2 + r^2 - 2qr\cos\beta$

Differentiating we get $-2ps(-\sin\alpha) = -2qr(-\sin\beta)\dfrac{d\beta}{d\alpha} \Rightarrow \dfrac{d\beta}{d\alpha} = \dfrac{ps}{qr}\;\dfrac{\sin\alpha}{\sin\beta}$

$\Rightarrow -\dfrac{ps}{qr}\;\dfrac{\cos\alpha}{\cos\beta} = \dfrac{ps}{qr}\;\dfrac{\sin\alpha}{\sin\beta} \;\;\Rightarrow \sin\alpha\cos\beta + \cos\alpha\sin\beta = 0 \Rightarrow \sin(\alpha+\beta) = 0 \Rightarrow \alpha + \beta = \pi$

Also , $\dfrac{dA}{d\beta} = \dfrac{1}{2}\; ps\; \dfrac{\sin(\alpha+\beta)}{\sin\beta} = 0 \;\Rightarrow\; \alpha + \beta = \pi$

If $\alpha + \beta < \pi$ then $\dfrac{dA}{d\beta} > 0$

If $\alpha + \beta > \pi$, then $\dfrac{dA}{d\beta} < 0$

$\therefore$ By 1st derivative test A has maxima when $\alpha + \beta = \pi \;\Rightarrow$ A, B, C, D are concyclic

27. Let $u = 2^x + 2^{-x}$

$u^3 = 8^x + 8^{-x} + 3(2^x)\ (2^{-x})\ (2^x + 2^{-x}) \ \Rightarrow u^3 - 3u = 8^x + 8^{-x}$

also, $4^x + 4^{-x} = u^2 - 2 \Rightarrow f(x) = u^3 - 3u - 4\ (u^2 - 2) = u^3 - 4u^2 - 3u + 8$

Let $g(u) = u^3 - 4u^2 - 3u + 8$; $u \geq 2$

$g'(u) = 3u^2 - 8u - 3 = (3u + 1)\ (u - 3)$

putting $g'(u) = 0$; we get $u = 3$

$g''(u) = 6u - 8 \Rightarrow g''(3) = 1 > 0$

$\Rightarrow u = 3$ is point of minima $\Rightarrow g(3) = 27 - 36 - 9 + 8 = -10 \Rightarrow$ minimum $f(x) = -10$

28. Let $f(x) = \dfrac{\log_e(2x-1)}{\log_e x}$ for $x > 1$

Now $f'(x) = \dfrac{(2x-1)\ \log_e(2x-1)\ \ -2x\log_e x}{x(2x-1)\ \ \{\log_e(2x-1)\}^2}$

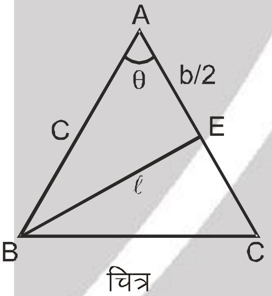
Let $g(x) = (2x - 1) \log_e(2x - 1) - 2x\log_e x$

$\Rightarrow g'(x) = 2\log_e(2x - 1) - 2\log_e x + 2 - 2 = 2 \log_e \left(2 - \dfrac{1}{x}\right) > 0$ for $x > 1$

$\Rightarrow$ for $x > 1$, we have $g(x) > g(1) \Rightarrow g(x) > 0 \ \Rightarrow f'(x) > 0$ for $x > 1 \Rightarrow$ f(x) is increasing for $x > 1$

$4 > 3 > 2 \Rightarrow f(4) > f(3) > f(2) \Rightarrow \dfrac{\log_e 7}{\log_e 4} > \dfrac{\log_e 5}{\log_e 3} > \dfrac{\log_e 3}{\log_e 2}$.

29. $\Delta = \dfrac{1}{2}$ bc sin $\theta = \dfrac{1}{2} c^2 \sin\theta$ ($\because$ b = c)



चित्र

In Δ ABE, using cosine rule, $\ell^2 = c^2 + \dfrac{b^2}{4} - bc\cos\theta = \dfrac{5c^2}{4} - c^2\cos\theta$

$$c^2 = \frac{4\ell^2}{5 - 4\cos\theta}$$

$\therefore \quad \Delta = 2\ell^2 . \dfrac{\sin\theta}{5 - 4\cos\theta}$

$\dfrac{d\Delta}{d\theta} = \dfrac{2\ell^2.((5-4\cos\theta)\cos\theta - \sin\theta \quad (4\sin\theta))}{(5-4\sin\theta)^2}$

$$\begin{array}{cc} + & - \\ \hline \multicolumn{2}{c}{\cos^{-1} 0.8} \end{array}$$

signs of $\dfrac{d\Delta}{d\theta} = \dfrac{2\ell^2.(5\cos\theta - 4)}{(5-4\sin\theta)^2}$

$\therefore \quad$ For Δ to be maximum, $\cos\theta = 0.8$

30. $y = 1 - x^2$

Consider point P $(x_0, 1 - x_0^2)$ $0 < x_0 \le 1$

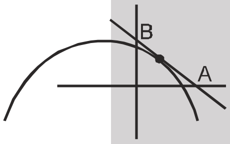
equation of tangent at P is $y - (1 - x_0^2) = -2x_0 (x - x_0)$

intersection with x-axis at $\Rightarrow A \equiv \left(x_0 + \frac{(1 - x_0^2)}{2x_0}, \quad 0 \right)$

intersection with y-axis at B(0, $2x_0^2 + (1 - x_0^2)$ area of ΔOAB $\Delta = \frac{1}{2} \frac{(x_0^2 + 1)^2}{2x_0} = \frac{1}{4} \frac{(x_0^2 + 1)^2}{x_0}$

$\frac{dA}{dx_0} = \frac{(x_0^2 + 1)}{4x_0^2} [3x_0^2 - 1] = 0$ $\qquad \Rightarrow x_0 = \frac{1}{\sqrt{3}}$

$\frac{dA}{dx_0}$ changes sign from +ve to – ve



Figure

at $x_0 = \frac{1}{\sqrt{3}}$ So point of minimum $\qquad \Rightarrow A_{min} = \frac{4\sqrt{3}}{9}$

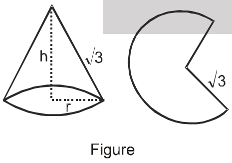
31. $3 = h^2 + r^2$

$\Rightarrow \quad r^2 = 3 - h^2$

$V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (3 - h^2) h$

$\frac{dV}{dh} = \frac{1}{3} \pi (3 - 3h^2)$

$\frac{dV}{dh} = 0 \qquad$ at h = 1



Figure

$\frac{d^2V}{dh^2} < 0 \qquad$ at h = 1

$\Rightarrow \quad V_{max} = \frac{2\pi}{3}$

32. $V = k\sqrt{\left(\frac{\lambda}{a}\right)+\left(\frac{a}{\lambda}\right)}$

V will be minimum when $\frac{\lambda}{a} + \frac{a}{\lambda}$ will be minimum

A.M. $\geq$ G.M.

$$\frac{\frac{\lambda}{a}+\frac{a}{\lambda}}{2} \geq \sqrt{\frac{\lambda}{a}\times\frac{a}{\lambda}}$$

$$\frac{\lambda}{a}+\frac{a}{\lambda} \geq 2$$

$\Rightarrow$ minimum of $\frac{\lambda}{a}+\frac{a}{\lambda} = 2$

$V_{min} = K\sqrt{2}$ which is independent of a.

33. $P \equiv (R\cos\theta, R\sin\theta)$ $\Rightarrow$ $a(R\cos\theta)^2 + 2b(R\cos\theta)(R\sin\theta) + a(R\sin\theta)^2 = c$

$R^2[a\cos^2\theta + b\sin2\theta + a\sin^2\theta] = c$ $\Rightarrow$ $R^2 = \frac{c}{a+b\sin2\theta}$

for minimum distance $\sin2\theta = 1$

$\therefore$ $R^2 = \frac{c}{a+b}$ $\Rightarrow$ $R = \sqrt{\frac{c}{a+b}}$

34. Let $\phi(x) = x + \sqrt{1+x^2}$

$\phi'(x) = 1 + \frac{x}{\sqrt{1+x^2}}$

If $x < 0$, $-|x| = x$

$$\phi'(x) = \frac{\sqrt{1+x^2}-|x|}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}-\sqrt{x^2}}{\sqrt{1+x^2}} > 0$$

If $x > 0$, $\phi'(x) > 0$

Hence $\phi(x)$ is increasing

As we know $e^x \geq x + 1 \Rightarrow \phi(e^x) \geq \phi(x+1)$

$$e^x + \sqrt{1+e^{2x}} \geq x + 1 + \sqrt{1+(x+1)^2}$$

35. Let $x > -1$

Consider $f(x) = (1 + x)\ell n(1 + x) - \tan^{-1} x$

$f'(x) = \ell n(1 + x) + 1 - \frac{1}{1+x^2}$

$f''(x) = \frac{(1+x)^2 + 3x^2 + x^4}{(1+x)\ \ (1+x^2)^2} > 0 \quad \Rightarrow f'(x)$ is increasing

For $x < 0$, $f'(x) < f'(0) \Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow f(x) > f(0) \Rightarrow f(x) > 0$

$\Rightarrow (1 + x)\ell n(1 + x) - \tan^{-1} x > 0$

$\ell n(1 + x) > \frac{\tan^{-1} x}{x+1}$

For $x > 0$, $f'(x) > f'(0) \Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing $\Rightarrow f(x) > f(0) \Rightarrow f(x) > 0 \Rightarrow \ell n(1 + x) > \frac{\tan^{-1} x}{x+1}$

Hence larger of these is $\ell n(1 + x)$.

36. $\cos x \in (0, 1) \Rightarrow f'(\cos x) < 0$, $f''(\cos x) > 0$

$g'(x) = \sin x \cos x \left(\frac{f'(\sin x)}{\sin x} - \frac{f'(\cos x)}{\cos x}\right)$.

Consider $\phi(t) = \frac{f'(t)}{t}$, $t \in (0, 1)$

$\phi'(t) = \frac{f''(t)t - f'(t)}{t^2}$

$\because$ $f''(\sin x) > 0 \Rightarrow f''(t) > 0$

$f'(\sin x) < 0 \Rightarrow f'(t) < 0 \Rightarrow \phi'(t) > 0\ t \in (0, 1)$

$\phi(t)$ is increasing. For $x \in \left(0, \frac{\pi}{4}\right)$ $\cos x > \sin x$

$\phi(\cos x) > \phi(\sin x)\ \frac{f'(\cos x)}{\cos x} > \frac{f'(\sin x)}{\sin x}$

$\Rightarrow g'(x) < 0 \Rightarrow g(x)$ is decreasing. Similarly, for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ $g(x)$ is increasing

37. Let $P(x) = x^{2013} - x^{2012} - 1007x^2 + 1007x + k$

as P(x) is a polynomial function in x hence it is everywhere continous and differentiable

also $P(0) = 0 + k = k$; $P(1007^{1/2011}) = k$

hence by Rolle's theorem $P'(x) = f(x) = 0$

for atleast one real value of 'x' in given interval.

39. Clearly g(x) satisfies condition in LMVT $\Rightarrow \frac{g(5) - g(0)}{5 - 0} = g'(c)$, $c \in (0, 5)\ \frac{\frac{f(5)}{6} - \frac{f(0)}{1}}{5} = g'(c)\ \ -\frac{5}{6} = g'(c)$

39. Let two consecutive zero of f(x) be a and b $f(a) = 0 = f(b)$.

If possible, suppose g(x) has no zero. Define $\phi(x) = \dfrac{f(x)}{g(x)}$

$\phi(x)$ satisfies conditions in Rolle's theorem, $\Rightarrow \phi'(c) = 0$ for at least one $c \in (a, b)$

$\Rightarrow f'(c)\, g(c) - f(c)\, g'(c) = 0$

Which is a contradiction to given condition $f(x)\, g'(x) \neq f'(x)\, g(x)$

Hence our supposition that g(x) has no zero is wrong $\Rightarrow$ g(x) has at least one zero.

40. Let $f'(x) = \phi'(x + a) - \phi'(x) \Rightarrow f(x) = \phi(x + a) - \phi(x) + k$

$f(0) = \phi(a) - \phi(0) + k$

$f(2a) = \phi(3a) - \phi(2a) + k \Rightarrow f(0) = f(2a)$

By Rolle's theorem on [0, 2a], $f'(c) = 0$ for at least one $c \in (0, 2a)$

$\Rightarrow \phi'(x + a) = \phi'(x)$ has at least one root in (0, 2a)

41. $f(x) = 8ax - a \sin 6x - 7x - \sin 5x$

$f'(x) = 8a - 6a \cos 6x - 7 - 5\cos 5x = 8a - 7 - 6a \cos 6x - 5\cos 5x$

f(x) is an increasing function

$f'(x) > 0$

$\therefore \quad 8a - 7 > 6a + 5$

$\Rightarrow \quad 2a > 12$

$a > 6$

$a \in (6, \infty)$

42. $g'(x) = \dfrac{h'(x)}{h(x)}$ $g''(x) = \dfrac{h(x).h''(x) - (h'(x))^2}{(h(x))^2}$

$\Rightarrow \quad g''(x) < 0$

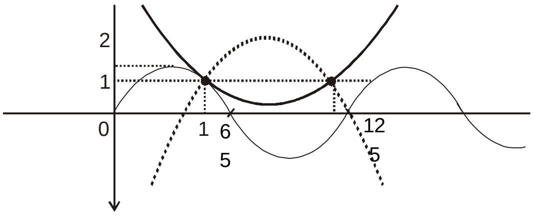
$\therefore$ g(x) is concave down on J.

43. $f''(x) = 2ax + 2(a + 2)$

Point of inflection is $x = -\dfrac{a+2}{a}$

$-\dfrac{a+2}{a} < 0 \Rightarrow a \in (-\infty, -2) \cup (0, \infty)$

44. Period of $y = 2\sin \dfrac{5\pi}{6} x$ is $\dfrac{2\pi}{5\pi/6} = \dfrac{12}{5}$ & $y = \alpha(x - 1)(x - 2) + 1$ which is a quadratic. So for given information (1, 1) is the common point to two curves and the possible graph would the



α can not be nagative α must be positive for these twp graphs to touch each other at (1, 1)

(Which can be the only possible point of contact) $\frac{dy}{dx}$ of both curves must be same.

$y = 2\sin\frac{5\pi}{6}x \quad \Rightarrow \quad \frac{dy}{dx} = \left(2\cos\frac{5\pi}{6}x\right).\frac{5\pi}{6}$

$\left(\frac{dy}{dx}\right)_{\text{at } x=1} = \left(2\cos\frac{5\pi}{6}\right).\frac{5\pi}{6} = \frac{5\pi}{6}.\cos\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{3}.\frac{\sqrt{3}}{2}$ ...(1)

$y = \alpha x^2 - 3\alpha x + 2\alpha + 1 \quad \Rightarrow \quad \frac{dy}{dx} = 2\alpha x - 3\alpha$

$\left(\frac{dy}{dx}\right)_{\text{at } x=1} = 2\alpha - 3\alpha = -\alpha$ ....(2)

for (1) & (2) $\alpha = \frac{5\pi}{2\sqrt{3}} \quad \Rightarrow \quad \frac{\sqrt{3}\alpha}{5\pi} = \frac{1}{2}$

45.✍ Let d be distance between (k, 0) and any point (x, y) on curve.

$d = \sqrt{(k-x)^2 + y^2}$

$d = \sqrt{-x^2 + 2(1-k)x + k^2}$ $\quad (\because y^2 = 2x - 2x^2)$.

Maximum $d = \sqrt{\frac{4(-1)k^2 - 4(1-k)^2}{4(-1)}}$ Maximum $d = \sqrt{2k^2 - 2k + 1}$

46_. $f(x) = (x + 1)(px^2 + (q - p)x + p)$

$g(x) = px^2 + (q - p)x + p$

$g(2) = 3p + 2q < 0$

$g(3) = 7p + 3q > 0$

$\Rightarrow$ one root is –1, one root lies between (2, 3), one root lies between $\left(\frac{1}{3}, \frac{1}{2}\right)$

47_.