PHYSICS

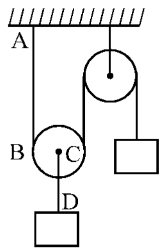
TARGET : JEE- Advanced 2021

CAPS-18

WAVE ON A STRING

SCQ (Single Correct Type) :

1. Both the strings, shown in figure are made of same material and diameter of CD is double that of AB. The pulleys are light. The speed of a transverse wave in the string AB is v_1 and in CD it is v_2, then $\frac{v_1}{v_2}$ is :



(A) 1 (B) 2 (C) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}$

Ans. (C)

Sol. $v = \sqrt{\frac{T}{\mu}}$ $v_1 = v_{AB} = \sqrt{\frac{T_{AB}}{\mu_{AB}}}$; $v_{CD} = \sqrt{\frac{T_{CD}}{\mu_{CD}}} = v_2$

$2T_{AB} = T_{CD}$; $\mu_{CD} = 4\mu_{AB}$

$\frac{v_1}{v_2} = \sqrt{\frac{T_{AB}}{T_{CD}}} \times \sqrt{\frac{\mu_{CD}}{\mu_{AB}}}$ $R_{CD} = 2R_{AB}$

$\frac{v_1}{v_2} = \sqrt{\frac{1}{2}} \times \sqrt{4} = \frac{2}{\sqrt{2}}$ $\mu_{CD} = [\pi\ (2R)2 \times 1]\rho$

$\frac{v_1}{v_2} = \sqrt{2}$ $\mu_{AB} = [\pi R^2 \times 1]\rho$ $\Rightarrow \mu_{CD} = 4\ \mu_{AB}$

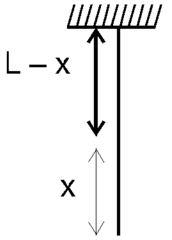
2. A heavy but uniform rope of length L is suspended from a ceiling. A particle is dropped from the ceiling at the instant when the bottom end is given the jerk. Where will the particle meet the pulse :

(A) at a distance lf $\frac{2L}{3}$ from the bottom (B) at a distance $\frac{L}{3}$ from the bottom

(C) at a distance $\frac{3L}{4}$ from the bottom (D) None of these

Ans. (B)

Sol.



For the pulse

$$V = \sqrt{\frac{\mu x g}{\mu}} = \sqrt{xg} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \sqrt{xg} \Rightarrow \int_o^x \frac{dx}{\sqrt{x}} = \sqrt{g}\int_o^t dt$$

$$t = 2\sqrt{\frac{x}{g}} \quad \text{———(1)}$$

for the particle

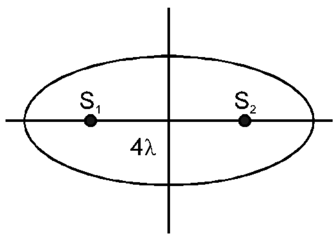
$$L - x = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2(L-x)}{g}} \quad \text{———(2)}$$

from equation (1) & (2)

$\Rightarrow \quad \therefore x = \frac{L}{3}$ from the bottom

3. S_1, S_2 are two coherent sources (having initial phase difference zero) of sound located along x-axis separated by 4λ where λ is wavelength of sound emitted by them. Number of maxima located on the elliptical boundary around it will be :

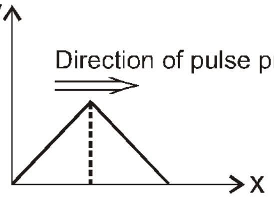


(A) 16 (B) 12 (C) 8 (D) 4

Ans. **(A)**

4. The wave function of a triangular wave pulse is defined by the relation below at time t = 0 sec.

$$y = \begin{cases} mx & \text{for } 0 \le x \le \frac{a}{2} \\ -m(x-a) & \text{for } \frac{a}{2} \le x \le a \\ 0 & \text{every where else} \end{cases}$$



The wave pulse is moving in the +X direction in a string having tension T and mass per unit length μ. The total kinetic energy present with the wave pulse is -

(A) $\frac{m^2Ta}{2}$ (B) m^2Ta (C) $\frac{3m^2Ta}{2}$ (D) None of these

Ans. **(A)**

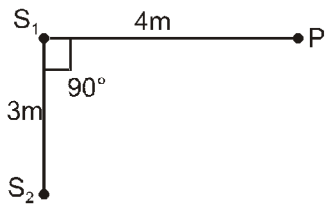
Sol. Kinetic energy present with the wave pulse :

$$\int_0^{a/2} \frac{1}{2}\mu dx\left(\frac{\partial y}{\partial t}\right)^2 + \int_{a/2}^{a} \frac{1}{2}\mu dx\left(\frac{\partial y}{\partial t}\right)^2$$

$\left(\frac{\partial y}{\partial t}\right) = \frac{mdx}{dt}$ for $0 \le x \le \frac{a}{2}$ where $dx/dt = \sqrt{\frac{T}{\mu}}$

$\left(\frac{\partial y}{\partial t}\right) = -\frac{mdx}{dt}$ for $\frac{a}{2} \le x \le a$

5. S_1 and S_2 are two coherent sources of sound of frequency 110Hz each. They have no initial phase difference. The intensity at a point P due to S_1 is I_0 and due to S_2 is $4I_0$. If the velocity of sound is 330 m/s then the resultant intensity at P is



(A) I_0 (B) $9I_0$ (C) $3I_0$ (D) $8I_0$

Ans. **(C)**

Sol. The wavelength of sound source = $\frac{330}{110}$ = 3 metre.

The phase difference betwen interfering waves at P is

$$= \Delta\phi = \frac{2\pi}{\lambda}(S_2P - S_1P) = \frac{2\pi}{3}(5-4) = \frac{2\pi}{3}$$

∴ Resultant intensity at P = $I_0 + 4I_0 + 2\sqrt{I_0}\sqrt{4I_0}\cos\frac{2\pi}{3} = 3I_0$

6. A standing wave pattern is formed on a string. One of the waves is given by equation $y_1 = a\cos(\omega t - kx + \pi/3)$ then the equation of the other wave such that at x = 0 a node is formed.

(A) $y_2 = a\sin(\omega t + kx + \frac{\pi}{3})$

(B) $y_2 = a\cos(\omega t + kx + \frac{\pi}{3})$

(C) $y_2 = a\cos(\omega t + kx + \frac{3\pi}{3})$

(D) $y_2 = a\cos(\omega t + kx + \frac{4\pi}{3})$

Ans. **(D)**

Sol. At x = 0 the phase difference should be π.

$\therefore$ the correct option is D.

Alternate solution

$y_2 = a\cos(\omega t + kx + \phi_0)$

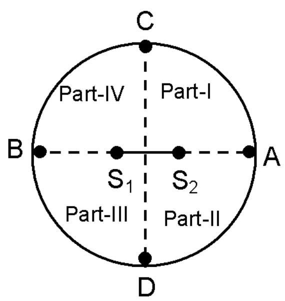
$\therefore y = y_1 + y_2 = a\cos(\omega t - kx + \frac{\pi}{3}) + a\cos(\omega t + kx + \phi_0)$

$$= 2a\cos\left[\omega t + \frac{\frac{\pi}{3} + \varphi_0}{2}\right] \times \cos\left[kx + \frac{\varphi_0 - \frac{\pi}{3}}{2}\right]$$

$\because y = 0$ at x = 0 for any t $\Rightarrow kx + \frac{\varphi_0 - \frac{\pi}{3}}{2} = \frac{\pi}{2}$ at x = 0

$\therefore \phi_0 = \frac{4\pi}{3}$. Hence $y_2 = a\cos(\omega t + kx + \frac{4\pi}{3})$

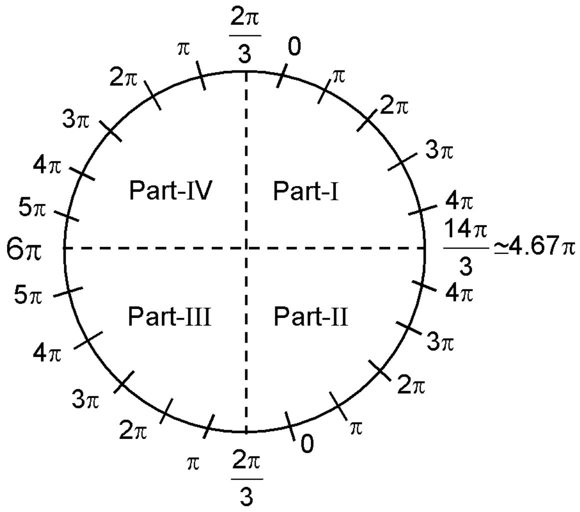
7. Two sound sources S_1 and S_2 are kept symmetrically about the centre of a circle as shown. The circle is divided in 4 parts. Both the sources are separated by a $8\lambda/3$ distance, where λ is wavelength of sound emitted by the sources. A detector is moving around the circle. How many times it will detect a minima in II part if S_1 is leading S_2 by phase of $2\pi/3$?



(A) 2 (B) 3 (C) 4 (D) 5

Ans. **(A)**

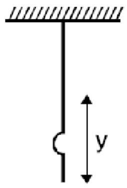
Sol. Phase difference equivalent to $\frac{8\lambda}{3}$ path difference

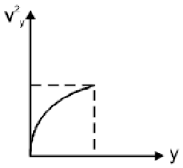


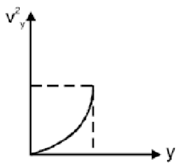
$\frac{2\pi}{\lambda} \times \frac{8\lambda}{3} = \frac{16\pi}{3}$

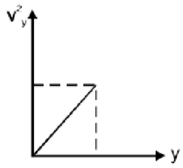
Part I, II, III and IV will be 2, 2, 3 and 3 minima respectively

8. A non–uniform rope of length hangs from a ceiling. Mass per unit length of rope (μ) changes as $\mu = \mu_0 e^y$, where y is the distance along the string from its lowest point. Then graph between square of velocity of wave (v_y^2) and y will be best represented as :



(A) 

(B) 

(C) 

(D) None of these

Ans. **(A)**

Sol. $v_y = \sqrt{\frac{T_y}{\mu_y}}$

$T_y = \left\{\int_0^y \mu_0 e^y dy\right\} g$

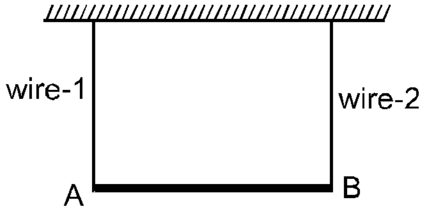
$T_y = \mu_0(e^y - 1).g$

$v_y = \sqrt{g - \frac{g}{e^y}}$

$v_y^2 = g(1 - e^{-y})$.

MCQ (One or more than one correct) :

9. A rod of length 0.3 m having variable linear mass density from A to B as $\lambda = \lambda_0 x$(x is distance from A in meter), where λ_0 = 100 kg/m is suspended by two light wires of same length. Ratio of their linear mass density is 2 : 9. Then which of the following is/are correct :



(A) Ratio of wave speed in wire–1 to wire–2 is 3 : 2

(B) Ratio of wave speed in wire–1 to wire–2 is 3 : 1

(C) Second harmonic in wire–1 has same frequency as third harmonic in wire–2

(D) Third overtone in wire–1 has same frequency as fifth overtone in wire–2

Ans. **(ACD)**

Sol. Centre of mass = $\dfrac{\int_0^L \lambda_0 x^2 dx}{\int_0^L \lambda_0 x dx} = \dfrac{\lambda_0 \frac{L^3}{3}}{\lambda_0 \frac{L^2}{2}} = \dfrac{2L}{3}$

Mass = $\int_0^L \lambda_0 x dx = \dfrac{\lambda_0 L^2}{2}$

$T_1 + T_2 = \dfrac{\lambda_0 L^2}{2} g$

Balancing torque about A,

$T_2 \times L = \dfrac{\lambda_0 L^2}{2} \times \dfrac{2L}{3} g$

$T_2 = \dfrac{\lambda_0 L^2 g}{3}$ $\quad T_1 = \dfrac{\lambda_0 L^2 g}{6}$

L = 0.3 m $\quad \lambda_0$ = 100 kg/m^2

T_2 = 30 N, $\quad T_1$ = 15 N

$V_1 = \sqrt{\dfrac{T_1}{\mu_1}}$ and $V_2 = \sqrt{\dfrac{T_2}{\mu_2}}$

$\dfrac{V_1}{V_2} = \sqrt{\dfrac{T_1 \mu_2}{T_2 \mu_1}} = \dfrac{3}{2}$

$\dfrac{n_1}{2\ell}\sqrt{\dfrac{15}{2}} = \dfrac{n_2}{2\ell}\sqrt{\dfrac{30}{9}}$ (ℓ is the length of the wire)

$\dfrac{n_1}{n_2} = \dfrac{2}{3}$

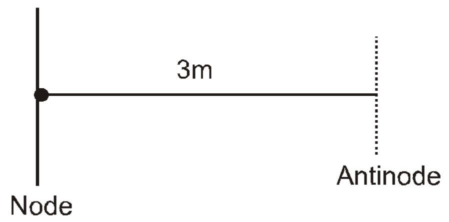
10. One end of a taut string of length 3m along the x-axis is fixed at x = 0. The speed of the waves in the string is 100m/s. The other end of the string is vibrating in the y-direction so that stationary waves are set up in the string. The possible waveform(s) of these stationary waves is (are)

(A) $y(t) = A \sin\frac{\pi x}{6}\cos\frac{50\pi t}{3}$ (B) $y(t) = A \sin\frac{\pi x}{3}\cos\frac{100\pi t}{3}$

(C) $y(t) = A \sin\frac{5\pi x}{6}\cos\frac{250\pi t}{3}$ (D) $y(t) = A \sin\frac{5\pi x}{2}\cos 250\pi t$

Ans. **(ACD)**

Sol. V = 100 m/s



Possible modes of vibration

$\ell = (2n+1)\frac{\lambda}{4}$

$\lambda = \frac{12}{(2n+1)}$ m

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{12/(2n+1)} = \frac{(2n+1)\pi}{6}$

$\omega = vk = 100(2n+1)\frac{\pi}{6} = \frac{(2n+1)\ 50\pi}{3}$

if n = 0 $k = \frac{\pi}{6}$ $\omega = \frac{50\ \pi}{3}$

n = 1 $k = \frac{5\ \pi}{6}$ $\omega = \frac{250\ \pi}{3}$

n = 7 $k = \frac{5\ \pi}{2}$ $\omega = 250\ \pi$

11. The vibrations of a string of length 600 cm fixed at both ends are represented by the equation

$$y = 4\sin\left(\pi\frac{x}{15}\right)\cos(96\pi t)$$

where x and y are in cm and t in seconds.

(A) The maximum displacement of a particle at x = 5 cm is $2\sqrt{3}$ cm.

(B) The nodes located along the string are 15n where integer n varies from 0 to 40.

(C) The velocity of the particle at x = 7.5 cm at t = 0.25 sec is zero

(D) The equations of the component waves whose superposition gives the above wave are

$$2\sin 2\pi\left(\frac{x}{30}+48t\right),\ 2\sin 2\pi\left(\frac{x}{30}-48t\right).$$

Ans. **(ABCD)**

Sol. $y = 4 \sin \cos 96 \pi t$

At x = 5 cm, $y = 4 \sin \cos (96 \pi t)$ and $y_{max} =$ cm

Positions of nodes is given by equation

$$\sin\left(\frac{\pi x}{15}\right) = 0 \quad \Rightarrow \quad 2\sqrt{3} = n\pi$$

$$\Rightarrow \quad x = 15n$$

At x = 7.5 cm and t = 0.25 sec.

Velocity of the particle = $\frac{\partial y}{\partial t} = -344 \pi \sin\left(\frac{\pi x}{15}\right) \sin (96 \pi t) = 0$

12. A horizontal stretched string, fixed at two ends, is vibrating in its fifth harmonic according to the equation, $y(x, t) = (0.01 \text{ m}) \sin [(62.8 \text{ m}^{-1}) x] \cos [(628 \text{ s}^{-1})t]$. Assuming $\pi = 3.14$, the correct statement(s) is (are) :

(A) The number of nodes is 5.

(B) The length of the string is 0.25 m.

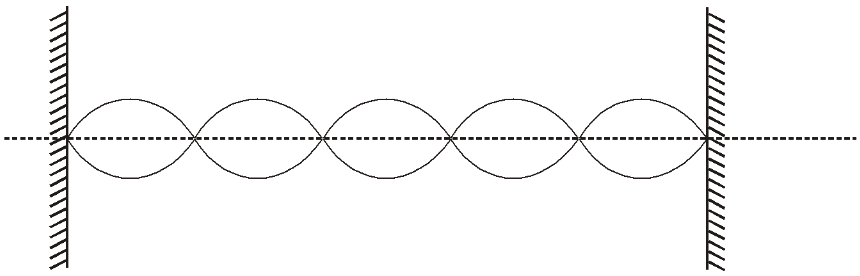
(C) The maximum displacement of the midpoint of the string its equilibrium position is 0.01 m.

(D) The fundamental frequency is 100 Hz.

Ans. **(BC)**

Sol. (A) There are 5 complete loops.

Total number of nodes = 6



(B) $\omega = 628 \text{ sec}^{-1}$

$$k = 62.8 \text{ m}^{-1} = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{1}{10}$$

$$v_w = \frac{\omega}{k} = \frac{628}{62.8} = 10 \text{ ms}^{-1}$$

$$L = \frac{5\lambda}{2} = 0.25$$

(C) 2A = 0.01 = maximum amplitude of antinode

(D) $f = \frac{v}{2\ell} = \frac{10}{2 \times 0.25} = 20$ Hz.

13. A wave given by $\xi = 10 \sin [80\pi t - 4\pi x]$ propagates in a wire of length 1m fixed at both ends. If another wave of similar amplitude is superimposed on this wave to produce a stationary wave then

(A) the superimposed wave is $\xi = -10 \sin [80\pi t + 4\pi x]$

(B) the maximum amplitude of the stationary wave is 20 m.

(C) the wave length of the wave is 0.5 m.

(D) the number of total nodes produced in the wire are 3.

Ans. **(ABC)**

Sol. $\ell = 1$ m $\varepsilon = 10 \sin (80\pi t - 4\pi x)$

For superimposed second wave is

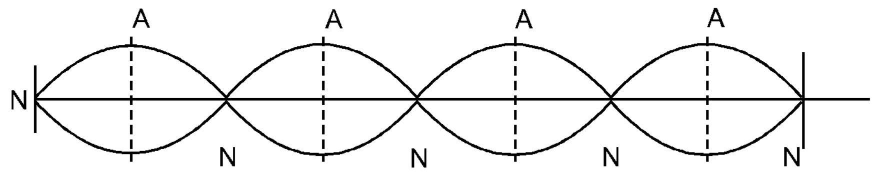
$\varepsilon_2 = 10\sin (80\pi t + 4\pi x)$

Amplitude of stationary wave = 2A = 2 × 10 = 20m

$K = 4\pi = \frac{2\pi}{\lambda}$

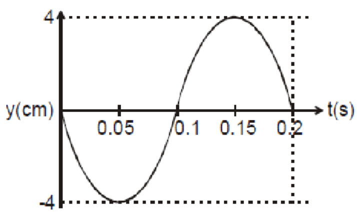
$\lambda = \frac{2\pi}{4\lambda} = \frac{1}{2} = 0.5$ m

$\ell = 1$m $(\ell = 2\lambda)$



Total (N = 5)

14. For a certain transverse standing wave on a long string, an antinode is formed at x = 0 and next to it, a node is formed at x = 0.10 m. the position y(t) of the string particle at x = 0 is shown in figure.



(A) Transverse displacement of the particle at x = 0.05m and t = 0.05 s is $-2\sqrt{2}$ cm.

(B) Transverse displacement of the particle at x = 0.04 m and t = 0.025 s is $-2\sqrt{2}$ cm.

(C) Speed of the travelling waves that interfere to produce this standing wave is 2 m/s.

(D) The transverse velocity of the string particle at $x = \frac{1}{15}$ m and t = 0.1 s is 20 π cm/s

Ans. **(ACD)**

Sol. $\frac{\lambda}{4} = 0.1 \Rightarrow \lambda = 0.4$ m from graph ⇒ T = 0.2 sec. and amplitude of standing wave is 2A = 4 cm.

Equation of the standing wave

$$y(x,t) = -2A\ \cos\left(\frac{2\pi}{0.4}x\right).\ \ \sin\left(\frac{2\pi}{0.2}t\right)\ \text{cm}$$

y (x = 0.05, t = 0.05) = $-2\sqrt{2}$ cm

y(x = 0.04, t = 0.025) = $-2\sqrt{2}$ cos 36°

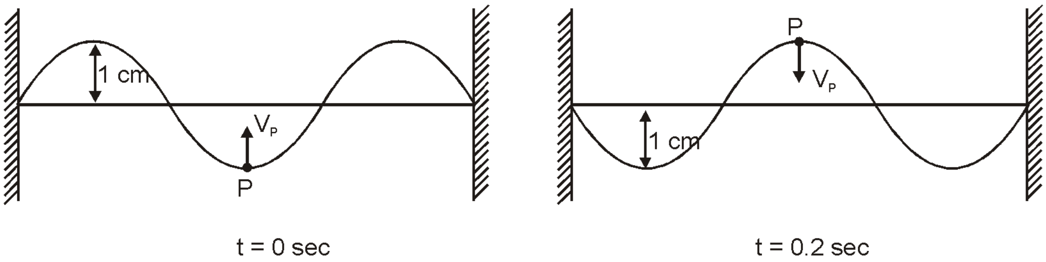
speed $= \frac{\lambda}{T} = 2\text{m/sec}.$

$$V_y = \frac{dy}{dt} = -2A \times \frac{2\pi}{0.2}\cos\left(\frac{2\pi\ x}{0.4}\right).\cos\left(\frac{2\pi t}{0.2}\right)$$

$$V_y = \left(x = \frac{1}{15}\text{m},\ t = 0.1\right) = 20\pi\ \text{cm/sec}.$$

Comprehension Type Question:

Stationary wave is setup in a uniform string clamped at both the ends. Length of the string is 0.3 m. Snapshot of the string is taken at two instants one at t = 0 sec and another at t = 0.2 sec. These two snapshots are shown below.



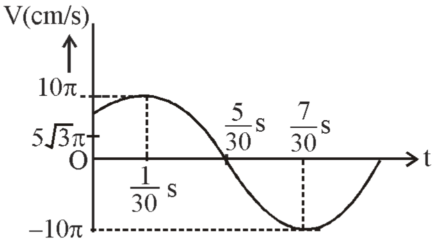
Velocity of point P (which is also the mid point of the string) is in upward direction (take upward direction to be positive) at t = 0 sec. At the instant snapshots are taken particles are at half of their respective maximum displacement from mean position. During this time interval particles have crossed their mean position only once. Answer the following questions for the given situation.

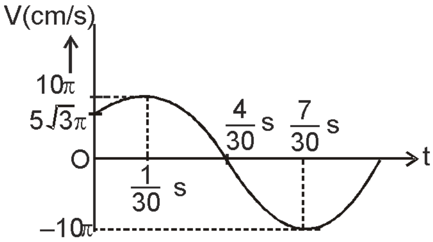
15. Velocity of travelling wave in the string is

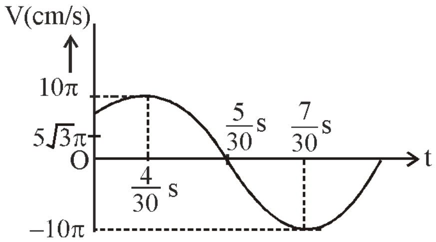
(A) 1 m/s (B) 0.5 m/s (C) 2 m/s (D) 0.25 m/s.

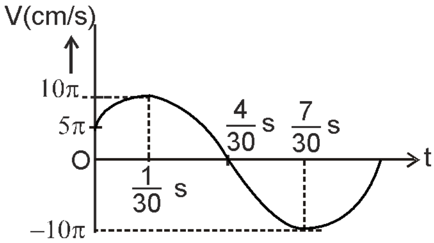
Ans. **(B)**

16. Velocity time graph of particle at mid point of the string (i.e., particle P)

(A) 

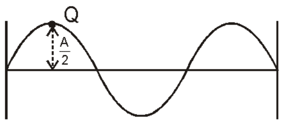
(B) 

(C) 

(D) 

Ans. **(B)**

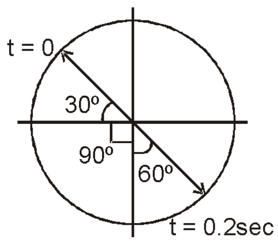
Sol. t = 0



Displacement equations of point Q $= A \sin\left(\omega t + \frac{5\pi}{6}\right)$

Equation of standing wave $y(x) = A(x) \sin\left(\omega t + \frac{5\pi}{6}\right) = A \sin kx. \sin\left(\omega t + \frac{5\pi}{6}\right)$

According to snapshots



$t = \frac{1}{5} = \frac{\pi}{\omega} \Rightarrow \omega = 5\pi$ rad/s

Time period $T = \frac{2\pi}{5\pi} = \frac{2}{5}$sec

wavelength $\lambda = 0.2$ m

wave velocity $v = \frac{\lambda}{T} = \frac{2}{10} \cdot \frac{5}{2} = \frac{1}{2}$m / s

Disp. equation for point P $y = A \sin\left(\omega t + \frac{11\pi}{6}\right)$

velocity equation for point P $V_p = \omega A \cos\left(\omega t + \frac{11\pi}{6}\right)$

here $\omega = 5\pi$rad/s $\quad$ A = 2 cm

Numerical based Questions :

17. A string of mass ' m ' and length ℓ, fixed at both ends is vibrating in its fundamental mode.The maximum amplitude is ' a ' and the tension in the string is ' T '. If the energy of vibrations of the string is $\frac{\pi^2 a^2 T}{\eta L}$. Find η

Ans. **4**

Sol. Energy E = $\int_0^{\ell} \frac{1}{2} dm v^2 = \int_0^{\ell} \frac{1}{2} dm A_x^{\ 2} \ \omega^2$

$= \int_0^{\ell} \frac{1}{2}\left(\frac{m}{\ell}\right) dx \ \ A^2 \sin^2 kx \ \ \omega^2$

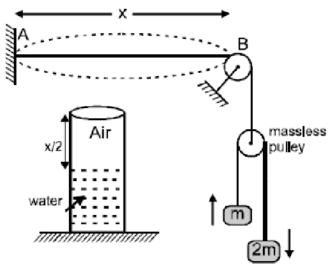
$= \int_0^{\ell} \frac{1}{2}\left(\frac{m}{\ell}\right) \ A^2 \omega^2 \sin^2 kx \ \ dx$

$= \frac{1}{4} m A^2 \omega^2$

As $\omega = 2\pi f = 2\pi \ \frac{v}{2\ell} = \frac{\pi}{\ell}\sqrt{\frac{T}{\mu}} \ = \ \frac{\pi}{\ell}\sqrt{\frac{T\ell}{m}}$

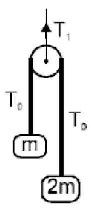
$\therefore$ Energy = $\frac{1}{4} \ m \ a^2 \frac{\pi^2}{\ell^2} \ \frac{T\ell}{m} \ = \frac{a^2 \pi^2 T}{4\ell}$

18. AB wire (length x) is vibrating in its fundamental mode. Wire AB is in resonance with resonance tube in which air column (lenght x/2) is also vibrating with its fundamental mode. Sound speed is 400 m/sec and linear mass density of AB wire is 10^{-4} kg/m and g = 10 m/sec^2, value of mass m = [$\beta(10^{-1})$] kg, then find value of β. Neglect the masses of wires in comparison to block's mass 'm'.



Ans. **(6)**

Sol.



$T_1 = 2T_0 = 2\left[\frac{2m(2m)}{m+2m}\right] g$

$T_1 = \frac{8m}{3} g = \frac{80m}{3}$ ....(i)

In resonance,

$f_{wire} = f_{tube}$

$$\frac{(1)V_1}{2\ell_1} = \frac{(1)V_2}{4\ell_2}$$

$$\frac{\left(\sqrt{\frac{T_1}{\mu}}\right)}{2(x)} = \frac{(400)}{4\left(\frac{x}{2}\right)}$$

$\Rightarrow$ $T_1 = \mu(16 \times 10^4)$

From (i)

$$\frac{80}{3}m = 10^{-4}(16 \times 10^4)$$

m = 0.6 kg.

Matrix Match Type :

19. In case of string waves, match the statements in column-I with the statements in column-II.

	Column-I		Column-II
(A)	A tight string is fixed at both ends and sustaining standing wave	(p)	At the middle, antinode is formed in odd harmonic
(B)	A tight string is fixed at one end and free at the other end	(q)	At the middle, node is formed in even harmonic
(C)	A tight string is fixed at both ends and vibrating in four loops	(r)	the frequency of vibration is 300% more than its fundamental frequency
(D)	A tight string is fixed at one end and free at the other end, vibrating in 2nd overtone	(s)	Phase difference between SHMs of any two particles will be either π or zero.
		(t)	The frequency of vibration is 400% more than fundamental frequency.

Ans. (A) p,q,s (B) s (C) q,r,s (D) s,t

Sol. (A) Number of loops (of length $\lambda/2$) will be even or odd and node or antinode will respectively be formed at the middle.

Phase difference between two particle in same loop will be zero and that between two particles in adjacent loops will be π.

(B) and (D) Number of loops will not be integral. Hence neither a node nor an antinode will be formed in in the middle.

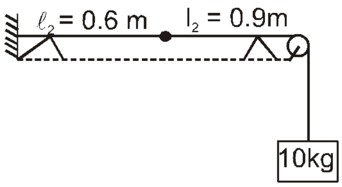
Phase of difference between two particle in same loop will be zero and that between two particles in adjacent loops will be π.

Subjective Type Questions :

20. An aluminium wire of cross-sectional area 1×10^{-6} m^2 is joined to a steel wire of same cross-sectional area. This compound wire is stretched on a sonometer, pulled by a weight of 10 kg. The total length of the compound wire between the bridges is 1.5 m, of which the aluminium is 0.6 m and the rest is steel wire. Transverse vibrations are set up in the wire by using an external force of variable frequency. Find the lowest frequency of excitation for which standing waves are formed such that the joint in the wire is a node. What is the total number of nodes observed at this frequency, excluding the two at the ends of the wire? The density of aluminium is 2.6×10^3 kg/m^3 and that of steel is 1.04×10^4 kg/m^3.

Ans. 162 vibrations/sec, 3

Sol.



A = 1×10^{-6} m, T = 10g = 100N

$\sigma_1 = 2.6 \times 10^3$kg/m^2 $\quad \sigma_2 = 1.04 \times 10^4$ kg /m^3

$$v_1 = \sqrt{\frac{100}{1\times10^{-6}\times2.6\times10^3}} = \frac{10^3}{\sqrt{26}}\text{m/sec}$$

$$v_2 = \sqrt{\frac{100}{1\times10^{-6}\times1.04\times10^4}} = \frac{10^3}{2\sqrt{26}}$$

$f_1 = f_2$

$$\frac{n_1\times10^3}{2\times0.6\sqrt{26}} = \frac{n_2\times10^3}{2\times2\times0.9\sqrt{26}} \left[\frac{n_1}{n_2}=\frac{1}{3}\right]$$

$$\text{Lowest frequency} = \frac{n_1v_1}{2L_1} = \frac{1\times10^3}{\sqrt{26}\times2\times0.6}$$

$$= \frac{10^4}{12\sqrt{26}} = 162 \text{ vibration/sec.}$$

$$\frac{n_1}{n_2} = \left(\frac{1}{3}\right)$$

One side 1 loop and other side 3 loop

So excluding the two nodes at ends total nodes is 3

21. A metal wire with volume density ρ and young's modulus Y is streatched between rigid supports. At temperature T, the speed of a transverse wave is found to be v_1. When temperature decreases $T - \Delta T$, the speed increases to v_2. Determine coefficient of linear expansion of wire.

Ans. $\alpha = \frac{\rho \ (v_2^2 - v_1^2)}{Y \ \Delta T}$

Sol

If temperature decreases,s tension in wire increases and v increases

$v_1 = \sqrt{\frac{F}{\mu}}$

$F = \mu v_1^2$

$\Delta F = YA \alpha \Delta T$

$v_2 = \sqrt{\frac{F + \Delta F}{\mu}} = \sqrt{\frac{F + YA \ \alpha \ \Delta T}{\mu}}$

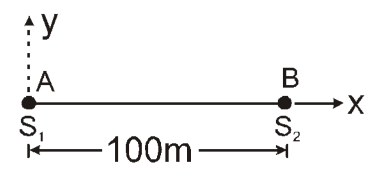
$v_2^2 = v_1^2 + \frac{YA \ \alpha \ \Delta T}{\mu}$

$\frac{YA \ \alpha \ \Delta T}{(m/\ell)} = v_2^2 - v_1^2$

$\frac{Y \ \alpha \ \Delta T}{\rho} = v_2^2 - v_1^2$

$\alpha = \frac{\rho \ (v_2^2 - v_1^2)}{Y \ \Delta T}$

22. In the figure shown A and B are two ends of a string of length 100m. S_1 and S_2 are two sources due to which points 'A' and 'B' oscillate in 'y' and 'z' directions respectively according to the equation y = 2 sin (100 π t + 30°) and z = 3 sin (100 π t + 60°) where t is in sec and y is in mm. The speed of propagation of disturbance along the string is 50 m/s. Find the instantaneous position vector (in mm) and velocity vector in (m/s) of a particle 'P' of string which is at 25m from A. You have to find these parameters after both the disturbances from S_1 and S_2 have reached 'P'. Also find the phase difference between the waves at the point 'P' when they meet at 'P' first time.



Ans. $\vec{r}_{(\text{in mm})} = 25000\hat{i} + 2\sin(100\pi + 30^0)\hat{j} + 3\sin(100\pi t + 60^0)\hat{k}$

$\vec{v}(\text{in m/s}) = 0.2\pi\cos(100\pi t + 30^0)\ \hat{i} + 0.3\pi\cos(100\pi t\hat{j} + 60^0)\hat{k}$

Phase difference at time 't' is 30^0 constant always after they meet at 'P'.

Sol. $y_p = 2\sin(100\pi t + 30^0)$

$$y = 2\sin\left[100\pi(t - \frac{25}{50}) + 30^0\right] = 2\sin(100pt - 50p + 30^0)$$

$$z = 3\sin\left[100\pi\left(t - \frac{75}{50}\right) + 60^0\right]$$

$$= 3\sin\left(100\pi t - 150\pi + 60^0\right) = 3\sin(100pt + 60^0)$$

$$\vec{r}_{(\text{in mm में})} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= 25000\hat{i} + 2\sin(100\pi + 30^0)\hat{j} + 3\sin(100\pi t + 60^0)\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 2\times 100\pi\cos(100\pi t + 30^0)\ \hat{i} + 3\times 100\pi\cos(100\pi t\hat{j} + 60^0)\hat{k}$$

$$\vec{v}(\text{in m/s}) = 0.2\pi\cos(100\pi t + 30^0)\ \hat{i} + 0.3\pi\cos(100\pi t\hat{j} + 60^0)\hat{k}$$

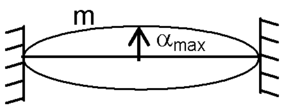
Phase difference at time 't' = 30^0 constant always after they meet at 'P'.

23. A string of mass m is fixed at both ends. The fundamental tone oscillations are excited with angular frequency ω and maximum displacement amplitude a_{max}. Find :

(a) the maximum kinetic energy of the string;

(b) the mean kinetic energy of the string averaged over one oscillation period.

Ans. (a) $T_{max} = 1/4\ m\ \omega^2\ a^2_{max}$; (b) $T = 1/8\ m\omega^2\ a^2_{max}$.

Sol. 

equation of standing wave :

$y = \alpha_{max}\sin kx\cos\omega t$

$V_y = \frac{dy}{dt} = -a_{max}\sin kx\sin\omega t$

$Vy = \frac{dy}{dt} = -\alpha_{max}\omega\sin kx\sin\omega t$

(a) For maximum velocity $\sin\omega t = 1$

then $V_y = -\alpha_{max}\omega\sin kx$

Then kinetic energy :

$$dk = \frac{1}{2}(dm)Vy^2$$

$$dk = \frac{1}{2}(\mu dx)\alpha^2_{max}\omega^2\sin^2 kx$$

$$\int dk = \frac{1}{2}\mu\alpha^2_{max}\omega^2\int_0^{\lambda/2}\sin^2 kx\,dx$$

$$k = \frac{1}{2}\mu\alpha^2_{max}\omega^2\left(\frac{1}{2}\right)$$

$K = \frac{1}{4}\mu\alpha_{max}^2\omega^2$

(b) Average kinetic energy for strip of length dx :

$<dk> = <\frac{1}{2}dmv^2> = <\frac{1}{2}(\mu dx)(\alpha_{max}\omega \sin kx \sin\omega t)^2>$

$<dk> = \frac{1}{2}\mu\alpha^2_{max}\omega^2 \sin^2 kxdx \int_0^{2\pi/\omega} \sin^2\omega t = \frac{1}{4}\alpha_{max}^2\omega^2 \sin^2 kxdx$

Averge ower all mole cubes :

$$<k> = \frac{1}{4}\mu\alpha_{max}^2\omega^2 < \sin^2 kx >$$

$$<k> = \frac{1}{8}\mu\alpha_{max}^2\omega^2 \qquad \text{Ans.}$$

From (i) and (ii) :

$$dE = dU + ak$$

$$dE = \frac{1}{2}\mu a^2\omega^2 [\sin^2 kx \sin^2\omega t + \cos^2 kx \cos^2\omega t]\, dx \qquad \ldots(iii)$$

Average value of dE at (dx) element.

$$<dE> = \frac{1}{2}\mu a^2\omega^2 \left[\sin^2 kx < \sin^2\omega t > + \cos^2 kx < \cos^2\omega t >\right]dx$$

$$<dE> = \frac{1}{4}\mu a^2\omega^2[\sin^2 kx + \cos^2 kx]dx$$

$$<E> = \frac{1}{4}\mu a^2\omega^2 \int_0^{\lambda/2} dx = \frac{1}{4}\mu a^2\omega^2 \frac{\lambda}{2}$$

$$<E> = \frac{1}{8}\mu a^2\omega^2\lambda \qquad \ldots.(iv)$$

Also $\rho = \frac{\mu}{s}$ and $\lambda = \frac{2\pi}{K}$ pat in ...(iv)

$$<E> = \frac{1}{4}\pi s\rho \frac{(a\omega)^2}{k} \qquad \text{Ans.}$$

Analysis :

on kinetic energy : from (i) :

$$dk = \frac{1}{2}\mu a^2\omega^2 \sin^2 kx \sin^2\omega t\, dx$$

At node point : sinkx = 0

then dk = 0 (always at rech time)

At antinode point : sinkx = ±1

$$\text{then } dk = \frac{1}{2}\mu a^2\omega^2 \sin^2\omega t\, dx$$

this equation suggest that kinetic energy will be

variable and it maximum when $\sin\omega t = 1 \Rightarrow \cos\omega t = 0$

then string particle will beat mean position.

On potential Energy from (ii)

$$dU = \frac{1}{2}\mu a^2\omega^2 \cos^2 kx \cos^2 \omega t\, dx$$

At node point : sinkx = 0 $\Rightarrow$ coskx = ±1

And hence potential energy will be maximum at node and minimum (0) at antinode always

At node :

$$dU = \frac{1}{2}\mu a^2\omega^2 \cos^2 \omega t\, dx$$

Potential energy at node will be maximum when

$\cos\omega t = 1$ and this instant all particles will be at extreme position

Analysis :

Energy between two node from (iii)

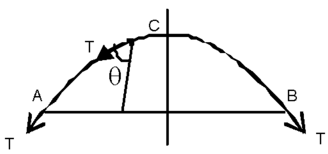
$$E = \frac{1}{2}\mu a^2\omega^2\left[\sin^2 \omega t \int_0^{\lambda/2} \sin^2 kx\, dx + \cos^2 \omega t \int_0^{\lambda/2} \cos^2 kx\, dx\right]$$

$$E = \frac{1}{2}\mu a^2\omega^2 \left[\frac{\lambda}{4}\sin^2 \omega t + \cos^2 \omega t \frac{\lambda}{4}\right]$$

$E = \frac{a^2\omega^2\lambda}{8}\mu$ = const dues not depend on time.

Energy is confined between two node.

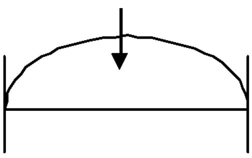
Explanation with mechanics



Here point A and B Due node points which displace ment are zero. Hence work done by tension will be zero. And then energy will be conserved.

For point C : Tension component work and hence energy wil charge of particle and each half cycle work done by tension will Zero. And energy will be conserved in each half cycle.

Method : 2



Since enegy between two nodes are conserved at reach instant of time. Now calculate energy when string is at mean position and at mean position potential energy is zero. Hence total energy will be equal to kinetic energy at mean position.

At mean position :

E = U + K

Here U = 0

Now $K = \Sigma\frac{1}{2}(dm)v^2$

$K = \Sigma \frac{1}{2}(\mu dm)v^2 \qquad \ldots(i)$

Now $\quad v = \frac{dE}{dt} = -a\omega \sin kx \sin \omega t$

from (i) :

$K = \Sigma \frac{1}{2}(\mu dx)a^2\omega^2 \sin^2 kx \sin^2 \omega t \qquad \ldots(ii)$

At mean position $\varepsilon = 0 \qquad \Rightarrow \cos\omega t = 0 \Rightarrow \omega t = \frac{\pi}{2}$

from (ii)

$K = \Sigma \frac{1}{2}\mu a^2\omega^2 \sin^2 kx dx$

$\frac{1}{2}\mu a^2\omega^2 \int_0^{\lambda/2} \sin^2 kx\, dx = \frac{a^2\omega^2\mu\lambda}{8}$

$E = K = \frac{a^2\omega^2\mu\lambda}{8}$ Put $\mu = \rho s$ and $\lambda = \frac{2\pi}{K}$

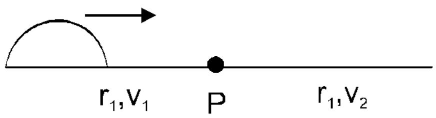
$E = \frac{1}{4}\pi s\rho \frac{(a\omega)^2}{k}$

24. A sinusoidal wave pulse is moving on a stretched string as shown. Strings on two sides of the joint P are of same material same tension, but of radii r_1 and $r_2 < r_1$. When the pulse passes through the point P, the average power of pulses transmitted and reflected are in ratio $\frac{5}{4}$. If the incident pulse is given by equation y = A sin(kx – ωt) where letters have usual meaning, find.

(i) ratio of speed of transmitted and reflected wave $\frac{\upsilon_2}{\upsilon_1}$,

(ii) ratio of radii of strings $\frac{r_2}{r_1}$

(iii) equation of the transmitted and reflected wave.



Ans. **(i) 5 (ii) 0.2 (iii) $y_r = \frac{5}{3}$A sin and $y_t = \frac{2}{5}$A sin (Kx + ωt).**

Sol. Power transmitted by a pulse

$P = \frac{1}{2}\mu A^2\omega^2\upsilon$

$= \frac{1}{2}\sqrt{\mu}\, A^2\, \omega^2 \sqrt{T}$

As given,

$$\frac{P_t}{P_r} = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\left(\frac{A_2}{A_1}\right)^2 \quad \text{...........(i)}$$

Amplitude of transmitted wave

$$A_2 = \frac{2\upsilon_2}{\upsilon_1 + \upsilon_2}A = \frac{2\sqrt{T/\mu_2}}{\sqrt{T/\mu_1} + \sqrt{T/\mu_2}}A = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_2} + \sqrt{\mu_1}}A$$

and $A_1 = \frac{\upsilon_2 - \upsilon_1}{\upsilon_1 + \upsilon_2}A = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_2} + \sqrt{\mu_1}}A$

$\therefore \quad \frac{A_2}{A_1} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} - \sqrt{\mu_2}} \quad$ ....(ii)

$\therefore$ From equation (i) and (ii)

$$\frac{5}{4} = \sqrt{\frac{\mu_2}{\mu_1}} \cdot \left(\frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} - \sqrt{\mu_2}}\right)^2$$

$\Rightarrow \quad \frac{5}{4} = x.\frac{4}{(1-x)^2} \qquad$ where $x = \sqrt{\frac{\mu_2}{\mu_1}}$

$\Rightarrow \quad x = 5 \qquad$ or $\qquad 0.2$

$\Rightarrow \quad \frac{\mu_2}{\mu_1} = 25 \qquad$ or $\qquad 0.04$

(i) $\quad \frac{\mu_2}{\mu_1} = \frac{\rho.\pi r_2^2}{\rho.\pi r_1^2} = \left(\frac{r_2}{r_1}\right)^2$

$\Rightarrow \quad \frac{r_2}{r_1} = \sqrt{\frac{\mu_2}{\mu_1}} = 0.2$

[Note : We have taken $\frac{\mu_2}{\mu_1} = 0.04$ as $r_1 > r_2$ is given in the question]

(ii) $\quad \frac{\upsilon_2}{\upsilon_1} = \sqrt{\frac{\mu_1}{\mu_2}} = \frac{1}{0.2} = 5$

(iii) The wave number,

$K = \frac{2\pi}{\lambda} = \frac{\omega}{\upsilon}$

$K_1 = K_r = K_i = K$

$K_2 = K_t = \frac{\omega}{\upsilon_2} = \frac{\omega}{\upsilon_1}.\frac{\upsilon_1}{\upsilon_2} = \frac{K}{5}$

There is no phase change on reflection.

$A_2 = \frac{2}{x+1}A = \frac{2}{1.2}A = \frac{5}{3}A$

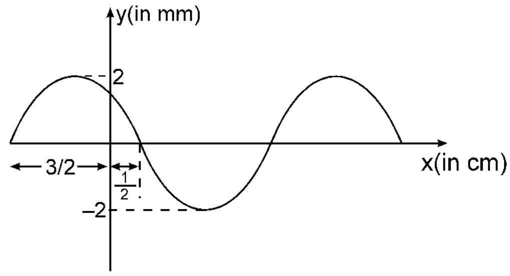
$A_1 = \frac{1-x}{1+x}A = \frac{0.8}{1.2}A = \frac{2}{3}A$

Hence, equation are

$y_r = A_2 \sin(K_2x - \omega t) = \frac{5}{3}A \sin\left(\frac{Kx}{5} - \omega t\right)$

and $y_t = A_1 \sin(K_1x + \omega t) = \frac{2}{3}A \sin(Kx + \omega t)$.

25. A standing wave pattern of maximum amplitude 2mm is obtained in a string whose shape at t = 0 is represented in the graph.



If the speed of the travelling wave in the string is 5 cm/s then find the component waves.

Sol. Equation of standing wave can be written as

$Y = 2 \sin(kx + \theta) \cos \omega t$

because the particles at t = 0 are at extreme position.

$k = \frac{2\pi}{\lambda} = \frac{2\pi}{4} = \frac{\pi}{2}$

From the graph it is clear that $x = \frac{1}{2}$, Amplitude = 0

$\therefore \quad 2 \sin\left[\frac{\pi}{2}.\frac{1}{2} + \theta\right] = 0 \Rightarrow \frac{\pi}{4} + \theta = 0$ or π.

we will select $\frac{\pi}{4} + \theta = \pi$ to suit the initial condition

$\therefore \quad \theta = \frac{3\pi}{4}$

$\therefore \quad y = 2 \sin\left(\frac{\pi}{2}x + \frac{3\pi}{4}\right) \cos\frac{5\pi}{2}t$

$= \sin\left(\frac{\pi}{2}x + \frac{3\pi}{4} + \frac{5\pi}{2}t\right) + \sin\left(\frac{\pi}{2}x + \frac{3\pi}{4} - \frac{5\pi}{2}t\right)$