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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-9

BINOMIAL THEOREM

ANSWER KEY OF CAPS-9

1.	(A)	**2.**	(A)	**3.**	(D)	**4.**	(C)	**5.**	(B)
6.	(D)	**7.**	(A)	**8.**	(B)	**9.**	(ABC)	**10.**	(AC)
11.	(ABC)	**12.**	(BCD)	**13.**	(ACD)	**14.**	(A)	**15.**	(B)
16.	(10)	**17.**	(5)	**18.**	(3)	**19.**	(960)		
20.	(– **22100**)	**21.**	(A)	**22.**	(C)	**23.**	(a) n=12, (b) $\left\{\frac{5}{7} \le x \le \frac{10}{9}\right\}$		

SCQ (Single Correct Type) :

1. If a, b, c, d be four consecutive coefficients in the binomial expansion of $(1 + x)^n$, then value of the expression $\left(\left(\frac{b}{b+c}\right)^2 - \frac{ac}{(a+b)(c+d)}\right)$ (where $x > 0$ and $n \in N$) is

(A) Positive (B) negative (C) zero (D) depend on n

Ans. **(A)**

Sol. $a = {}^nC_{r-1}, b = {}^nC_r, c = {}^nC_{r+1}, d = {}^nC_{r+2}$

$a + b = {}^{n+1}C_r, b + c = {}^{n+1}C_{r+1}, c + d = {}^{n+1}C_{r+2}$

$\frac{a+b}{a} = \frac{n+1}{r}$

$\Rightarrow \frac{a}{a+b} = \frac{r}{n+1}, \frac{b}{b+c} = \frac{r+1}{n+1}, \frac{c}{c+d} = \frac{r+2}{n+1}$

$\therefore \frac{a}{a+b}, \frac{b}{b+c}, \frac{c}{c+d}$ are in A.P.

AM > GM

$\frac{b}{b+c} > \sqrt{\frac{ac}{(a+b)(c+d)}}$

$\Rightarrow \left(\frac{b}{b+c}\right)^2 - \frac{ac}{(a+b)(c+d)} > 0$

2. The coefficient of x^{28} in the expansion of $(2 - x^3 + x^6)^{30}$ is

(A) 0 (B) 1 (C) 14 (D) 28

Ans. **(A)**

Sol. Since 3m + 6n = 28 has no solution in non-negative integers, coefficient of x^{28} is zero

3. If each coefficient in the expansion of the expression $x(1+x)^n$ $(n\in\mathbb{N})$ in powers of x is divided by the exponent of corresponding power, then the sum of the values thus obtained is equal to ______.

(A) $\frac{2^n}{n+1}$ (B) $\frac{2^n-1}{n+1}$ (C) $\frac{2^n+1}{n+1}$ (D) $\frac{2^{n+1}-1}{n+1}$

Ans. **(D)**

Sol. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+\ldots.+\frac{C_{n-1}}{n}+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$

4. The number of distinct terms in the expansion of $\left(x+y^2\right)^{13}+\left(x^2+y\right)^{14}$ is ______.

(A) 27 (B) 29 (C) 28 (D) 25

Ans. **(C)**

Sol. To get common terms in both the expansion

$x^{r_1}(y^2)^{13-r_1}=(x^2)^{r_2}(y)^{14-r_2}$

$r_1 = 2r_2$ and $26 - 2r_1 = 14 - r_2$

$r_1 = 8$, $r_2 = 4$

$\therefore$ Only one term is common

5. If $f(x)=\sum_{r=1}^{n}\left[r\left(n\cdot{}^{n-1}C_{r-1}-r.\,{}^{n}C_{r-1}\right)+(2r+1)\,{}^{n}C_r\right]$ then ______.

(A) $f(n)=n^2-1$ (B) $f(n)=(n+1)^2-1$ (C) $f(n)=(n+1)^2+1$ (D) $\sum_{n=1}^{10}f(n)=374$

Ans. **(B)**

Sol. $f(n)=\sum_{r=1}^{n}\left((r+1)^2\times{}^{n}C_r-r^2\times{}^{n}C_{r-1}\right)=(n+1)^n\times{}^{n}C_{n-1}=(n+1)^2-1$

6. If $\left(1+px+x^2\right)^n=1+a_1x+a_2x^2+\ldots..+a_{2n}x^{2n}$, where $n\in\mathbb{N}$, $p\in\mathbb{R}$. If p = −3 and n is even number, then the value of $a_1+3a_2+5a_3+7a_4+\ldots..+(4n-1)a_{2n}$ is ______.

(A) n (B) 2n −1 (C) 2n – 2 (D) 2n

Ans. **(D)**

Sol. $a_1+3a_2+5a_3+\ldots.+(4n-1)a_{2n}=\sum_{r=1}^{2n}(2r-1)a_r$

$=2\sum_{r=1}^{2n}ra_r-\sum_{r=1}^{2n}a_r$

$=2\left(n(p+2)^2\right)-\left((p+2)^n-1\right)=(2n-1)(p+2)^n+1$

$=2n$ (where p = –3, n is even number)

7. If $6^{83} + 8^{83}$ is divided by 49, then the remainder is

(A) 35 (B) 5 (C) 1 (D) 0

Ans. **(A)**

Sol. $(1 + 7)^{83} + (7 - 1)^{83} = (1 + 7)^{83} - (1 - 7)^{83}$

$= 2[{}^{83}C_1 \cdot 7 + {}^{83}C_3 \cdot 7^3 + \text{..........} + {}^{83}C_{83} \cdot 7^{83}] = (2 \cdot 7 \cdot 83) + 49I$ where I an integer

$\therefore \quad 14 \times 83 = 1162$

$$\frac{1162}{49} = 23\frac{35}{49}$$

$\therefore$ remainder is 35 Ans.

8. The sum of the series $(1^2 + 1).1! + (2^2 + 1).2! + (3^2 + 1).3! + \ldots\ldots + (n^2 + 1)\, n!$

(A) $(n + 1). (n + 2)!$ (B) $n.(n + 1)!$ (C) $(n + 1)\,(n + 1)!$ (D) none of these

Ans. **(B)**

Sol. $(1^2 + 1)\, 1! + (2^2 + 1)\, 2! + \text{........} + (n^2 + 1)\, n! \quad = \sum_{r=1}^{n} (r^2 + 1) r!$

Now, $(r^2 + 1) = A(r + 1)(r + 2) + B(r + 1) + C$

$\Rightarrow \quad A = 1, B = -3, C = 2$ (by comparing)

Now, $\sum_{r=1}^{n} (r^2 + 1) r! = \sum_{r=1}^{n} \left[(r+1)(r+2)r! - 3(r+1)r! + 2r!\right]$

$= \sum_{r=1}^{n} \left[(r+2)! - 3(r+1)! + 2r!\right]$

$= + 2\,[r! - (r + 1)!]$

$= \sum_{r=1}^{n} \left[(r+2)! - (r+1)!\right]$

$= \left[(n+2)! - 2!\right] + 2\left[1! - (n+1)!\right]$

$= (n + 2)! - 2(n + 1)!$

$= n\,(n + 1)!$

MCQ (One or more than one correct) :

9. If $J_m = \sum_{r=0}^{m-3} {}^mC_r\, {}^mC_{r+3}$ and m_1, m_2 are two values of m satisfying $5\, J_M = 3J_{M+1}$, then correct statement is/are (where [.] denotes greatest integer function

(A) $m_1 + m_2 = \frac{-8}{7}$ (B) $m_1 m_2 = \frac{46}{7}$

(C) $[(1 - m_1)(1 - m_2)] = 8$ (D) $[(1 + m_1)(1 + m_2)] = 8$

Ans. **(ABC)**

Sol. $5J_m = 3J_{m+1}$

$$\Rightarrow 5\sum_{r=0}^{m-3} {}^mC_r\, {}^mC_{r+3} = 3\sum_{r=0}^{m-2} {}^{m+1}C_r\, {}^{m+1}C_{r+3}$$

$$\Rightarrow 5 \times {}^{2m}C_{m-3} = 3 \times {}^{2m+2}C_{m-2}$$

$$\Rightarrow 7m^2 + 8m + 46 = 0$$

10. For natural number m, n if $(1 - y)^m (1 + y)^n = 1 + a_1 y + a^2y^2 + ...$ and $a_1 = a_2 = 10$ then

(A) m < n (B) m > n (C) m + n = 80 (D) m – n = 20

Ans. **(AC)**

Sol. $(1-y)^m(1+y)^n = 1 + (n-m)\,y + \left\{\frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn\right\} y^2 + ...$

11. Which of the following is/are true

(A) $5^6 - {}^5C_1.4^6 + {}^5C_2.3^6 - {}^5C_3.2^6 + {}^5C_4.1^6 = {}^6C_2.\underline{|5}$

(B) $6^5 - {}^6C_1.5^5 + {}^6C_2.4^5 - {}^6C_3.3^5 + {}^6C_4.2^5 - {}^6C_1.1^5 = 0$

(C) $6^6 - {}^6C_1.5^6 + {}^6C_2.4^6 - {}^6C_3.3^6 + {}^6C_4.2^6 - {}^6C_5.1^6 = 720$

(D) $6^5 - {}^6C_1.5^5 + {}^6C_2.4^5 - {}^6C_3.3^5 + {}^6C_4.2^5 - {}^6C_5.1^5 = {}^5C_2.\underline{|6}$

Ans. **(ABC)**

Sol. (a) Number of onto functions from a set containing 6 elements to a set containing 5 elements $= {}^6C_2 \times \underline{|5}$

(b) Number of onto functions from a set containing 5 elements to a set containing 6 elements = 0

(c) Number of onto functions from a set containing 6 elements to a set containing 6 elemets = $\underline{|6}$ = 720

12. In the expansion of $\left(x^3 + 3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$

(A) there appears a term with the power x^2

(B) there does not appear a term with the power x^2

(C) there appears a term with the power x^{-3}

(D) the ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

Ans. **(BCD)**

Sol. $\left(x^3 + 3\cdot 2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11} = \left(x^3 + 3\cdot 2^{\log_2 (x^3)^{-1}}\right)^{11} = \left(x^3 + \frac{3}{x^3}\right)^{11}$

general term = ${}^{11}C_r\,(x^3)^{11-r}\left(\frac{3}{x^3}\right)^r = {}^{11}C_r \cdot 3^r \cdot x^{33-6r}$

Now, $33 - 6r = 2 \Rightarrow r = \frac{31}{6} \Rightarrow$ No term appear with power x^2

$33 = -3 \Rightarrow r = 6 \Rightarrow$ term appear with power of x^{-3}

for x^3, $r = 5$ and for x^{-3}, $r = 6$

$\Rightarrow$ ratio of coefficient of x^3 to that of $x^{-3} = \frac{{}^{11}C_5 \cdot 3^5}{{}^{11}C_6 \cdot 3^6} = \frac{1}{3}$.

13. If $(9+\sqrt{80})^n = I + f$ where I, n are integers and $0 < f < 1$, then

(A) I is an odd integer

(B) I is an even integer

(C) $(I + f)(1 - f) = 1$

(D) $1 - f = (9-\sqrt{80})^n$

Ans. **(ACD)**

Sol. $\left(9+4\sqrt{5}\right)^n = 3 + f$

let $\left(9-4\sqrt{5}\right)^n = f\,'$

Comprehension Type Question:

Comprehension # 1

If $(1 + x + x^2)^{2n} = 1 + a_1x + a_2x^2 + \ldots\ldots\ldots a_{4n}x^{4n}$, then

14. The value of $\sum_{r=0}^{n-1} a_{2r}$, is

(A) $\frac{9^n+1-2a_{2n}}{4}$ (B) $\frac{9^n+1+2a_{2n}}{4}$ (C) $\frac{9^n+1-2a_n}{4}$ (D) $\frac{9^n-1-2a_{2n}}{4}$

Ans. **(A)**

15. What value a_2 takes ?

(A) ${}^{2n}C_2$ (B) ${}^{2n+1}C_2$ (C) ${}^{2n-1}C_2$ (D) ${}^{n}C_2$

Ans. **(B)**

Sol. $(1 + x + x^2)^{2n} = 1 + a_1x + a_2x^2 + \ldots\ldots + a_{4n}x^{4n}$.

$\sum_{r=0}^{n-1} a_{2r} = a_0 + a_2 + a_4 + \ldots.. + a_{2n-2}$

put $x = 1 \Rightarrow 3^{2n} = 1 + a_1 + a_2 + \ldots\ldots\ldots + a_{4n}$ ....(i)

put $x = -1 \Rightarrow 1 = 1 - a_1 + a_2 + \ldots\ldots\ldots\ldots + a_{4n}$ ....(ii)

(i) + (ii)

$\Rightarrow 1 + 3^{2n} = 2(1 + a_2 + a_4 + \ldots\ldots\ldots + a_{4n})$

$\Rightarrow (1 + a_2 + a_4 + \ldots\ldots\ldots\ldots + a_{2n-2}) + a_{2n} + (a_{2n+2} + \ldots\ldots\ldots + a_{4n-2} + a_{4n}) = \frac{1+9^n}{2}$

We know that $a_{4n} = 1$, $a_{4n-2} = a_2$ ...... $a_{2n-2} = a_{2n+2}$ …… etc

$\Rightarrow \quad 2(1 + a_2 + a_4 + ... + a_{2n-2}) + a_{2n} = \dfrac{1+9^n}{2}$

$\Rightarrow \quad 1 + a_2 + a_4 + ... + a_{2n-2} = \dfrac{9^n+1-2a_{2n}}{4}$

$(1 + x + x^2)^{2n} = 1 + a_1x + a_2x^2 + ...+ a_{4n}\,x^{4n}$ by differentiation

$2n(1 + x + x^2)^{2n-1}(1 + 2x) = a_1 + 2a_2x + 3a_3x^2 + ... + 4na_{4n}\,x^{4n-1}$.

Again by differentiation

$2n\left[(2n-1)(1+x+x^2)^{2n-2}(1+2x)^2+(1+x+x^2)^{2n-1}\cdot 2\right] = 2a_2 + 6a_3x + ... + 4na_{4n}\,x^{4n-1}$.

put $\quad x = 0 \Rightarrow 2a_2 = 2n[(2n-1) + 2]$

$a_2 = n(2n + 1) = \dfrac{2n(2n+1)}{2} = {}^{2n+1}C_2$.

Numerical based Questions :

16. If the coefficient of x^5 in $(1 + x)^{21} + (1 + x)^{22} + \ldots + (1+x)^{30}$ is $\lambda\,{}^nC_r + \mu\,{}^mC_r$, λ, μ, n, r, m being integers and ${}^nC_r$, ${}^mC_r$ are binomial coefficients, then the value of $\lambda(n + r) + \mu(m + r)$ is

Ans. (10)

Sol. ${}^{21}C_5 + {}^{22}C_5 + ... + {}^{30}C_5 + {}^{21}C_6 - {}^{21}C_6$

$= {}^{31}C_6 - {}^{21}C_6$

$\lambda(n + r) + \mu(m + r) = (31 + 6) + (-1)(21 + 6)$

$= 31 - 21 = 10$

17. The value of $\displaystyle\sum_{r=0}^{9}\frac{{}^{10}C_r}{{}^{10}C_r+{}^{10}C_{r+1}}$ is equal to

Ans. (5)

Sol. $S = \displaystyle\sum_{r=0}^{9}\frac{{}^{10}C_r}{{}^{10}C_r+{}^{10}C_{r+1}}$

$= \displaystyle\sum_{r=0}^{9}\frac{{}^{10}C_r}{{}^{11}C_{r+1}}$

$= \displaystyle\sum_{r=0}^{9}\frac{r+1}{11} = 5$

18. When the terms in the binomial expansion of $\left(\sqrt{x}+\frac{1}{2x^{1/4}}\right)^n$ are arranged in decreasing powers of x, the coefficients of the first three terms are in arithmetic progression. The number of terms in the expansion with integer powers of x is _____.

Ans. **(3)**

Sol. $1, \frac{{}^nC_1}{2}, \frac{{}^nC_2}{2^2}$ are in AP $\Rightarrow$ n = 8

$T_{r+1} = {}^8C_r \frac{1}{2^r} \times x \frac{16-3r}{4}$

So $4 | 16 - 3r (0 \le r \le 8)$ only if $r \in \{0, 4, 8\}$

19. Find the coefficient of x^7 in $(1 - x + 2x^3)^{10}$.

Ans. **960**

Sol. Coefficient of x^7 in $(1 - x + 2x^3)^{10}$

general term : ${}^{10}C_r (1-x)^{10-r} \cdot (2x^3)^r$; ${}^{10}C_r (1-x)^{10-r} \cdot 2^r \cdot x^{3r}$.

When r = 0, coefficient of x^7 in ${}^{10}C_0 (1-x)^{10} \Rightarrow - {}^{10}C_7$.

When r = 1, coefficient of x^4 in ${}^{10}C_1 (1-x)^9 \cdot 2 \Rightarrow 20 ({}^9C_4)$

When r = 2, coefficient of x^1 in ${}^{10}C_2 (1-x)^8 \cdot 2^2 \Rightarrow 180 (-{}^8C_1)$

$\Rightarrow$ coefficent of $x^7 = (-{}^{10}C_7 + 20 \cdot {}^9C_4 - 180 \cdot {}^8C_1)$

$= (-120 + 2520 - 1440) = 960$. **Ans.**

20. Find the coeffcient of x^{49} in the polynomial

$\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\cdot\frac{C_2}{C_1}\right)\left(x-3^2\cdot\frac{C_3}{C_2}\right)\ldots\ldots\ldots\ldots\left(x-50^2\cdot\frac{C_{50}}{C_{49}}\right)$ where $C_r = {}^{50}C_r$.

Ans. **– 22100**

Sol. $\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\cdot\frac{C_2}{C_1}\right)\ldots\ldots\left(x-50^2\cdot\frac{C_{50}}{C_{49}}\right)$

Coefficient of x^{49} in this expansion is

$\frac{-C_1}{C_0} - 2^2\cdot\frac{C_2}{C_1} \ldots\ldots - (+50)^2 \cdot \frac{C_{50}}{C_{49}}$

$-T_r = (r)^2 \cdot \frac{{}^{50}C_r}{{}^{50}C_{r-1}} = (r^2) \cdot \frac{51-r}{r} = r(51-r)$

$-T_r = 51r - r^2$

$-S = 51\sum_{r=1}^{50} r - \sum_{r=1}^{50} r^2$

$$-S = 51 \cdot \frac{50\times 51}{2} - \frac{50(51)(101)}{6} = \frac{50\times 51}{2}\left[51 - \frac{101}{3}\right]$$

$$= \frac{50\cdot 51}{2}\left[\frac{153-101}{3}\right] = \frac{50\times 51}{6}\times 52$$

$S = -1300 \times 17 = -22100$

Matrix Match Type :

21. Match the following:

Column – I	**Column – II**
(A) Number of distinct terms in the expansion of $(x+y-z)^{16}$ is	(p) 2^{12}
(B) Number of terms in the expansion of $\left(x+\sqrt{x^2-1}\right)^6 + \left(x-\sqrt{x^2-1}\right)^6$ is	(q) 97
(C) The number of irrational terms in $\left(\sqrt[8]{5}+\sqrt[6]{2}\right)^{100}$ is	(r) 4
(D) The sum of numerical coefficients in the expansion of $\left(1+\frac{x}{3}+\frac{2y}{3}\right)^{12}$ is	(s) 153

Options:

(A) A → (s); B → (r); C → (q); D → (p) (B) A → (s); B → (r); C → (p); D → (q)

(C) A → (r); B → (s); C → (q); D → (p) (D) A → (s); B → (q); C → (r); D → (p)

Ans. **(A)**

Sol. (A) ${}^{(16+3-1)}c_{3-1} = {}^{18}C_2 = \frac{18\times 17}{2} = 153$ (B) $\frac{n}{2}+1 = \frac{6}{2}+1 = 4$

(C) $\frac{100}{24} = 4 + F(T_1, T_{25}, T_{49}, T_{73}$ are rationals) (D) Put $x = y = z \Rightarrow \left(1+\frac{1}{3}+\frac{2}{3}\right)^{12} = 2^{12}$

22. Match the following:

Column – I	**Column – II**
(A) The minimum value of a + b + c + d if $\log_3(a+b)+\log_3(c+d)\ge 4$ is	(p) 18
(B) The number of distinct terms in the expansion of $\left(x+y+\frac{1}{x}+\frac{1}{y}\right)^{14}$ is	(q) 225
(C) The remainder when $(23)^{86}$ is divided by 100 is	(r) 89
(D) If the third term in the expansion of $\left(\frac{1}{x}+x^{\log_{10}x}\right)^5$ is 1000 and x > 1, then the value of x is	(s) 100

(A) A → (p); B → (q); C → (s); D → (r) (B) A → (p); B → (r); C → (q); D → (s)

(C) A → (p); B → (q); C → (r); D → (s) (D) A → (q); B → (p); C → (r); D → (s)

Ans. (C)

Sol. (A) $(a+b)(c+d) \geq 81$

Applying AM GM $\geq$ GM $\dfrac{(a+b)+(c+d)}{2} \geq \left((a+b)(c+d)\right)^{\frac{1}{2}}$

Minimum value of $a+b+c+d$ is 18

(B) $\dfrac{(x+y)^{14}(1+xy)^{14}}{(xy)^{14}}$

Both factors in the numerator have 15 independent terms total number of terms 15 × 15 = 225

(C) $(23)^{86} = (529)^{43} = (530-1)^{43}$

$= \left[(530)^{43} - {}^{43}C_1(530)^{42} + {}^{43}C_2(530)^{41} - \ldots - 1\,{}^{43}C_{41}(530)^2\right] + (43 \times 530) - 1$

Hence the last two digit of $(23)^{86}$ are last two digits of (43×530) – 1 that is 89

(D) $T_3 = {}^5C_2 \dfrac{1}{x^3}\left(x^{\log_{10} x}\right)^2 = 1000$

Taking log on the base 10 both side $-3\log_{10} x + 2(\log_{10} x)^2 - 2 = 0$

Put $\log_{10} x = t$

$2t^2 - 3t - 2 = 0$

$t = -\dfrac{1}{2}, 2$

$x > 1, \log_{10} x = 2 \qquad \Rightarrow x = 100$

Subjective based Questions:

23. (a) Find the index of n of the binomial $\left(\dfrac{x}{5} + \dfrac{2}{5}\right)^n$ if the 9 the term of the expansion has numerically greatest coefficient ($n \in N$).

(b) For what positive value of x is the fourth term in the expansion of $(5 + 3x)^9$ is the greatest.

Ans. (a) n = 12, (b) $\left\{\dfrac{5}{7} \leq x \leq \dfrac{10}{9}\right\}$

Sol. (a) $\left(\dfrac{x}{5} + \dfrac{2}{5}\right)^n$

$T_9 > T_{10}$ and $T_9 > T_8$

${}^nC_8\left(\dfrac{2}{5}\right)^8 > {}^nC_9\left(\dfrac{2}{5}\right)^9\left(\dfrac{1}{5}\right)^{n-9}$ and ${}^nC_8\left(\dfrac{1}{5}\right)^{n-8}\left(\dfrac{2}{5}\right)^8 < {}^nC_7\left(\dfrac{2}{5}\right)^7\left(\dfrac{1}{5}\right)$

$\left(\dfrac{1}{5}\right){}^nC_8 < {}^nC_9\dfrac{2}{5}$

$\frac{1}{5}.\frac{9}{n-8} > \frac{2}{5}$ $\qquad\qquad$ $\frac{n-7}{8}.\frac{2}{5} > \frac{1}{5}$

$9 > 2n - 16$ $\qquad\qquad$ $2n - 14 > 8$

$25 > 2n$ $\qquad\qquad$ $2n > 22$

$n < 12.5$ $\qquad\qquad$ $n > 11$

$n = 12$

(b) $(5 + 3x)^{10}$

$T_4 = {}^{10}C_3\, 5^7\, (3\,|\,x\,|)^3$

$T_4 > T_3$

${}^{10}C_3\, 5^7\, (3\,|\,x\,|\,)^3 > {}^{10}C_2\, 5^8\, (3\,|\,x\,|)^2$

$\frac{8}{3} \times 3\,|\,x\,| > 5$

$|\,x\,| > \frac{5}{8}$ $\quad$ x is a +ve quantity

$x > \frac{5}{8}$

$T_4 > T_5$

${}^{10}C_3\, 5^7\, (3\,|\,x\,|\,)^3 > {}^{10}C_4\, 5^6\, (3\,|\,x\,|)^4$

$\frac{4}{7} \times 5 > 3\,|\,x\,|$

$|\,x\,| > \frac{20}{21}$

$\therefore\ \frac{5}{8} < x < \frac{20}{21}$

24. Show that $(1\cdot2)\, {}^nC_2 + (2\cdot3)^n\, C_3 + \ldots\ldots + (n-1)\cdot n\, {}^nC_n = n\,(n-1)2^{n-2}$

Sol. $(1\cdot 2)\, {}^nC_2 + (2\cdot 3)\, {}^nC_3 + \ldots\ldots. + (n-1)\, n\, {}^nC_n$

$= \sum_{r=2}^{n} (r-1)\, r\, {}^nC_r = \sum_{r=2}^{n} \frac{(r-1)\, r\, n!}{r!\,(n-r)!} = \sum_{r=2}^{n} \frac{n!}{(r-2)!\,(n-r)!}$

$= \sum_{r=2}^{n} n(n-1)\left(\frac{(n-2)\,!}{(r-2)!\ (n-r)\,!}\right) = n(n-1)\left[\sum_{r=2}^{n} {}^{n-2}C_{r-2}\right]$

$= n(n-1)\left[{}^{n-2}C_0 + {}^{n-2}C_1 + \ldots. + {}^{n-2}C_{n-2}\right]$

$= n(n-1)\cdot 2^{n-2}.$

25. If $a_1, a_2, a_3, \ldots\ldots, a_n$ are in A.P. and S_n is the sum of first n term's, show that

$$\sum_{k=0}^{n} {}^{n}C_k S_k = 2^{n-2}(na_1 + S_n)$$

Sol. $\sum_{k=0}^{n} {}^{n}C_k \cdot S_k$

$S_k = \frac{k}{2}\left[2a_1 + (k-1)\,d\right]$

$$\Rightarrow \quad \sum_{k=0}^{n} {}^{n}C_k \cdot S_k = \sum_{k=0}^{n}\left[\frac{1}{2}kC_k\left[(2a_1 - d) + kd\right]\right]$$

$$= \sum_{k=0}^{n}\left[\frac{1}{2}(2a_1 - d)\,kC_k + \frac{1}{2}\ d \cdot k^2 C_k\right]$$

$$= \frac{1}{2}(2a_1 - d)\left(n \cdot 2^{n-1}\right) + \frac{1}{2}d\left(n \cdot 2^{n-1} + \left(n(n-1) \cdot 2^{n-2}\right)\right)$$

26. Show that $S_1 = p^2 - q^2 = \left(x^2 - y^2\right)^n$

Sol. $S_1 = \sum_{r=1}^{2n} {}^{4n}C_{2r-1}\,(-1)^{r-1}$

$$= {}^{4n}C_1 - {}^{4n}C_3 + {}^{4n}C_5 - \ldots\ldots\ldots + (-1)^{2n-1}\,{}^{4n}C_{4n-1}$$

We know that

$(1 + x)^{4n} = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_{4n} \cdot x^{4n}$

by putting $x = i$ and $x = -i$

$(1 + i)^{4n} = C_0 + iC_1 + i^2C_2 + \ldots.. + i^{4n}\,C_{4n}$ ....(i)

$(1 - i)^{4n} = C_0 - iC_1 + i^2C_2 - \ldots\ldots + i^{4n}\,C_{4n}$ ....(ii)

(i) – (ii)

$\Rightarrow \quad 2\,(i\,C_1 + i^3C_3 + i^5C_5 + \ldots\ldots + i^{4n-1}C_{4n-1}) = (1 + i)^{4n} - (1 - i)^{4n}$

$2i\,(C_1 - C_3 + C_3 - \ldots\ldots\ldots - C_{4n-1}) = \left((1+i)^2\right)^{2n} - \left((1-i)^2\right)^{2n}$

$S_1 = \dfrac{(2i)^{2n} - (-2i)^{2n}}{2i} = 0.$

27. In the expansion of $(x + y)^n$, if the sum of odd term's be p and sum of even term's be q, then prove that

(i) $p^2 - q^2 = \left(x^2 - y^2\right)^n$

(ii) $4pq = (x + y)^{2n} - (x - y)^{2n}$

Sol. $(x + y)^n = C_0x^n + C_1 x^{n-1}y + C_2 x^{n-2}y^2 + ... + C_n x^0 \cdot y^n$

$(x - y)^n = C_0 x^n - C_1x^{n-1}y + C_2 x^{n-2}y^2 + ... + (-1)^n C_nx^0 \cdot y^n$

$(x + y)^n + (x - y)^n = 2p$ ....(i)

$(x + y)^n - (x - y)^n = 2q$ ......(ii)

Now,

$4p^2 - 4q^2 = [(x + y)^n + (x - y)^n]^2 - [(x + y)^n - (x - y)^n]^2$

$4(p^2 - q^2) = 4 \cdot (x^2 - y^2)^n$

$\Rightarrow$ $p^2 - q^2 = (x^2 - y^2)^n$

Again $(2p)(2q) = [(x + y)^n + (x - y)^n] [(x + y)^n - (x - y)^n]$

$4pq = (x + y)^{2n} - (x - y)^{2n}$.