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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-25

Conic Sections

ANSWER KEY OF CAPS-25

1.	(D)	**2.**	(B)	**3.**	(D)	**4.**	(D)	**5.**	(A)
6.	(CD)	**7.**	(ACD)	**8.**	(ACD)	**9.**	(AB)	**10.**	(ABCD)
11.	(D)	**12.**	(C)	**13.**	(D)	**14.**	(55)	**15.**	(1)
16.	(25)	**17.**	(1)	**18.**	(4)	**19.**	(2)	**20.**	(7)
21.	(5)	**22.**	(B)						

SCQ (Single Correct Type) :

1. If α, β are eccentric angles of the extremities of a focal chord of an ellipse, then eccentricity of the ellipse is

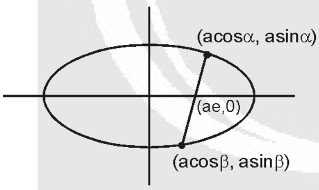
(A) $\dfrac{\cos\alpha+\cos\beta}{\cos(\alpha+\beta)}$ (B) $\dfrac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}$ (C) sec α + sec β (D) $\dfrac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$

Ans. **(D)**

Sol. Let equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (a > b)

Here chord is given by

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$



it passes through (ae, 0)

$$\therefore\ e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$e=\frac{2\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)}{2\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)}\times\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$$

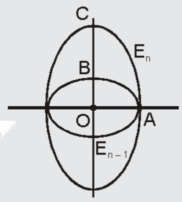
2. A series of concentric ellipses E_1, E_2, ..., E_n are drawn such that E_n touches the extremities of the major axis of E_{n-1} and the foci of E_n coincide with the extremities of minor axis of E_{n-1}. If the eccentricity of the ellipses is independent of n, then the value of the eccentricity, is

(A) $\frac{\sqrt{5}}{3}$ (B) $\frac{\sqrt{5}-1}{2}$ (C) $\frac{\sqrt{5}+1}{2}$ (D) $\frac{1}{\sqrt{5}}$

Ans. **(B)**

Sol. The figure shows two ellipses E_{n-1} and E_n.

The eccentricity is given to be independent of n, implies that the ratio of minor axis to the major axis, is same for all the ellipses.



For ellipse E_{n-1}, let

minor axis = b, major axis = a

For ellipse E_n, we have

minor axis = a, major axis = $\frac{OB}{e} = \frac{b}{e}$

[$\because$ B is the focus of E_n]

assuming e to be the eccentricity. Thus, we have

$$\frac{b}{a} = \frac{a}{b/e} \Rightarrow e = \frac{b^2}{a^2} = 1 - e^2 \Rightarrow e^2 + e - 1 = 0$$

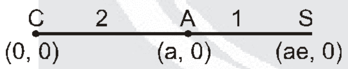
gives $e = \frac{\sqrt{5}-1}{2}$ [$\because$ e must be + ve]

3. The transverse axis of a hyperbola is of length 2a and a vertex divides the segment of the axis between the centre and the corresponding focus in the ratio 2 : 1, the equation of the hyperbola is :

(A) $4x^2 - 5y^2 = 4a^2$ (B) $4x^2 - 5y^2 = 5a^2$ (C) $5x^2 - 4y^2 = 4a^2$ (D) $5x^2 - 4y^2 = 5a^2$

Ans. **(D)**

Sol.



Clearly $\frac{2ae}{3} = a \Rightarrow e = \frac{3}{2}$

$\therefore$ $S = \left(\frac{3a}{2}, 0\right)$

Directrix is $x = \frac{2a}{3}$

$\therefore$ equation of hyperbola will be $\left(x - \frac{3a}{2}\right)^2 + y^2 = \frac{9}{4}\left(x - \frac{2a}{3}\right)^2$

Which reduces to $5x^2 - 4y^2 = 5a^2$

4. If AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔOAB (O is the origin) is an equilateral triangle, then the eccentricity 'e' of the hyperbola satisfies

(A) $e > \sqrt{3}$ (B) $1 < e < 2\frac{2}{\sqrt{3}}$ (C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$

Ans. **(D)**

Sol. Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & If PQ is any double ordinate then

P = (h,k), Q = (h, – k) & O (0, 0) is the origin

$\because$ Δ POQ is equlateral

$\Rightarrow$ $OP^2 = OQ^2 = PQ^2$

$\Rightarrow$ $h^2 = 3k^2$ .... (i)

also (h, k) lies on given hyperbola

$\therefore$ $\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$ ...... (ii)

(i) & (ii)

$\Rightarrow$ $k^2 = \frac{a^2b^2}{3b^2 - a^2} > 0 \Rightarrow 3b^2 - a^2 > 0 \Rightarrow \frac{b^2}{a^2} > \frac{1}{3}$

$\therefore$ $e^2 = 1 + \frac{b^2}{a^2} > 1 + \frac{1}{3}$ $\therefore$ $e^2 > \frac{4}{3}$ or $e > \frac{2}{\sqrt{3}}$

5. An ellipse whose major axis is parallel to X –axis such that the segments of the focal chords are 1 and 3 units. The lines ax + by + c = 0 are the chords of the ellipse such that a,b,c are in A.P. and bisected by the point at which they intersect. The equation of its auxiliary circle is $x^2 + y^2 + 2\alpha x + 2\beta y - 2\alpha - 1 = 0$ then _____.

Equation of the auxiliary circle is

(A) $x^2 + y^2 - 2x + 4y + 1 = 0$ (B) $x^2 + y^2 + 2x + 2y - 3 = 0$

(C) $x^2 + y^2 + 2x + 4y + 1 = 0$ (D) $x^2 + y^2 - 4x + 2y - 3 = 0$

Ans. **(A)**

Sol. Let ellipse be $\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2} = 1$

$\because PS + PS' = 2A = 1 + 3$ $\Rightarrow A = 2$

$\Rightarrow ax + \frac{(a+c)}{2}y + c = 0 \Rightarrow a(2a + y) + c(2 + y) = 0$

$\Rightarrow y = -2, x = 1$

$\Rightarrow$ centre (h,k) = (1,2)

$\therefore$ Equation of auxillary circle is

$(x-1)^2 + (y+2)^2 = 4$

$\Rightarrow x^2 + y^2 - 2x + 4y + 1 = 0$

MCQ (One or more than one correct) :

6. A point moves such that the sum of the squares of its distances from the two sides of length 'a' of a rectangle is twice the sum of the squares of its distances from the other two sides of length 'b'. The locus of the point can be :

(A) a circle (B) an ellipse (C) a hyperbola (D) a pair of lines

Ans. **(CD)**

Sol. Let the two sides of the rectangle lie along x-axis & y axis as shown

Given that

$(PA)^2 + (PB)^2 = 2\ (PC^2 + PD^2)$

$\Rightarrow \quad k^2 + (k - b)^2 = 2\ (h^2 + (a - h)^2)$

$\Rightarrow \quad 2k^2 - 2kb + b^2 = 4h^2 - 4ah + 2a^2$

Replacing h by x and k by y

$\Rightarrow \quad 2y^2 - 2by + b^2 = 4x^2 - 4ax + 2a^2$

$2(y^2 - by) + b^2 = 4(x^2 - ax) + 2a^2$

$2\left(y - \frac{b}{2}\right)^2 + \frac{b^2}{2} = 4\left(x - \frac{a}{2}\right)^2 + a^2$

$4\left(x - \frac{a}{2}\right)^2 - 2\left(y - \frac{b}{2}\right)^2 = \frac{b^2}{2} - a^2$

Hence it is a hyperbola or pair of lines if $\frac{b^2}{2} - a^2 \neq 0$ or $\frac{b^2}{2} - a^2 = 0$ respectively.

7. Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

Ans. **(ACD)**

Sol. (A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ ....(i) i.e. $t + \frac{1}{t} = \frac{2x}{a}$

$y = \frac{b}{2}\left(t - \frac{1}{t}\right)$ ....(ii) i.e. $t - \frac{1}{t} = \frac{2y}{b}$

adding $2t = \frac{2x}{a} + \frac{2y}{b}$ $\quad t = \frac{x}{a} + \frac{y}{b} = \frac{bx + ay}{ab}$

Put in (i) $\quad x = \frac{a}{2}\left(\frac{x}{a} + \frac{y}{b} + \frac{ab}{bx + ay}\right)$ i.e. $b^2x^2 - a^2y^2 = a^2b^2$

whereas (B) represents an ellipse

(C) $\quad x = e^t + e^{-t}$ .....(i)

$y = e^t - e^{-t}$

i.e. $\quad x + y = 2e^t \qquad e^t = \dfrac{x+y}{2}$ put in (i)

$x = \dfrac{x+y}{2} + \dfrac{2}{x+y} \qquad$ i.e. $\quad y^2 - x^2 = 4 \quad$ Hyperbola

(D) $\quad x^2 - 6 = 2\cos t$

$y^2 + 2 = 4\cos^2\dfrac{t}{2} = 2\,(1 + \cos t) = 2\left(1 + \dfrac{x^2-6}{2}\right)$

$= 2\left(\dfrac{2+x^2-6}{2}\right) \qquad y^2 + 2 = x^2 - 4 \qquad$ Hyperbola

8. Let C_1 : $9x^2 - 16y^2 - 18x + 32y - 23 = 0$ and C_2 : $25x^2 + 9y^2 - 50x - 18y + 33 = 0$ are two conics then

(A) eccentricity of C_1 is $\dfrac{5}{4}$. (B) eccentricity of C_2 is $\dfrac{5}{3}$.

(C) area of the quadrilateral with vertices at the foci of the conics is $\dfrac{8}{9}$.

(D) latus rectum of C_1 is greater than latus rectum of C_2.

Ans. **(ACD)**

Sol. $9\,[x^2 - 2x] - 16\,[y^2 - 2y] = 23$

$9\,[(x-1)^2 - 1] - 16\,[(y-1)^2 - 1] = 23$

$9\,(x-1)^2 - 16\,(y-1)^2 = 16$

$\dfrac{(x-1)^2}{\left(\frac{4}{3}\right)^2} - (y-1)^2 = 1 \quad$ which is a hyperbola

$C_1 : \dfrac{(x-1)^2}{\left(\frac{4}{3}\right)^2} - (y-1)^2 = 1 \qquad$ ....(1)

$C_2 : \dfrac{(x-1)^2}{\left(\frac{1}{5}\right)^2} + \dfrac{(y-1)^2}{\left(\frac{1}{3}\right)^2} = 1 \qquad$ ....(2)

Let $\quad x - 1 = X$ and $y - 1 = Y$

$\therefore \quad E : \dfrac{X^2}{\left(\frac{1}{5}\right)^2} + \dfrac{Y^2}{\left(\frac{1}{3}\right)^2} = 1$; with $e_E = \dfrac{4}{5}$ $\left[e_E^2 = 1 - \dfrac{9}{25} = \dfrac{16}{25} \Rightarrow e_E = \dfrac{4}{5}\right]$

and H : $\frac{X^2}{\left(\frac{4}{3}\right)^2} - Y^2 = 1$; with $e_H = \frac{5}{4}$ $\left[e_H^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e_H = \frac{5}{4}\right]$

foci of ellipse $\left(0, \frac{4}{15}\right)$ and $\left(0, \frac{-4}{15}\right)$

focii of hyperbola $\left(\frac{5}{3}, 0\right)$ and $\left(\frac{-5}{3}, 0\right)$

Hence area = $\frac{1}{2}$ $d_1 d_2$ = $\frac{1}{2} \cdot \frac{8}{15} \cdot \frac{10}{3} = \frac{8}{9}$ $\Rightarrow$ (A). **Ans.**

9. A hyperbola centred at C has one focus at P(6, 8) . If its directrices are 3x + 4y +10 = 0 and 3x + 4y –10 = 0 , then _____.

(A) CP =10

(B) eccentricity = $\sqrt{5}$

(C) CP = 8

(D) eccentricity = $\frac{\sqrt{5}}{2}$

Ans. **(AB)**

Sol. AM of distances of focus from two directrices is CP, that is 10

$$\frac{CP}{\text{distance between two directrices}} = \frac{(\text{eccentricity})^2}{2}$$

$$\Rightarrow \frac{10}{4} = \frac{e^2}{2} \Rightarrow e = \sqrt{5}$$

10. Two ellipses $\frac{x^2}{\cos^2\alpha} + \frac{y^2}{\sin^2\alpha} = 1$ and $\frac{x^2}{\sin^2\alpha} + \frac{y^2}{\cos^2\alpha} = 1\left(0 < \alpha < \frac{\pi}{4}\right)$ intersect at four points P,Q, R, S then which of the following statement(s) is /are true?

(A) PQRS is a square with length of the side sin 2α

(B) PQRS lie on a circle whose centre is origin and with radius $\frac{\sin 2\alpha}{\sqrt{2}}$

(C) eccentricity of the two given ellipses are same

(D) there are two points on $\frac{x^2}{\sin^2\alpha} + \frac{y^2}{\cos^2\alpha} = 1$ whose reflection in y = x lie on the same ellipse

Ans. **(ABCD)**

Sol. Since $0 < \alpha < \frac{\pi}{4}, \cos^2\alpha > \sin^2\alpha$

$\Rightarrow$ eccentricity of the two given ellipse, are same.

$$\frac{x^2}{\cos^2\alpha} + \frac{y^2}{\sin^2\alpha} = \frac{x^2}{\sin^2\alpha} + \frac{y^2}{\cos^2\alpha}$$

$$\Rightarrow \left(\frac{1}{\cos^2\alpha} - \frac{1}{\sin^2\alpha}\right)x^2 + \left(\frac{1}{\sin^2\alpha} - \frac{1}{\cos^2\alpha}\right)y^2 = 0$$

Since sum of the coefficients of and is zero

POR and QOS are perpendicular equations of these lines are $y = \pm x$

$\therefore$ P(cos α sin, sin α cos α), and S(cos α sin$-$ sin α cos α)

$\therefore$ PS = sin2, and hence b, d also correct

Comprehension Type Question:

Comprehension # 1

Paragraph for question nos. 11 to 13

The graph of the conic $x^2 - (y - 1)^2 = 1$ has one tangent line with positive slope that passes through the origin. the point of tangency being (a, b). Then

11. The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(A) $\frac{5\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

Ans. **(D)**

12. Length of the latus rectum of the conic is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) none

Ans. **(C)**

13. Eccentricity of the conic is

(A) $\frac{4}{3}$ (B) $\sqrt{3}$ (C) 2 (D) none

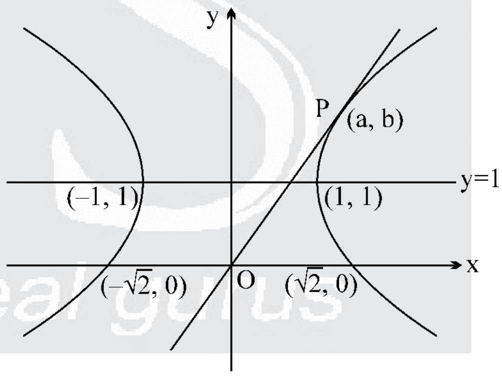
Ans. **(D)**

Sol. (i) differentiate the curve

$2x - 2(y - 1)\frac{dy}{dx} = 0$

$= \left.\frac{dy}{dx}\right]_{a,b} = \frac{a}{b-1} = \frac{b}{a}$ $\left(m_{OP} = \frac{b}{a}\right)$

$a^2 = b^2 - b$ ....(1)



Also (a, b) satisfy the curve

$a^2 - (b - 1)^2 = 1$

$a^2 - (b^2 - 2b + 1) = 1$

$a^2 - b^2 + 2b = 2$

$\therefore$ $-b + 2b = 2$ $\Rightarrow$ $b = 2$ {putting $a^2 - b^2 = -b$ from (1)}

$\therefore$ $a = \sqrt{2}$ $(a \neq -\sqrt{2})$

$\therefore$ $\sin^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{4}$ Ans.

(ii) Length of latus rectum = $\frac{2b^2}{a}$ = 2a = distance between the vertices = 2

(note that the hyperbola is rectangular)

(iii) Curve is a rectangular hyperbola $\Rightarrow$ e = $\sqrt{2}$ Ans.

Numerical based Questions :

14. Common tangents are drawn to the parabola $y^2 = 4x$ & the ellipse $3x^2 + 8y^2 = 48$ touching the parabola at A & B and the ellipse at C & D, then the area of the quadrilateral ABCD is $\lambda\sqrt{2}$ the λ is equal to

Ans. 55

Sol. $y^2 = 4x$, $\frac{x^2}{16} + \frac{y^2}{6} = 1$

For common tangent

$\frac{1}{m} = \pm\sqrt{6m^2 + 6}$

$\Rightarrow$ m = $\pm\frac{1}{2\sqrt{2}}$

$\therefore$ Equation of tangent

y = $\frac{1}{2\sqrt{2}}x + 2\sqrt{2}$ given $\frac{xx_1}{16} + \frac{yy_1}{6} = 1$

$yy_1 = 2a(x + x_1)$

$\Rightarrow$ $x_1 = 8$, $y_1 = 4\sqrt{2}$ $x_3 = 2$ $y_3 = \frac{3}{\sqrt{2}}$

$\therefore$ $x_2 = 8$, $y_1 = -4\sqrt{2}$ $x_4 = -2$ $y_4 = -\frac{3}{\sqrt{2}}$

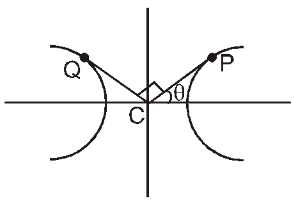
$\therefore$ A = $\frac{1}{2} \times \left[8\sqrt{2} + \frac{6}{\sqrt{2}}\right] \times (10)$

= $55\sqrt{2}$

15. If two points P & Q on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then $\frac{1}{CP^2} + \frac{1}{CQ^2} = \lambda\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$ where λ is :

Ans. 1

Sol.



CP $\equiv \frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r_1$ where CP = r_1 $\therefore$ $P(r_1\cos\theta, r_1\sin\theta)$

Similarly $Q\left(r_2\cos\left(\frac{\pi}{2}+\theta\right), r_2\sin\left(\frac{\pi}{2}+\theta\right)\right)$

$Q(-r_2\sin\theta, r_2\cos\theta)$ P & Q lies on Hyperbola

$\therefore\ r_1^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)=1$

$\therefore\ r_1^2=\frac{a^2b^2}{(b^2\cos^2\theta-a^2\sin^2\theta)}$ & $r_2^2=\frac{a^2b^2}{(b^2\sin^2\theta-a^2\cos^2\theta)}$

$\therefore\ \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{b^2-a^2}{a^2b^2}=\frac{1}{a^2}-\frac{1}{b^2}$ H.P.

16. If the distance between the centres of the hyperbolas :

$x^2-16xy-11y^2-12x+6y+21=0$ ..... (i)

$9x^2-16y^2-18x-32y-151=0$ ..... (ii)

is d then $125d^2=$ ....

Ans. **25**

Sol. Let $f(x, y)=x^2-16xy-11y^2-12x+6y+21$

& $g(x, y)=9x^2-16y^2-18x-32y-151$

$\frac{\partial f}{\partial x}=0 \quad \Rightarrow 2x-16y-12=0$ ...... (i)

$\frac{\partial f}{\partial y}=0 \quad \Rightarrow -16x-22y+6=0$ ....(ii)

Solving (i) & (ii) we get $C_1\left(\frac{6}{5},\frac{-3}{5}\right)$

Where C_1 is the centre of 1st hyperbola

Similarly $C_2=(1,-1)$

given that $C_1C_2=d \Rightarrow \frac{1}{25}+\frac{4}{25}=d^2=\frac{1}{5}$

$\therefore\ 125d^2=25$

17. If $(a^2, a-2)$ be a point interior to the region of the parabola $y^2=2x$ bounded by the chord joining the points (2, 2) and (8, – 4), then the number of all possible integral values of a is :

Ans. **1**

Sol. I.F. $(a^2, a-2)$

$S \equiv y^2-2x \qquad S \equiv y^2-2x$

Equation of line AB

$y-2=\frac{-6}{6}(x-2)$

$y-2=-x+2$

$L \equiv x+y-4=0$

$S_1 \equiv (a-2)^2-2a^2<0$

$a^2 + 4 - 4a - 2a^2 <$ $\Rightarrow$ $a^2 + 4a - 4 > 0$

$-4a - a^2 + 4 < 0$ $\qquad a^2 + 4a + 4 > 8$

$L_1 < 0$ $\qquad (a + 2)^2 > 8$

$a^2 + a - 6 < 0$ $\qquad a + 2 > 2\sqrt{2} \quad a + 2 < -2\sqrt{2}$

$-3 < a < 2$ $\qquad a > -2 + 2\sqrt{2} \quad a < -2\sqrt{2} - 2 \Rightarrow \qquad -2 + 2\sqrt{2} < a < 2$

so integral value of a is equal to 1 only.

18. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then maximum value of ab is _____.

Ans. **(4)**

Sol. Equation of tangent is $y = 2x \pm \sqrt{4a^2 + b^2}$

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$

This tangent passes through (–2, 0)

So, $0 = -4 \pm \sqrt{4a^2 + b^2} \Rightarrow 4a^2 + b^2 = 16$

Using $AM \geq GM$, we get $\frac{4a^2 + b^2}{2} \geq \sqrt{4a^2b^2} \Rightarrow ab \leq 4$

19. If the length of the latus rectum of a standard hyperbola of eccentricity 2 is equal to the limit of the series $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \ldots.$ and r is the radius of the director circle of its conjugate hyperbola, then r^2 = _____.

Ans. **(2)**

Sol. $t_n = \frac{6(2n+1)}{n(n+1)(2n+1)} = 6\left[\frac{1}{n} - \frac{1}{n+1}\right] \Rightarrow S_\infty = 6$

Let hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then latus rectum $= \frac{2b^2}{a} = 6$

$\Rightarrow b^2 = 3a$ and $e^2 = 1 + \frac{b^2}{a^2} = 4 \Rightarrow b^2 = 3a^2$

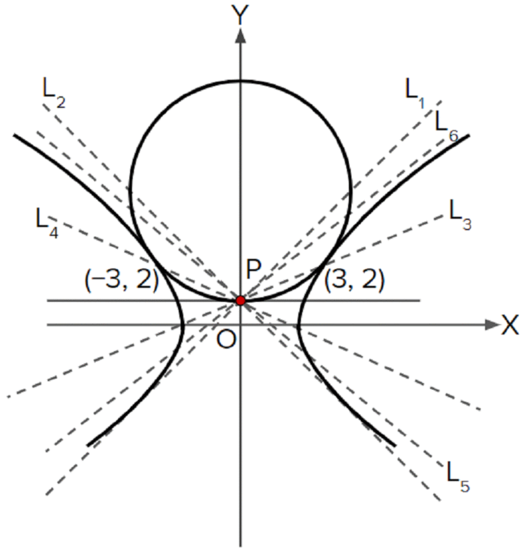
So, $a^2 = 1, b^2 = 3$

$\therefore$ Radius of director circle of conjugate hyperbola is given by

$r^2 = b^2 - a^2 = 2$

20. Let $C_1 : x^2 - y^2 = 5$ and $C_2 : x^2 + y^2 - 8y + 3 = 0$ be the equations of a hyperbola and a circle respectively. The curves C_1 and C_2 touch each other at (±3, 2) . P(0, 4 − $\sqrt{13}$) is a point on the curve C_2 . Let a line through P meet C_1 at m number of points and C_2 at n number of points. If (m+ n) = 3, then the number of such straight lines is_____.

Ans. **(7)**



Sol.

L_1, L_2 are tangents from P to hyperbola.

L_3, L_4 are two lines passing through point 3, 2 .

L_5, L_6 are two lines through P and parallel to asymptotes.

L_7 is tangent to circle at P.

21. Coordinates of the vertices B and C are (2, 0) and (8, 0) respectively. The vertex A is varying in such a way that $4\tan\frac{B}{2}\tan\frac{C}{2}=1$. If the locus of A is an ellipse then the length of its semi major axis is _____.

Ans. **(5)**

Sol. $4\tan\frac{B}{2}\text{tam}\frac{C}{2}=1$

$$\Rightarrow \sqrt{\frac{(s-c)(s-a)(s-a)(s-b)}{s(s-b)\,s(s-c)}}=\frac{1}{4}$$

$$\Rightarrow \frac{s-a}{s}=\frac{1}{4}\Rightarrow \frac{25-a}{a}=\frac{5}{3}\Rightarrow b+c=\frac{5}{3}\times 6=10$$

$\left(\because a=\overline{BC}=6\right)$

$\therefore$ Locus of A is

$$\frac{(x-5)^2}{25}+\frac{y^2}{16}=1$$

Matrix Match Type :

22. Match the following:

For the ellipse $(3x-6)^2+(3y-9)^2=\frac{4}{169}(5x+12y+6)^2$

Column-I	Column-II
(a) Length major axis	(p) $\frac{24}{5}$
(b) Length minor axi	(q) $\frac{16}{5}$
(c) Length of Latus Rectum	(r) $\frac{16}{3}$
(d) Distance between directrices	(s) $\frac{72}{5}$
	(t) $\frac{48}{5}$

(A) a → r; b → q; c → r; d → s

(B) a → t; b → q; c → r; d → s

(C) a → t; b → s; c → r; d → s

(D) a → t; b → q; c → q; d → s

Ans. (B)

Sol. $(x-2)^2+(y-3)^2=\frac{4}{9}\left(\frac{5x+12y+6}{13}\right)^2$

$e=\frac{2}{3}$, focus(2, 3), directrix = 5x + 12y + 6 = 0

Distance between focus, directrix $=\frac{a}{e}-ae=a\left(\frac{3}{2}-\frac{2}{3}\right)=a\left(\frac{5}{6}\right)=\frac{52}{13}=4$

$a=\frac{24}{5}, 2a=\frac{48}{5}$

$b^2=\frac{576}{25}\left(1-\frac{4}{9}\right)=\frac{576}{5\times 9}=\frac{64}{5}$

$2b=\frac{2\times 8}{\sqrt{5}}=\frac{16}{\sqrt{5}}$

$LR=\frac{2b^2}{9}=2\frac{(64)}{5\left(\frac{24}{5}\right)}=\frac{16}{3}$

Distance between directrices $=2\frac{a}{e}=\frac{48}{5\left(\frac{2}{3}\right)}=\frac{72}{5}$

Subjective Type Questions :

23. If S and H be the focii of an ellipse and any point A be taken on the curve and the chords ASB, BHC, CSD and DHE be drawn and eccentric angles of A, B, C, D, E, ... be θ_1, θ_2, q_3, q_4, ... Prove that $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2} = \tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = ...$

Sol. Equation of chord AB where A = (acos θ_1, b sin θ_1) and B = (a cos θ_2, b sin θ_2) is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

it passes through S (ae, 0) then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{e-1}{e+1}$ ....(1)

Similarly chord BHC joining θ_2 and θ_3 passes through H (– ae, 0)

$$\therefore \quad \tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2} = \frac{e+1}{e-1}$$

or $\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2} = \frac{e-1}{e+1}$ ....(2)

and chord CSD joining θ_3 and θ_4 passes through S (ae, 0)

$$\therefore \quad \tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = \frac{e-1}{e+1} \quad ....(3)$$

Hence, from (1), (2), (3), ... we get

$$\tan\frac{\theta_1}{\theta_2}\tan\frac{\theta_2}{2} = \cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2} = \tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = ...\text{etc.}$$

24. A variable point P on an ellipse of eccentricity e is joined to joined to its foci S, S'. Prove that the locus of the incentre of ΔPSS' is an ellipse of eccentricity $\sqrt{\frac{2e}{1+e}}$.

Sol. Let the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ......(1)$$

Here S (ae, 0) and S'(– ae, 0)

Let P(a cos θ, b sin θ) be a variable point on ellipse (1).

Let I (α, β) be the incentre of ΔPS'S

Now S'S = 2ae, PS' = a(1 + e cos θ), PS = a(1 – e cos θ)

Since I (α, β) is the incentre of ΔPS'S

$$\therefore \quad \alpha = \frac{2ae.\,a\cos\theta + a(1+e\cos\theta)\,ae + a(1-e\cos\theta)(-ae)}{2ae + a(1+e\cos\theta) + a(1-e\cos\theta)}$$

$$= \frac{2a^2e\cos\theta(1+e)}{2a(e+1)} = ae\cos\theta \quad ....(2)$$

and $\beta = \dfrac{2ae\,b\sin\theta + a(1+e\cos\theta)\cdot 0 + a(1-e\cos\theta)\cdot 0}{2ae + a(1+e\cos\theta) + a(1-e\cos\theta)}$

$= \dfrac{2aeb\sin\theta}{2a(e+1)} = \dfrac{e}{e+1} b\sin\theta$ ....(3)

Eliminating θ between equations (2) and (3), we get

$$\frac{\alpha^2}{a^2e^2} + \frac{\beta^2}{\frac{e^2b^2}{(1+e)^2}} = 1$$

Hence locus of P(α, β) is $\dfrac{x^2}{a^2e^2} + \dfrac{y^2}{\left(\frac{eb}{1+e}\right)^2} = 1$

Eccentricity of ellipse (4)

$$= \sqrt{1-\left(\frac{eb}{1+e}\right)^2 \cdot \frac{1}{a^2e^2}} = \sqrt{1-\frac{b^2}{a^2(1+e)^2}} = \sqrt{1-\frac{1-e^2}{(1+e)^2}} = \sqrt{1-\frac{1-e}{1+e}} = \sqrt{\frac{2e}{1+e}}$$

25. Let P and Q be two points on the ellipse $x^2 + 4y^2 = 16$ whose eccentric angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ respectively. The tangent at P and the normal at Q cut each other at R and the normal at Q cuts the ellipse again at M. Find the area of the triangle PRM.

Sol. Here the equation of the given ellipse is $x^2 + 4y^2 = 16$

Co-ordinate of $P \equiv \left(4\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right) = \left(2\sqrt{2}, \sqrt{2}\right)$

Co-ordinate of $P \equiv \left(4\cos\frac{3\pi}{4}, 2\sin\frac{3\pi}{4}\right)$

$= \left(-2\sqrt{2}, \sqrt{2}\right)$

Now equation of the tangent at point P is i.e. PR

$xx_1 + 4yy_1 = 16$

$\Rightarrow 2\sqrt{2}x + 4\sqrt{2}y = 16$

$\Rightarrow x + 2y = 4\sqrt{2}$

Now equation of the normal at point Q is i.e. QR

$ax\sec\theta - by\ \ \text{cosec}\theta = a^2 - b^2$

$\Rightarrow 4x\left(-\sqrt{2}\right) - 2y\left(\sqrt{2}\right) = 16 - 4$

$\Rightarrow -4 \times \sqrt{2} - 2\sqrt{2}y = -12$

$\Rightarrow 2x + y = -3\sqrt{2}$

Solving (1) and (2) we obtain the co-ordinate of R i.e. $\left(-\frac{10\sqrt{2}}{3}, \frac{11\sqrt{2}}{3}\right)$

Now solving (2) with the equation of ellipse

$$x^2 + 4(2x + 3\sqrt{2})^2 = 16$$

$$\Rightarrow x^2 + 4(4x^2 + 18 + 12\sqrt{2}x) = 16$$

$$\Rightarrow 17x^2 + 48\sqrt{2}x + 56 = 0$$

$$\Rightarrow x = \frac{-48\sqrt{2} \pm \sqrt{\left(48\sqrt{2}\right)^2 - 4 \times 17 \times 56}}{2 \times 17}$$

$$\Rightarrow x = \frac{-48\sqrt{2} + \sqrt{\left(48\sqrt{2}\right)^2 - 4 \times 17 \times 56}}{2 \times 17}$$

$$\Rightarrow x = \frac{-48\sqrt{2} + 20\sqrt{2}}{34}, \frac{-48\sqrt{2} - 20\sqrt{2}}{34}$$

$\Rightarrow x = \frac{-14}{17}\sqrt{2}$ or $-2\sqrt{2}$. ($-2\sqrt{2}$ corresponds to point Q)

$$\Rightarrow x = -\frac{14\sqrt{2}}{17}$$

By putting the value of x in (2) we get $y = \frac{-23\sqrt{2}}{17}$

Coordinate of M is $\left(\frac{-14\sqrt{2}}{17}, \frac{23\sqrt{2}}{17}\right)$

$$= \frac{1}{2}\begin{vmatrix} 2\sqrt{2} & \sqrt{2} & 1 \\ -2\sqrt{2} & \sqrt{2} & 1 \\ \frac{-14\sqrt{2}}{17} & \frac{-23\sqrt{2}}{17} & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ -2 & 1 & 1 \\ \frac{-14}{17} & \frac{-23}{17} & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{2}{17} \times 2(17 + 23) = \frac{4}{17} \times 40 = \frac{160}{17} \text{ sq. units}$$

26. A rectangular hyperbola, with centre C, is intersected by a circle of radius r in four points p, Q, R and S. Prove that $CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2$

Sol. Let the equation of the circle be $x^2 + y^2 = r^2$. The rectangular hyperbola intersects the circle in four points. Let (h, k) be the centre of the hyperbola and let ($r \cos\theta_i$, $r \sin\theta_i$) I = 1, 2, 3, 4 be the four points of intersection namely P,Q,R and S. Hence by using the fact that mean point of the points of intersection of circle and rectangular hyperbola is mid point of the line segment joining the centre of circle and that of rectangular hyperbola, we have $\frac{h}{2} = \frac{\sum r \cos\theta_i}{4}$

$$\Rightarrow \; 2h = r\sum_{i=1}^{4}\cos\theta_i \qquad \text{and } \; 2k = r\sum_{i=1}^{4}\sin\theta_i$$

Now $\quad CP^2 + CQ^2 + CR^2 + CS^2 = \sum_{i=1}^{4}\left[\left(h - r\cos\theta_i\right)^2 + \left(k - r\cos\theta_i\right)^2\right]$

$$= \sum_{i=1}^{4}\left(h^2 + r^2\cos^2\theta_i - 2rh\cos\theta_i + k^2 + r^2\sin^2\theta_i - 2rk\sin\theta_i\right)$$

$$= \sum_{i=1}^{4}\left(h^2 + k^2 + 2rh\cos\theta_i - 2rk\sin\theta_i + r^2\right)$$

$$= 4r^2 + 4h^2 + 4k^2 - 2hr\sum_{i=1}^{4}\sin\theta_1$$

$$= 4r^2 + 4h^2 + 4k^2 - 2h\cdot 2h - 2k\cdot 2k = 4r^2$$