competishun
Power of real gurus

MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-3

Function

ANSWERKEY (CAPS - 03)

1. $2\pi^3$ **2.** $\frac{-1}{2}$ **3.** y **4.** n = ± 1, ± 3, ± 5, ± 15

5. $x \in R \sim (0, 1] \cup \{2, 3, 4, 5, 6, 7\} \cup (7, 8)$ **6.** (D) **7.** (A) **8.** (B)

9. (C) **10.** (AD) **11.** (BD) **12.** (ABD) **13.** (ABCD)

14. (AC) **15.** (ABCD) **16.** (ABCD) **17.** (ABCD) **18.** (9.00)

19. (4.00) **20.** (12) **21.** (17) **22.** (B) **23.** (A)

24. (A) **25.** (C) **26.** (B)

27. (A) Q, R; (B) P, Q; (C) Q, R, T; (D) P, S **28.** (A) P, R; (B) P, R, S; (C) P, R; (D) R, T

Subjective Type Questions :

1. Find fundamental period of function $f(x) = 4\cos^4\left(\frac{x-\pi}{4\pi^2}\right) - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

Ans. $2\pi^3$

Sol. $f(x) = 4\cos^4\left(\frac{x-\pi}{4\pi^2}\right) - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

$= \left[2\cos^2\left(\frac{x-\pi}{4\pi^2}\right)\right]^2 - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

$= \left[1+\cos\left(\frac{x-\pi}{2\pi^2}\right)\right]^2 - 2\cos\left(\frac{x-\pi}{2\pi^2}\right) = 1 + \cos^2\left(\frac{x-\pi}{2\pi^2}\right)$

$\Rightarrow$ period of $f(x) = \left(\frac{\pi}{1/2\pi^2}\right) = 2\pi^3$

2. Consider the function $f(x) = \frac{3x+a}{x^2+3}$ which has the greatest value $\frac{3}{2}$.

Find the minimum value of $f(x)$.

Ans. $\frac{-1}{2}$

Sol. We have $f(x) = \frac{3x+a}{x^2+3}$

Now, $\frac{3}{2} = \frac{3x+a}{x^2+3} \Rightarrow 3x^2 - 6x + 9 - 2a = 0$

This must be a perfect square, so

$D = 0 \Rightarrow 36 = 12(9-2a) \Rightarrow a = 3$

$F = F(x) = \frac{3x+3}{x^2+3}$

Solving by usual method Range $\left[\frac{-1}{2}, \frac{3}{2}\right]$ $\Rightarrow$ Minimum value = $\frac{-1}{2}$

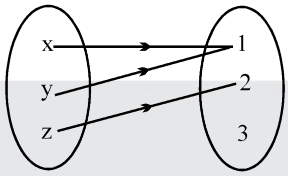
Power of real gurus

3. Let f be one-one function with domain {x, y, z} and range {1, 2, 3}. It is given that exactly one of the following statements is true and the remaining two are false.

$f(x) = 1$; $f(y) \neq 1$; $f(z) \neq 2$. Determine $f^{-1}(1)$

Ans. y

Sol. **Case - I**



When $f(x) = 1$ is true.

and $f(y) \neq 1$, $f(z) \neq 2$ are both false

Therefore $f(x) = 1$, $f(y) = 1$, $f(z) = 2$

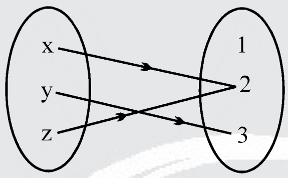
Which is not possible as f is one-one

Case - II

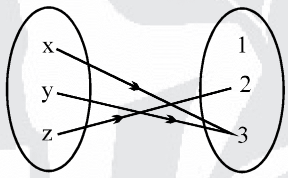
When $f(y) \neq 1$ is true and $f(x) = 1$, $f(z) \neq 2$ are both false.

Therefore $f(x) \neq 1$, $f(z) = 2$, $f(y) = 3$ i.e., $f(x) = 2$ or 3

If $f(x) = 2$



If $f(x) = 3$



Which is not possible as f is one-one

Case III :

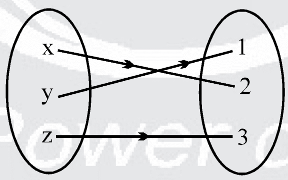
when $f(z) \neq 2$ is true and $f(y) \neq 1$, $f(x) = 1$ are both false Therefore

$f(y) = 1$, $f(z) = 3$, $f(x) \neq 1$ i.e., $f(x) = 2$

which is possible as f is one-one.

Thus $f(x) = 2$, $f(y) = 1$, $f(z) = 3$

$\Rightarrow \quad f^{-1}(1) = y$



Power of real gurus

4 . For what integral value of n, is 3π period of the function $\cos(nx)\sin\left(\dfrac{5x}{n}\right)$?

Ans. $n = \pm 1, \pm 3, \pm 5, \pm 15$

Sol. Let $f(x) = \cos nx \sin\left(\dfrac{5x}{n}\right)$

f(x) is periodic

$\therefore$ $f(x + \lambda) = f(x)$ where λ is period.

$\Rightarrow \cos(nx + n\lambda)\sin\left(\dfrac{5x + 5\lambda}{n}\right) = \cos nx \sin\left(\dfrac{5x}{n}\right)$

at $x = 0, \cos n\lambda \sin\left(\frac{5\lambda}{n}\right) = 0$

If $\cos n\lambda = 0 \Rightarrow n\lambda = r\pi + \frac{\pi}{2}$, $r \in I$

$\Rightarrow n(3\pi) = r\pi + \frac{\pi}{2}$ $\{\because \lambda = 3\pi\}$

$\Rightarrow 3n - r = \frac{1}{2}$ (Impossible)

Again let $\sin\left(\frac{5\lambda}{n}\right) = 0$

$\therefore \frac{5\lambda}{n} = p\pi$ $(p \in I)$ $\Rightarrow \frac{5(3\pi)}{n} = p\pi$ $(\because \lambda = 3\pi)$ $\Rightarrow n = \frac{15}{p}$

For $p = \pm 1, \pm 3, \pm 5, \pm 15$

$\therefore n = \pm 15, \pm 5, \pm 3, \pm 1$ $(\therefore n \in I)$

Hence $n = \pm 1, \pm 3, \pm 5, \pm 15$

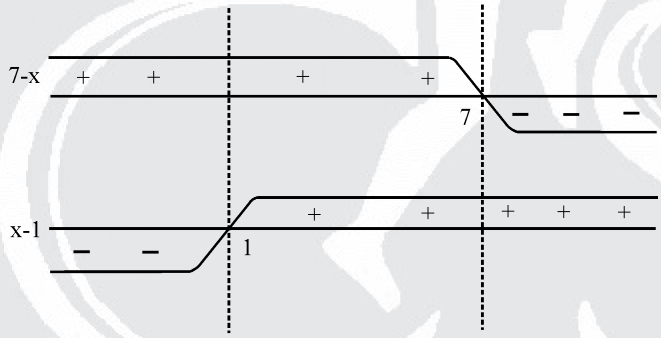
5. Find the domain of definition of the function $f(x) = \frac{1}{[|x-1|] + [|7-x|] - 6}$, where [.] denotes the greatest integer function

Ans. $x \in R \sim (0, 1] \cup \{2, 3, 4, 5, 6, 7\} \cup (7, 8)$

Sol. Since $f(x) = \frac{1}{[|x-1|] + [|7-x|] - 6}$

f(x) is defined when

$[|x-1|] + [|7-x|] - 6 \neq 0 \Rightarrow [|x-1|] + [|7-x|] \neq 6$



For $x \leq 1$

$[1 - x] + [7 - x] \neq 6$

$\Rightarrow 1 + [-x] + 7 + [-x] \neq 6$ $\Rightarrow 2[-x] \neq -2$

or $[-x] \neq -1$ $\Rightarrow x \notin (0, 1]$ ....(1)

For $1 < x \leq 7$:

$[x - 1] + [7 - x] \neq 6$

$\Rightarrow [x] - 1 + 7 + [-x] \neq 6$ $\Rightarrow [x] + [-x] \neq 0 \Rightarrow x \notin I$

$\Rightarrow x \notin \{2, 3, 4, 5, 6, 7\}$ ....(2)

For $x > 7$

$[x - 1] + [x - 7] \neq 6$

$\Rightarrow [x] - 1 + [x] - 7 \neq 6$ $\Rightarrow 2[x] \neq 14$

$\Rightarrow [x] \neq 7$ $\Rightarrow x \notin (7, 8)$ ....(3)

Combing (1), (2), (3),

Domain of f(x) is $x \in R \sim (0, 1] \cup \{2, 3, 4, 5, 6, 7\} \cup (7, 8)$

Power of real gurus

SCQ (Single Correct Type) :

6. Consider the function f (x) : A→A which satisfy the condition f (f (x)) = x , then number of such functions for A = {1, 2, 3, 4, 5} are

(A) 54 (B) 52 (C) 41 (D) 26

Ans. **(D)**

Sol. $1 + {}^5C_2 + {}^5C_4 . 3$

Two cases arise

Case 1 : $f(\alpha) = \alpha$

Case 2 : $f(\alpha) = \beta \quad \Rightarrow \quad f(\beta) = \alpha$

7. Let $f(x) = \begin{cases} 2x + a\ , & x \geq -1 \\ bx^2 + 3, & x < -1 \end{cases}$ and $g(x) = \begin{cases} x+4\ , & 0 \leq x \leq 8 \\ -3x-2, & -2 < x < 0 \end{cases}$ g(f(x)) is not defined if

(A) $a \in (10, \infty)$, $b \in (5, \infty)$ (B) $a \in (4, 10)$, $b \in (5, \infty)$

(C) $a \in (10, \infty)$, $b \in (0, 1)$ (D) $a \in (4, 10)$, $b \in (1, 5)$

Ans. **(A)**

Sol. g(f(x)) is not defined if (i) −2 + a > 8 and (ii) b + 3 > 8

a >10 and b > 5

8. For $x \neq 0, 1$ define $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = \frac{1}{1-x}$, $f_5(x) = \frac{x-1}{x}$.

Let h be a function such that $f_3ohof_2 = f_4$ then h is equal to

(A) f_5 (B) f_4 (C) f_3 (D) f_2

Ans. **(B)**

Sol. $f_3ohof_2 = f_4$

$\Rightarrow \quad h = f_3^{-1}of_4of_2^{-1} = f_3of_4of_2 = f_4$

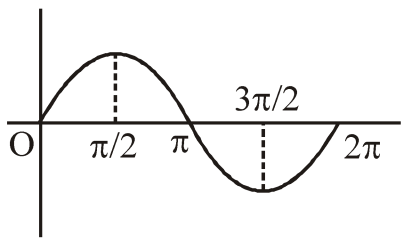
9. Let
$$f(x) = \max.\{\sin t : 0 \leq t \leq x\}$$
$$g(x) = \min.\{\sin t : 0 \leq t \leq x\}$$
$$\text{and } h(x) = [f(x) - g(x)]$$

where [ ] denotes greatest integer function, then the range of h(x) is

(A) {0, 1} (B) {1, 2} (C) {0, 1, 2} (D) {–3, –2, –1, 0, 1, 2, 3}

Ans. **(C)**

Sol.
$$f(x) = \sin x\ ; \quad 0 \leq x < \frac{\pi}{2}$$
$$= 1\ ; \quad \frac{\pi}{2} \leq x \leq 2\pi$$
$$g(x) = 0\ ; \quad 0 \leq x \leq \pi$$
$$= \sin x\ ; \quad \pi < x < \frac{3\pi}{2}$$
$$= -1\ ; \quad \frac{3\pi}{2} \leq x \leq 2\pi$$



$$h(x) = 0\ ; \quad 0 \leq x < \frac{\pi}{2}$$

Power of real gurus

$= \quad 1 \quad ; \qquad \frac{\pi}{2} \le x < \frac{3\pi}{2}$

$= \quad 2 \quad ; \qquad x \ge \frac{3\pi}{2}$

Hence the range of h(x) is $\{0, 1, 2\}$

MCQ (One or more than one correct) :

10. Let $f : X \to Y$ where $3^{(f(x))^2} + 2^{-x} = 4$ and f(x) is one-one and onto function and $f(x) \ge 0$ if

(A) $X = (0, \infty), Y = \left(1, \sqrt{\log_3 4}\right)$ (B) $X = \left[-\log_2 3, \infty\right), Y = \left[0, \infty\right)$

(C) $X = (0, \infty), Y = \left(0, \sqrt{\log_3 4}\right]$ (D) $X = \left[-\log_2 3, \infty\right), Y = \left[0, \sqrt{\log_3 4}\right)$

Ans. **(AD)**

Sol. $3^{f^2} + 2^{-x} = 4$

$\Rightarrow 3^{f^2} = 4 - 2^{-x}$

$\Rightarrow f = \sqrt{\log_3\left(4 - 2^{-x}\right)}$

Domain :

$4 - 2^{-x} > 0$ & $\log_3(4 - 2^{-x}) \ge 0$

$2^{-x} < 4$ & $4 - 2^{-x} \ge 1$

$x > -2$ & $2^{-x} \le 3$

$-x \le \log_2 3$

$-x \ge -\log_2 3$

Domain: $[-\log_2 3, \infty]$

f(x) being composition of one-one function is one-one

Range :

$-\infty < 4 - 2^{-x} < 4$

For $x \in$ Domain, we have

$\Rightarrow 0 \le \sqrt{\log_3\left(4 - 2^{-x}\right)} < \sqrt{\log_3 4}$

Similarly, if $x \in (0, \infty)$ then range is $\left(1, \sqrt{\log_3 4}\right)$

11. Let $f : R \to R$ be a function defined by $f(x+1) = \frac{f(x) - 5}{f(x) - 3}\ \forall x \in R$. Then which of the following statement(s) is/are true_____.

(A) f (x + 2) = f (x) (B) f (x + 4) = f (x) (C) f (x + 6) = f (x) (D) f (x + 8) = f (x)

Ans. **(BD)**

Sol. $f(x+1) = \frac{f(x) - 5}{f(x) - 3}$ ....(1)

$\Rightarrow f(x) = \frac{3f(x+1) - 5}{f(x+1) - 1}$

Replace x by x – 1

$\Rightarrow f(x-1) = \frac{3f(x-5)}{f(x) - 1}$ .....(2)

Replace x by x + 1 in (1)

$f(x+2) = \frac{f(x+1) - 5}{f(x+1) - 3} = \frac{2f(x) - 5}{f(x) - 2}$ .....(3)

Power of real gurus

Similarly, $f(x-2)=\dfrac{2(x)-5}{f(x)-2}$

$\therefore \quad f(x + 2) = f(x - 2) \quad \Rightarrow f(x + 4) = f(x)$

12. Let $f(x) = a_1 \cos(\alpha_1 + x) + a_2 \cos(\alpha_2 + x) + ... + a_n \cos(\alpha_n + x)$. If f(x) vanishes for x = 0 and $x = x_1$ (where $x_1 \neq k\pi$, $k \in Z$), then______.

(A) $a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + ...+a_n \cos \alpha_n = 0$

(B) $a_1 \sin \alpha_1 + a_2 \cos \alpha_2 + ...+a_n \sin \alpha_n = 0$

(C) f(x) = 0 has only two solutions 0, x_1

(D) f(x) is identically zero $\forall x$

Ans. **(ABD)**

Sol. $f(0) = 0 \quad \Rightarrow a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + ...+ a_n \cos \alpha_n = 0$

$f(x_1) = 0 \quad \Rightarrow (a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + ...+ a_n \cos \alpha_n) \cos x_1$

$+ (a_1 \sin \alpha_1 + a_2 \sin \alpha_2 + ...+ a_n \sin \alpha_n) \sin x_1 = 0)$

$\Rightarrow \quad a_1 \sin \alpha_1 + a_2 \sin \alpha_2 + ...+ a_n \sin \alpha_n = 0 \quad \because \left(x_1 \neq n\pi\right)$

$\therefore \quad a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + ...+ a_n \cos \alpha_n = 0$

and $a_1 \sin \alpha_1 + a_2 \sin \alpha_2 + ...+a_n \sin \alpha_n = 0$

$\Rightarrow f(x) = 0 \ \forall x$

13. Which of the following function(s) have no domain?

(A) $f(x) = \log_{x-1}(2 - [x] - [x]^2)$ where [x] denotes the greatest integer function.

(B) $g(x) = \cos^{-1}(2-\{x\})$ where {x} denotes the fractional part function.

(C) $h(x) = ln\, ln(\cos x)$

(D) $f(x) = \dfrac{1}{\sec^{-1}\left(\operatorname{sgn}\left(e^{-x}\right)\right)}$

Ans. **(ABCD)**

Sol. (A) obviously x > 1

for 1 < x < 2 , [x] = 1 $\Rightarrow$ N = 0

for $x \geq 2$, N < 0 $\Rightarrow$ no domain

(B) $0 \leq \{x\} < 1$ hence $-1 < -\{x\} \leq 0$ hence $1 < 2 - \{x\} \leq 2$

$\Rightarrow \quad \cos^{-1}(2 - \{x\})$ does not exist.

(C) $ln(\cos x)>0 \Rightarrow \cos x>1 \Rightarrow$ Not possible $\Rightarrow$ No Domain

(D) $e^{-x} \in (0, \infty) \Rightarrow \operatorname{sgn}(e^{-x}) = 1 \quad \Rightarrow \sec^{-1} = 0 \quad \Rightarrow$ Not defined

Power of real gurus