MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-4

Inverse Trigonometric Function & Solution of Triangle

SCQ (Single Correct Type) :

1. The value of

$\sin^{-1}(\cos 2) - \cos^{-1}(\sin 2) + \tan^{-1}(\cot 4) - \cot^{-1}(\tan 4) + \sec^{-1}(\text{cosec}\,6) - \text{cosec}^{-1}(\sec 6)$ is

(A) 0 (B) 3π (C) $8 - 3\pi$ (D) $5\pi - 16$

Ans. (D)

Sol. Given expression = $\left\{\frac{\pi}{2} - \cos^{-1}(\cos 2)\right\} - \left\{\frac{\pi}{2} - \sin^{-1}(\sin 2)\right\}$

$+ \left\{\frac{\pi}{2} - \cot^{-1}(\cot 4)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(\tan 4)\right\} + \left\{\frac{\pi}{2} - \text{cosec}^{-1}(\text{cosec}\,6)\right\} - \left\{\frac{\pi}{2} - \sec^{-1}(\sec 6)\right\}$

$= \sin^{-1}(\sin 2) - \cos^{-1}(\cos 2) + \tan^{-1}(\tan 4) - \cot^{-1}(\cot 4) + \sec^{-1}(\sec 6) - \text{cosec}^{-1}(\text{cosec}\,6)$

$= (\pi - 2) - 2 + (4 - \pi) - (4 - \pi) + (2\pi - 6) - (6 - 2\pi) = \pi - 4 + 4\pi - 12 = 5\pi - 16.$]

2. Let $\alpha = \cot^{-1}\left(\frac{\pi}{3}\right)$, $\beta = \sin^{-1}\left(\frac{\pi}{4}\right)$ and $\gamma = \sec^{-1}\left(\frac{2\pi}{3}\right)$, then the correct order sequence is

(A) $\alpha < \gamma < \beta$ (B) $\beta < \alpha < \gamma$ (C) $\gamma < \beta < \alpha$ (D) $\alpha < \beta < \gamma$

Ans. (D)

Sol. As $\frac{\pi}{3} > 1 \Rightarrow \cot^{-1}\frac{\pi}{3} < \cot^{-1}1 = \frac{\pi}{4} \Rightarrow \cot^{-1}\frac{\pi}{3} < \frac{\pi}{4} = 0.7856$ ....(i)

As $\frac{\pi}{4} > \frac{1}{\sqrt{2}} \Rightarrow \sin^{-1}\frac{\pi}{4} > \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4} \Rightarrow \sin^{-1}\frac{\pi}{4} > \frac{\pi}{4} = 0.7856$ ....(ii)

As $\frac{2\pi}{3} > 2 \Rightarrow \sec^{-1}\frac{2\pi}{3} > \sec^{-1}2 = \frac{\pi}{3} \Rightarrow \sec^{-1}\frac{2\pi}{3} > \frac{\pi}{3}$ 1.0476 ....(iii)

Clearly $\alpha < \beta < \gamma$.]

3. Let $f(x) = \frac{2}{\pi}\left(\sin^{-1}[x] + \tan^{-1}[x] + \cot^{-1}[x]\right)$ where $[x]$ denotes greatest integer less than or equal to x. If A and B denote the domain and range of $f(x)$ respectively, then the number of integers in $(A \cup B)$, is

(A) 1 (B) 2 (C) 3 (D) 4

Ans. (D)

Sol. For domain of $f(x)$, we must have $-1 \le [x] \le 1 \Rightarrow -1 \le x < 2$, so set $A = [-1, 2)$

$f(x) = \frac{2}{\pi}\left(\sin^{-1}[x] + \frac{\pi}{2}\right)$ $\left(\text{As } \tan^{-1}[x] + \cot^{-1}[x] = \frac{\pi}{2} \ \forall x \in A\right)$

So, set $B = \{0, 1, 2\}$ = Range of $f(x)$

Now $A \cup B = [-1, 2) \cup \{0, 1, 2\} = [-1, 2]$

Hence number of integers in $(A \cup B)$ = 4 **Ans.**

4. The radii of the escribed circles of ΔABC are r_a, r_b and r_c respectively. If $r_a + r_b = 3R$ and $r_b + r_c = 2R$, then the smallest angle of the triangle is

(A) $\tan^{-1}\left(\sqrt{2}-1\right)$ (B) $\frac{1}{2}\tan^{-1}\left(\sqrt{3}\right)$ (C) $\frac{1}{2}\tan^{-1}\left(\sqrt{2}+1\right)$ (D) $\tan^{-1}\left(2-\sqrt{3}\right)$

Ans. **(B)**

Sol. We have $r_a + r_b = 3R \Rightarrow \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = 3R = \frac{3abc}{4\Delta}\left(R = \frac{abc}{4\Delta}\right)$

$\Rightarrow \frac{\Delta(s-b+s-a)}{(s-a)(s-b)} = \frac{3abc}{4\Delta} \Rightarrow \frac{c\Delta}{(s-a)(s-b)} = \frac{3abc}{4\Delta} \Rightarrow \frac{\Delta^2}{(s-a)(s-b)} = \frac{3ab}{4}$

$\Rightarrow 4s(s-c) = 3ab \Rightarrow (a+b+c)(a+b-c) = 3ab$

$\Rightarrow (a+b)^2 - c^2 = 3ab \Rightarrow a^2 + b^2 - c^2 = ab$

$\Rightarrow c^2 = a^2 + b^2 - ab$

Note : Angles A, C, B are in A.P. can be converted into more than one

$\Rightarrow a^2 + b^2 - 2ab\cos C = a^2 + b^2 - ab$ (As $c^2 = a^2 + b^2 - 2ab\cos C$)

$\Rightarrow \cos C = \frac{1}{2} \Rightarrow \angle C = 60°$ ....(1)

|||ly from $r_b + r_c = 2R$

$\Rightarrow \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = 2R \Rightarrow \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{2abc}{4\Delta} \Rightarrow \frac{2\Delta^2}{(s-b)(s-c)} = bc$

$\Rightarrow 2s(s-a) = bc \Rightarrow (b+c+a)(b+c-a) = 2bc \Rightarrow (b+c)^2 - a^2 = 2bc$

$\Rightarrow b^2 + c^2 = a^2 \Rightarrow \angle A = 90° \Rightarrow \angle B = 30°$]

5. If $\cos^{-1}\frac{x}{a} - \sin^{-1}\frac{y}{b} = \theta$ (a, b $\neq$ 0), then the maximum value of $b^2x^2 + a^2y^2 + 2ab\, xy \sin\theta$ equals

(A) ab (B) $(a+b)^2$ (C) $2(a+b)^2$ (D) a^2b^2

Ans. **(D)**

Sol. We have $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \frac{\pi}{2} + \theta \Rightarrow \frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = -\sin\theta$

$\Rightarrow \frac{xy}{ab} + \sin\theta = \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}$

On squaring both the sides, we get

$\Rightarrow \frac{x^2y^2}{a^2b^2} + \sin^2\theta + \frac{2xy}{ab}\sin\theta = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}$

$\Rightarrow b^2x^2 + a^2y^2 + 2ab\, xy\sin\theta = a^2b^2\cos^2\theta \le a^2b^2$]

6. In ΔABC, the bisector of the angle A meets the side BC at D and the circumscribed circle at E, then DE equals

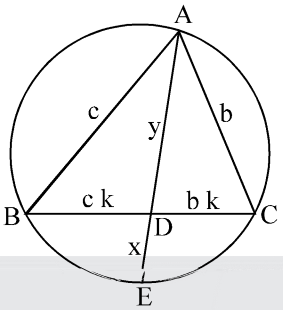
(A) $\frac{a^2\sec\frac{A}{2}}{2(b+c)}$ (B) $\frac{a^2\sin\frac{A}{2}}{2(b+c)}$ (C) $\frac{a^2\cos\frac{A}{2}}{2(b+c)}$ (D) $\frac{a^2\operatorname{cosec}\frac{A}{2}}{2(b+c)}$

Ans. **(A)**

Sol. We have $ck + bk = a \Rightarrow k = \frac{a}{b+c}$

Also $xy = bck^2$

$$x\left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)=\frac{bc\ \ a^2}{(b+c)^2}$$



Hence $x=\dfrac{a^2\ \sec\frac{A}{2}}{2\ (b+c)}$ = DE

7. If $\left[\cos^{-1}x\right]+\left[\cot^{-1}x\right]=0$, where [.] denotes the greatest integer function, then the complete set of values of *x* is

(A) (cos 1, 1] (B) (cos 1, cot 1) (C) (cot 1, 1] (D) [0, cot 1)

Ans. **(C)**

Sol. $\left[\cos^{-1}x\right]\geq 0\,\forall x\in[-1,1]$ and $\left[\cot^{-1}x\right]\geq 0\,\forall x\in R$

Hence $\left[\cos^{-1}x\right]+\left[\cot^{-1}x\right]=0$

$\Rightarrow\quad \left[\cos^{-1}x\right]=\left[\cot^{-1}x\right]=0$

If $\quad \left[\cos^{-1}x\right]=0$

$\Rightarrow\quad x\in[\cos 1,1]$

If $\quad \left[\cot^{-1}x\right]=0$

$\Rightarrow\quad x\in(\cot 1,\infty)\quad\Rightarrow\quad x\in(\cot 1,1)$

8. f : R→(0,π / 2] and f(x) = $\cos^{-1}(x^2 - ax + a + 1)$ is sujective than a∈

(A) $\left\{\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right\}$ (B) $\left\{\frac{1-\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right\}$

(C) $\left(-\infty,\frac{1-\sqrt{5}}{2}\right)\cup\left(\frac{1+\sqrt{5}}{2},\infty\right)$ (D) None of these

Ans. **(A)**

Sol. D = 0

9. In a ΔABC, if $b^2 = c^2 = 1999a^2$, then $\dfrac{\cot B+\cot C}{\cot A}$ is equal to____.

(A) $\frac{1}{999}$ (B) $\frac{1}{1999}$ (C) 999 (D) 1999

Ans. **(D)**

Sol. $b^2 + c^2 = 1999a^2 \qquad\Rightarrow b^2 + c^2 - a^2 = 1998a^2$

$$\frac{\cot B+\cot C}{\cot A}=\frac{\sin(B+C)}{\sin C\sin B\times\dfrac{\cos A}{\sin A}}=\frac{\sin^2 A}{\sin B\sin C\cos A}$$

$$= \left(\frac{a^2}{bc}\right)\left(\frac{2bc}{b^2+c^2-a^2}\right)$$

$$= \frac{2a^2}{1998a^2} = \frac{1}{999}$$

10. If 2a cos B = c in a ΔABC with sides a, b, c then $\tan\frac{A}{2}\left(\tan\frac{A}{2}+2\tan\frac{C}{2}\right)$ is equal to_____.

(A) 1 (B) 21 (C) $\frac{1}{2}$ (D) $\frac{1}{3}$

Ans. **(A)**

Sol. 2sinA cosB = sinC

sin (A + B) + sin (A – B) = sinC

Consider sin (A – B) = 0, that is A = B and C = π – 2A

$$\therefore \frac{C}{2} = \cot A = \frac{1-\tan^2\frac{A}{2}}{2\tan\frac{A}{2}}$$

11. In ΔABC, if BC = 1, $\sin\frac{A}{2} = x_1$, $\sin\frac{B}{2} = x_2$, $\cos\frac{A}{2} = x_3$, and $\cos\frac{B}{2} = x_4$ with

$\left(\frac{x_1}{x_2}\right)^{2010} - \left(\frac{x_3}{x_4}\right)^{2009} = 0$, then length of AC is equal to_____.

(A) > 1 (B) < 1 (C) $\frac{1}{2}$ (D) 1

Ans. **(D)**

Sol. In given ΔABC, both $\frac{A}{2}$ and $\frac{B}{2} \in \left(0, \frac{\pi}{2}\right)$

sin x, is increasing in $\left(0, \frac{\pi}{2}\right)$ and cos x is decreasing in $\left(0, \frac{\pi}{2}\right)$

$\therefore$ If $\frac{A}{2} > \frac{B}{2}$ $\Rightarrow \sin\frac{A}{2} > \sin\frac{B}{2}$ $\Rightarrow x_1 > x_2$

$\cos\frac{A}{2} < \cos\frac{B}{2} \Rightarrow x_3 > x_4$

$\therefore \left(\frac{x_1}{x_2}\right)^{2010} \neq \left(\frac{x_3}{x_4}\right)^{2009}$

Similarly, for $\frac{A}{2} < \frac{B}{2}$

$\therefore \frac{A}{2} = \frac{B}{2} \Rightarrow x_4 = x_2$ and $\frac{1}{x^3} = \frac{1}{x^4}$

$\therefore$ BC = AC = 1

MCQ (One or more than one correct) :

12. Let f : R $\to$ R defined by f (x) = $\cos^{-1}$ (–{ – x })

where {x} is fractional part function. Then which of the following is/are correct?

(A) f is many one but not even function.

(B) Range of f contains two prime numbers.

(C) f is aperiodic.

(D) Graph of f does not lie below x–axis.

Ans. **(ABD)**

Sol. We have $f(x) = \cos^{-1}(-\{-x\})$

$D_f = R$

As $0 \le \{-x\} < 1 \ \forall x \in R$

$\Rightarrow \quad -1 < -\{-x\} \le 0$

So $\quad R_f = \left[\frac{\pi}{2}, \pi\right)$

Clearly, f is neither even nor odd.

But $\quad f(x+1) = f(x) \quad \Rightarrow \quad$ f is periodic with period 1.

13. Which of the following statement(s) is/are **TRUE**?

(A) Domain of $y = \cos^{-1}(e^x)$ is same as range of $y = -\sqrt{-x}$.

(B) Number of elements common in the range of function $y = \tan^{-1}(\text{sgn } x)$ and $y = \cot^{-1}(\text{sgn } x)$ is only 1 (where sgn x denotes signum function of x.)

(C) The function $y = \text{sgn}(\cot^{-1}x)$ and $y = 1$ are identical functions.

(D) Number of integers in the solution set of $1 < \log_2(\tan^{-1}x) < 2$ is 4.

Ans. **(ABC)**

Sol. (A) For domain of $y = \cos^{-1}(e^x)$, $\quad 0 < e^x \le 1 \Rightarrow x \le 0$ or $x \in (-\infty, 0]$

For range of $y = -\sqrt{-x}$, $\quad -y = \sqrt{-x}$

$\sqrt{-x} \ge 0 \Rightarrow -y \ge 0 \Rightarrow y \in (-\infty, 0]$

$\therefore \quad$ Domain of $y = \cos^{-1}(e^x)$ is identical with range of $y = -\sqrt{-x}$.

(B) $y = \tan^{-1}(\text{sgn } x) = \begin{cases} \tan^{-1}(-1), & \forall x < 0 \\ \tan^{-1} 0, & \text{for } x = 0 \\ \tan^{-1}(1), & \forall x > 0 \end{cases}$

$\Rightarrow \quad$ Range of $y = \tan^{-1}(\text{sgn } x)$ is $\left\{\frac{-\pi}{4}, 0, \frac{\pi}{4}\right\}$

Also, $y = \cot^{-1}(\text{sgn } x) = \frac{\pi}{2} - \tan^{-1}(\text{sgn } x)$

$\Rightarrow \quad$ Range of $y = \cot^{-1}(\text{sgn } x)$ is $\left\{\frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\right\}$

$\therefore \quad$ Number of elements common in the range of $y = \tan^{-1}(\text{sgn } x)$ and $y = \cot^{-1}\text{sgn } x$ is only one i.e $\frac{\pi}{4}$.

(C) $y = \text{sgn}(\cot^{-1}x) = 1 \quad \forall x \in R$

$\therefore \quad y = \text{sgn}(\cot^{-1}x)$ and $y = 1$ are identical functions.

(D) $1 < \log_2(\tan^{-1}x) < 2 \Rightarrow 2^1 < \tan^{-1}x < 2^2$

$\Rightarrow \quad$ Not possible. $\left(\text{As } \frac{-\pi}{2} < \tan^{-1} x < \frac{\pi}{2} < 2\right)$

14. Let a function $f : A \to B$ be defined as $f(x) = \sin^{-1}(\tan x) - \text{cosec}^{-1}(\cot x)$.

Which of the following statement(s) is/are **TRUE** for the function f (x)?

(A) f (x) is periodic with fundamental period π.

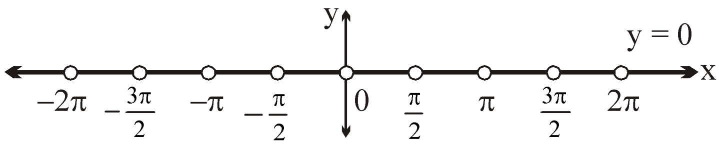
(B) The function f (x) is non-invertible.

(C) The composite function $f(f(x))$ is not defined.

(D) The function f (x) is an even function.

Ans. **(BCD)**

Sol. We have $f(x) = \sin^{-1}(\tan x) - \operatorname{cosec}^{-1}(\cot x) = \sin^{-1}(\tan x) - \sin^{-1}\left(\frac{1}{\cot x}\right) = 0 \quad \forall x \neq \frac{n\pi}{2}$ where $n \in I$.



Graph of y = f (x)

Clearly $y = f(x)$ is periodic function with fundamental period $\frac{\pi}{2}$.

The graph of $y = f(x)$ is union of line segments on x axis and not a straight line.

The function $y = f(x)$ is many-one, so non-invertible.

Also $y = f(x)$ is even and odd both simultaneously.

As $f(0)$ is not defined, so $f(f(x))$ is not defined.

15. Which of the following expression(s) have their value equal to four times the area of the triangle ABC? (All symbols used have their usual meaning in a triangle)

(A) $rs + r_1(s - a) + r_2(s - b) + r_3(s - c)$

(B) $\dfrac{(a+b+c)^2}{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}$

(C) $(a^2 + b^2 - c^2)\tan B$

(D) $b^2 \sin 2C + c^2 \sin 2B$

Ans. **(ABD)**

Sol. **(A)** $r = \frac{\Delta}{s}$; $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$ $\Rightarrow$ **(A) is correct**

(B) $\sum \cot\frac{A}{2} = \frac{s}{\Delta}[(s-a)+(s-b)+(s-c)] = \frac{s}{\Delta}[s] = \frac{4s^2}{4\Delta} = \frac{(a+b+c)^2}{4\Delta}$

$\therefore \quad \dfrac{(a+b+c)^2}{\sum \cot\frac{A}{2}} = 4\Delta \quad \Rightarrow$ **(B) is correct**

(C) Using $a^2 + b^2 - c^2 = 2ab\cos C$

given $(a^2 + b^2 - c^2)\tan B = 2ab\cos C\,\frac{\sin B}{\cos B} \neq 4\Delta \quad \Rightarrow$ **(C) is NOT correct**

Note: C could not be correct if $\tan B \to \tan C$

(D) $b^2 \sin 2C + c^2 \sin 2B$

using $b = k\sin B$, $\quad b[k\sin B\cdot 2\sin C\cos C] + c\,k\sin C\; 2\sin B\cos B$

$2bc\sin B\cos C + 2bc\sin C\cos B$

$2bc(\sin B\cos C + \cos C\sin B)$

$2bc\sin(B + C) = 2bc\sin A = 4\Delta \quad \Rightarrow$ **(D) is correct**

16. Following the usual notations, in a triangle ABC, if $(\sqrt{3}-1)\,a = 2b$ and $A = 3B$, then C cannot be equal to______.

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$ (C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{6}$

Ans. **(ABD)**

Sol. $(\sqrt{3}-1)a = 2b \Rightarrow \frac{a}{2} = \frac{b}{\sqrt{3}-1}$

$\Rightarrow \quad 3 - 4\sin^2 B = \frac{2}{\sqrt{3}-1} = \sqrt{3}+1 \quad \Rightarrow \quad \sin^2 B = \frac{2-\sqrt{3}}{4}$