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Power of real gurus

MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-20

VECTOR

ANSWER KEY OF CAPS-20

1.	(D)	**2.**	(A)	**3.**	(B)	**4.**	(C)	**5.**	(A)
6.	(A)	**7.**	(D)	**8.**	(C)	**9.**	(D)	**10.**	(ABCD)
11.	(BD)	**12.**	(CD)	**13.**	(BCD)	**14.**	(1)	**15.**	(5)
16.	(3)	**17.**	(2)	**18.**	(5)	**19.**	(6)	**20.**	(18)
21.	(36)	**22.**	(D)	**23.**	$\lambda = \frac{1}{2}$				

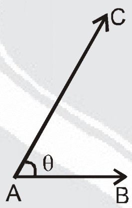
SCQ (Single Correct Type) :

1. The vector $\hat{i} + x\hat{j} + 3\hat{k}$ is rotated through an angle of $\cos^{-1}\frac{11}{14}$ and doubled in magnitude, then it becomes $4\hat{i} + (4x-2)\hat{j} + 2\hat{k}$. The value of 'x' is:

(A) $-\frac{2}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) 2

Ans. (D)

Sol. $|\overrightarrow{AC}|^2 = |2\overrightarrow{AB}|^2$



$\Rightarrow$ $|4\hat{i} + (4x-2)\hat{j} + 2\hat{k}|^2 = 4\,|(\hat{i} + x\hat{j} + 3\hat{k})|^2$

$\Rightarrow$ $16 + (4x - 2)^2 + 4 = 4\,(1 + x^2 + 9)$

$\Rightarrow$ $20 + 16x^2 + 4 - 16x = 4 + 4x^2 + 36$

$\Rightarrow$ $12x^2 - 16x - 16 = 0$

$\Rightarrow$ $3x^2 - 4x - 4 = 0$

$\Rightarrow$ $x = 2\,,\ -\frac{2}{3}$ ...(i)

angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{2\,|\overrightarrow{AB}|^2}$$

$$\Rightarrow \frac{11}{14} = \frac{(\hat{i} + x\hat{j} + 3\hat{k}).\,(4\hat{i} + (4x-2)\hat{j} + 2\hat{k})}{2(1+x^2+9)} = \frac{4 + x(4x-2) + 6}{2x^2 + 20}$$

$\Rightarrow$ $11x^2 + 110 = 70 + 28x^2 - 14x$

$\Rightarrow$ $17x^2 - 14x - 40 = 0$

$\therefore$ $x = 2,\ -\frac{20}{17}$ ....(ii)

from (i) and (ii)

$x = 2$

2. If the acute angle that the vector, $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ makes with the plane of the two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $\hat{i}-\hat{j}+2\hat{k}$ is $\cot^{-1}\sqrt{2}$ then :

(A) $\alpha(\beta+\gamma)=\beta\gamma$ (B) $\beta(\gamma+\alpha)=\gamma\alpha$

(C) $\gamma(\alpha+\beta)=\alpha\beta$ (D) $\alpha\beta+\beta\gamma+\gamma\alpha=0$

Ans. **(A)**

Sol. $\perp^r$ vector to plane

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$\vec{n} = 5\hat{i}-5\hat{j}-5\hat{k}$$

$$\therefore \quad \cos(90-\theta) = \frac{(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \cdot \frac{(\hat{i}-\hat{j}-\hat{k})}{\sqrt{3}}$$

$$\sin\theta = \frac{\alpha-\beta-\gamma}{\sqrt{3}\sqrt{\alpha^2+\beta^2+\gamma^2}} \qquad \left(\because\ \theta=\cot^{-1}\sqrt{2}\ \Rightarrow \cot\theta=\sqrt{2}\ \Rightarrow \sin\theta=\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{\alpha-\beta-\gamma}{\sqrt{3}\sqrt{\alpha^2+\beta^2+\gamma^2}} \qquad \Rightarrow \qquad \alpha(\beta+\gamma)=\beta\gamma$$

3. If $\bar{p},\bar{q},\bar{r}$ are three mutually perpendicular vectors of the same magnitude and if a vector $\bar{x}$ satisfies the equation $\bar{p}\times\left[(\bar{x}-\bar{q})\times\bar{p}\right]+\bar{q}\times\left[(\bar{x}-\bar{r})\times\bar{q}\right]+\bar{x}\times\left[(\bar{x}-\bar{p})\times\bar{r}\right]=\bar{0}$, then vector $\bar{x}$ is equal to _____.

(A) $\frac{1}{2}(\bar{p}+\bar{q}-2\bar{r})$ (B) $\frac{1}{2}(\bar{p}+\bar{q}+\bar{r})$ (C) $\frac{1}{3}(\bar{p}+\bar{q}+\bar{r})$ (D) $\frac{1}{3}(2\bar{p}+\bar{q}-\bar{r})$

Ans. **(B)**

Sol. Without loss of generality, we can assume that

$\bar{p}=|\bar{p}|\,\hat{i},\ \bar{q}=|\bar{p}|\,\hat{j},\ \bar{r}=|\bar{p}|\,\hat{k}$

The equation becomes $3|\bar{p}|^2\,\bar{x}-|\bar{p}|^2\,(\bar{p}+\bar{q}+\bar{r})-|\bar{p}|^2\,\bar{x}=0$

$$\therefore\ |\bar{p}|^2\,(2\bar{x}-(\bar{p}+\bar{q}+\bar{r}))=0 \Rightarrow \bar{x}=\frac{\bar{p}+\bar{q}+\bar{r}}{2}$$

4. Let $\bar{a}$ and $\bar{b}$ be two vectors of equal magnitude of 5 units each. Let p and $\bar{p}$ be $\bar{q}$ be vectors such that $\bar{p}=\bar{a}+\bar{b}$ and $\bar{q}=\bar{a}-\bar{b}$. If $|\bar{p}\times\bar{q}| = 2\left\{\lambda-\left(\bar{a}-\bar{b}\right)^2\right\}^{\frac{1}{2}}$, then value of λ is _____.

(A) 25 (B) 125 (C) 625 (D) None of these

Ans. **(C)**

Sol. $\bar{p}\times\bar{q}=\left(\bar{a}+\bar{b}\right)\times\left(\bar{a}-\bar{b}\right)=2\left(\bar{b}\times\bar{a}\right)$

$\Rightarrow|\bar{p}\times\bar{q}|^2=\left\{|\bar{b}|^2|\bar{a}|^2-\left(\bar{b}\cdot\bar{a}\right)^2\right\}$

$\Rightarrow |\bar{p}\times\bar{q}|^2 = 4\left\{625-\left(\bar{b}\cdot\bar{a}\right)^2\right\}$

$\Rightarrow |\bar{p}\cdot\bar{q}| = 2\left\{625-\left(\bar{a}\cdot\bar{b}\right)^2\right\}^{\frac{1}{2}}$

$\Rightarrow \lambda = 625$

5. $|\bar{a}|=|\bar{b}|=|\bar{c}|=|\bar{a}+\bar{b}|=1, \bar{a}\cdot\bar{c}=0$ and $\bar{a}=\frac{\hat{i}}{\sqrt{2}}+\frac{\hat{j}}{\sqrt{2}}$, where $\bar{a}, \hat{k}$ and $\bar{b}$ are linearly dependent vectors. If for some $\bar{c}, \left|\bar{b}\times\bar{c}\right| = \left|\bar{a}\times\bar{c}\right|$ and $\bar{c}=p\hat{i}+q\hat{j}+r\hat{k}$, then $16\left(p^4+q^4+r^4\right)$ is _____.

(A) 8 (B) 21 (C) $\frac{21}{3}$ (D) 4

Ans. **(A)**

Sol. $\bar{b}=\left(\bar{b}\cdot\hat{a}\right)\hat{a}+\left(\bar{b}\cdot\hat{k}\right)\hat{k}=-\frac{1}{2}\bar{a}+\frac{\sqrt{3}}{2}\hat{k}$

Now, $\left|\bar{b}\times\bar{c}\right|=\left|\bar{a}\times\bar{c}\right| \Rightarrow \bar{c}\perp\bar{b}$ and $\bar{c}\perp\bar{a}\ \left(\bar{a}\cdot\bar{c}=0\right)$

$\therefore \bar{c}=\pm\frac{\bar{a}\times\bar{b}}{\left|\bar{a}\times\bar{b}\right|}$ (as $|\bar{c}|$ = 1)

$=\pm\left(\frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}\right)$

$\therefore 16\left(p^4+q^4+r^4\right)=8$

6. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three unit vectors such that $\left|\bar{a}+\bar{b}+\bar{c}\right|=\sqrt{3}$. If $\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{b}\times\bar{c}\right)+\left(\bar{b}\times\bar{c}\right)\cdot\left(\bar{c}\times\bar{a}\right)+\left(\bar{c}\times\bar{a}\right)\cdot\left(\bar{a}\times\bar{b}\right)=\lambda$, then the maximum value of λ is

(A) 0 (B) 1 (C) $\sqrt{3}$ (D) 2

Ans. **(A)**

Sol. $\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{b}\times\bar{c}\right)=\left(\bar{b}\cdot\bar{c}\right)\left(\bar{a}\cdot\bar{b}\right)-\bar{a}\cdot\bar{c}$

$\left(\bar{b}\times\bar{c}\right)\cdot\left(\bar{c}\times\bar{a}\right)=\left(\bar{a}\cdot\bar{c}\right)\left(\bar{b}\cdot\bar{c}\right)-\bar{a}\cdot\bar{b}$

$\left(\bar{c}\times\bar{a}\right)\cdot\left(\bar{a}\times\bar{b}\right)=\left(\bar{a}\cdot\bar{b}\right)\left(\bar{a}\cdot\bar{c}\right)-\bar{b}\cdot\bar{c}$

Given that $\left|\bar{a}+\bar{b}+\bar{c}\right|=\sqrt{3}$

$\Rightarrow\left(\bar{a}+\bar{b}+\bar{c}\right)\cdot\left(\bar{a}+\bar{b}+\bar{c}\right)=3\Rightarrow\bar{a}\cdot\bar{b}+\bar{b}\cdot\bar{c}+\bar{c}\cdot\bar{a}=0$

$\Rightarrow\lambda=\left(\bar{a}\cdot\bar{b}\right)\left(\bar{b}\cdot\bar{c}\right)+\left(\bar{b}\cdot\bar{c}\right)\left(\bar{c}\cdot\bar{a}\right)+\left(\bar{c}\cdot\bar{a}\right)\left(\bar{a}\cdot\bar{b}\right)-\bar{a}\cdot\bar{b}-\bar{b}\cdot\bar{c}-\bar{c}\cdot\bar{a}$

$\Rightarrow\lambda=\left(\bar{a}\cdot\bar{b}\right)\left(\bar{b}\cdot\bar{c}\right)+\left(\bar{b}\cdot\bar{c}\right)\left(\bar{c}\cdot\bar{a}\right)+\left(\bar{c}\cdot\bar{a}\right)\left(\bar{a}\cdot\bar{b}\right)$

$\Rightarrow\lambda\le 0$ (since x + y + z = 0, xy + yz + xz ≤ 0)

$\therefore\ \lambda_{max}=0$ only when $\bar{a}\cdot\bar{b}=\bar{b}\cdot\bar{c}=\bar{a}\cdot\bar{c}=0$

7. Let $\overline{r}$ be the position vector of a variable point in the Cartesian OXY plane such that $\overline{r}\cdot\left(10\hat{j}-8\hat{i}-\overline{r}\right)=40$ and $P_1=\max\left\{\left|\overline{r}+2\hat{i}-3\hat{j}\right|^2\right\}$, $P_2=\min\left\{\left|\overline{r}+2\hat{i}-3\hat{j}\right|^2\right\}$. P^2 is equal to _____.

(A) 9 (B) $2\sqrt{2}-1$ (C) $6\sqrt{2}+1$ (D) $9-4\sqrt{2}$

Ans. **(D)**

Sol. $\overline{r}=x\hat{i}+y\hat{j}$

$x^2+y^2+8x-10y+40=0$

$C=(-4,5), r=1$

$P_1=\max\left\{(x+2)^2+(y-3)^2\right\}$

$P_2=\min\left\{(x+2)^2+(y-3)^2\right\}$

$\therefore$ P = (–2, 3)

$CP=2\sqrt{2}$

Then $P_2=\left(2\sqrt{2}-1\right)^2$, $P_1=\left(2\sqrt{2}+1\right)^2$

8. Let $\overline{r}$ be the position vector of a variable point in the Cartesian OXY plane such that $\overline{r}\cdot\left(10\hat{j}-8\hat{i}-\overline{r}\right)=40$ and $P_1=\max\left\{\left|\overline{r}+2\hat{i}-3\hat{j}\right|^2\right\}$, $P_2=\min\left\{\left|\overline{r}+2\hat{i}-3\hat{j}\right|^2\right\}$. P_1 + P^2 is equal to ____.

(A) 2 (B) 10 (C) 18 (D) 5

Ans. **(C)**

Sol. $\overline{r}=x\hat{i}+y\hat{j}$

$x^2+y^2+8x-10y+40=0$

$C=(-4,5), r=1$

$P_1=\max\left\{(x+2)^2+(y-3)^2\right\}$

$P_2=\min\left\{(x+2)^2+(y-3)^2\right\}$

$\therefore$ P = (–2, 3)

$CP=2\sqrt{2}$

Then $P_2=\left(2\sqrt{2}-1\right)^2$, $P_1=\left(2\sqrt{2}+1\right)^2$

$P_1+P_2=18$

9. Given $|\vec{a}|=|\vec{b}|=1$ and $|\vec{a}+\vec{b}|=\sqrt{3}$. If $\vec{c}$ -is a vector such that $\vec{c}-\vec{a}-2\vec{b}=3\left(\vec{a}\times\vec{b}\right)$, then $\left(\vec{c}\cdot\vec{b}\right)$ is equal to

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

Ans. **(D)**

Sol. $\left|\vec{a}+\vec{b}\right|=\sqrt{3}$

$\Rightarrow 1+1+2\left(\vec{a}\cdot\vec{b}\right)=3$

$\Rightarrow (\vec{a}\cdot\vec{b}) = \frac{1}{2}$

Given, $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a}\times\vec{b})$ -

Taking dot products with $\vec{b}$ we get

$\vec{c}\cdot\vec{a} - \vec{a}\cdot\vec{b} - 2 = 0$

$\Rightarrow \vec{c}\cdot\vec{b} = 2 + \frac{1}{2} = \frac{5}{2}$

MCQ (One or more than one correct) :

10. If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{b} = y\hat{i} + z\hat{j} + x\hat{k}$ and $\vec{c} = z\hat{i} + x\hat{j} + y\hat{k}$, then $\vec{a}\times(\vec{b}\times\vec{c})$ is

(A) parallel to $(y - z)\ \hat{i} + (z - x)\ \hat{j} + (x - y)\ \hat{k}$ (B) orthogonal to $\hat{i} + \hat{j} + \hat{k}$

(C) orthogonal to $(y + z)\ \hat{i} + (z + x)\ \hat{j} + (x + y)\ \hat{k}$ (D) orthogonal to $x\hat{i} + y\ \hat{j} + z\hat{k}$

Ans. **(ABCD)**

Sol. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (xy + yz + zx)\ \{(y - z)\hat{i} + (z - x)\hat{j} + (x - y)\hat{k}\}$

$\therefore$ all the option are correct

11. The value(s) of $\alpha \in [0, 2\pi]$ for which vector $\vec{a} = \hat{i} + 3\hat{j} + (\sin 2\alpha)\hat{k}$ makes an obtuse angle with the z-axis and the vectors $\vec{b} = (\tan\alpha)\hat{i} - \hat{j} + 2\sqrt{\sin\frac{\alpha}{2}}\,\hat{k}$ and $\vec{c} = (\tan\alpha)\ \hat{i} + (\tan\alpha)\ \hat{j} - 3\sqrt{\operatorname{cosec}\frac{\alpha}{2}}\,\hat{k}$ are orthogonal, is/are:

(A) $\tan^{-1} 3$ (B) $\pi - \tan^{-1} 2$ (C) $\pi + \tan^{-1} 3$ (D) $2\pi - \tan^{-1} 2$

Ans. **(BD)**

Sol. Since $\vec{a}$ makes obtuse angle with z-axis

$\therefore \quad \frac{\sin 2\alpha}{\sqrt{1 + 9 + \sin^2 2\alpha}} < 0 \quad$ i.e. $\quad \sin 2\alpha < 0$

$\therefore \quad$ either $\frac{\pi}{2} < \alpha < \pi$ or $\frac{3\pi}{2} < \alpha < 2\pi$ ......(i)

since $\vec{b}$ and $\vec{c}$ are orthogonal

$\therefore \quad \tan^2\alpha - \tan\alpha - 6 = 0 \quad$ i.e. $\quad \tan\alpha = 3, -2$ ......(ii)

from (i) and (ii), we get

$\tan\alpha = -2$

$\therefore \alpha = \pi - \tan^{-1} 2 \quad$ or $\alpha = 2\pi - \tan^{-1} 2$

12. Let $\bar{a}$ and $\bar{c}$ be unit vectors such that $\bar{a}\times\bar{b} = 2\bar{a}\times\bar{c}$, where $|\bar{b}| = 4$. The angle between $\bar{a}$ and $\bar{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. If $\bar{b} - 2\bar{c} = k\bar{a}$, then k is equal to _____.

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) –4 (D) 3

Ans. **(CD)**

Sol. $\bar{a}\cdot\bar{c} = |a||c|\cos\left(\cos^{-1}\frac{1}{4}\right) \Rightarrow \bar{a}\cdot\bar{c} = \frac{1}{4}$ ...(1)

Taking dot product by $\bar{a},\ \bar{b},\ \bar{c}$ on $\bar{b}-2\bar{c}=k\bar{a}$, we have

$\bar{a}\cdot\bar{b}-2(\bar{a}\cdot\bar{c})=k(\bar{a}\cdot\bar{a})$

$\Rightarrow \bar{a}\cdot\bar{b}-\frac{1}{2}=k \quad \Rightarrow \bar{a}\cdot\bar{b}=k+\frac{1}{2}$ ....(2)

Similarly $\bar{b}\cdot\bar{c}=8-\frac{k^2}{2}-\frac{k}{4}$ .....(3) and $\bar{b}\cdot\bar{c}-2=k(\bar{a}\cdot\bar{c})$ ....(4)

From (2), (3) and (4) we get $8-\frac{k^2}{2}-\frac{k}{4}-2=k\left(\frac{k}{4}\right)$

$\Rightarrow k=3,-4$

13. If $(\vec{a}\times b)\times(\vec{c}\times\vec{d})\cdot(\vec{a}\times\vec{d})=0$, then which of the following may be true?

(A) $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are necessarily coplanar (B) $\vec{a}$ lies in the plane of $\vec{c}$ and $\vec{d}$

(C) $\vec{b}$ lies in the plane of $\vec{c}$ and $\vec{d}$ (D) $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{d}$

Ans. **(BCD)**

Sol. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\cdot(\vec{a}\times\vec{d})=0$

$\Rightarrow\left(\left[\vec{a}\vec{c}\vec{d}\right]\vec{b}-\left[\vec{b}\vec{c}\vec{d}\right]\vec{a}\right)\cdot(\vec{a}\times\vec{d})=0$

$\Rightarrow\left[\vec{a}\vec{c}\vec{d}\right]\left[\vec{b}\vec{a}\vec{d}\right]=0$

Either $\vec{c}$ or $\vec{b}$ must lie in the plane of $\vec{a}$ and $\vec{d}$

Numerical based Questions :

14. If $\vec{d}$ is a unit vector and $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors, then the value of

$$\left|\frac{(\bar{a}\cdot\bar{d})(\bar{b}\times\bar{c})+(\bar{b}\cdot\bar{d})(\bar{c}\times\bar{a})+(\bar{c}\cdot\bar{d})(\bar{a}\times\bar{b})}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}\right|$$ is equal to ______.

Ans. **(1)**

Sol. Let $\bar{d}=x(\bar{b}\times\bar{c})+y(\bar{c}\times\bar{a})+z(\bar{a}\times\bar{b})$

Taking dot product with $\bar{a},\bar{b},\bar{c}$ one by one

$\bar{a}\cdot\bar{d}=x\left[\bar{a}\ \bar{b}\ \bar{c}\right]+0+0\Rightarrow x=\frac{\bar{a}\cdot\bar{d}}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$-

Similarly $y=\frac{\bar{b}\cdot\bar{d}}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]},\ z=\frac{\bar{c}\cdot\bar{d}}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$

$$\therefore\ \bar{d}=\left|\frac{(\bar{a}\cdot\bar{d})(\bar{b}\times\bar{c})+(\bar{b}\cdot\bar{d})(\bar{c}\times\bar{a})+(\bar{c}\cdot\bar{d})(\bar{a}\times\bar{b})}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}\right|$$

$$\therefore\ \left|\frac{(\bar{a}\cdot\bar{d})(\bar{b}\times\bar{c})+(\bar{b}\cdot\bar{d})(\bar{c}\times\bar{a})+(\bar{c}\cdot\bar{d})(\bar{a}\times\bar{b})}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}\right|=1$$

15. Let $(\hat{p}\times\bar{q})\times\hat{p}+(\hat{p}\cdot\bar{q})\bar{q}=(x^2+y^2)\bar{q}+(14-4x-6y)\hat{p}$, where $\hat{p}$ and $\bar{q}$ are two non-collinear vectors (and $\hat{p}$ is a unit vector) and x, y are scalars. Find the value of (x+y).

Ans. **(5)**

Sol. $(\hat{p}\times\bar{q})\times\hat{p}+(\hat{p}\cdot\bar{q})\bar{q}=(x^2+y^2)\bar{q}+(14-4x-6y)\hat{p}$

$(\hat{p}\times\hat{p})\bar{q}-(\bar{q}\cdot\hat{p})\hat{p}+(\hat{p}\cdot\bar{q})\bar{q}=(x^2+y^2)\bar{q}+(14-4x-6y)\hat{p}$

Since $\hat{p}$ and $\bar{q}$ are non-zero non-collinear,

$(\hat{p}\cdot\hat{p}+\hat{p}\cdot\bar{q})\bar{q}=(x^2+y^2)\bar{q}$ .....(1)

$-(\bar{q}\cdot\hat{p})\hat{p}=(14-4x-6y)\hat{p}$ .....(2)

$\therefore\ 1+\hat{p}\cdot\bar{q}=x^2+y^2$ .....(3)

and $\hat{p}\cdot\bar{q}=4x+6y-14$ .....(4)

From (3) and (4) we get $x^2+y^2-4x-6y+13=0$

$(x-2)^2+(y-3)^2=0\Rightarrow x=2$ and $y=3$

16. Let $\bar{r}=(\bar{a}\times\bar{b})\sin x+(\bar{b}\times\bar{c})\cos y+(\bar{c}\times\bar{a})$, where $\bar{a},\bar{b}$ and $\bar{c}$ are non-zero non-coplanar vectors. If $\bar{r}$ is orthogonal to $3\bar{a}+5\bar{b}+2\bar{c}$, then the value of $\sec^2 y+\operatorname{cosec}^2 x+\sec y\operatorname{cosec} x$ is ______.

Ans. **(3)**

Sol. $\bar{r}\cdot(3\bar{a}+5\bar{b}+2\bar{c})=0\Rightarrow\bar{a}\cdot(\bar{b}\times\bar{c})[2\sin x+3\cos y+5]=0$

$\Rightarrow\ 2\sin x+3\cos y=-5\quad(\because\bar{a}\cdot(\bar{b}\times\bar{c})\neq0)$

$\Rightarrow\ \sin x=-1,\cos y=1$

$\Rightarrow\ \operatorname{cosec} x=-1,\ \sec y=-1$

17. Given four non zero vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$. The vectors $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar but not collinear pair by pair and vector $\vec{d}$ is not coplanar with vectors $\vec{a},\vec{b}$ and $\vec{c}$ and $(\vec{a}\,\hat{}\,\vec{b})=(\vec{b}\,\hat{}\,\vec{c})=\frac{\pi}{3}$, $(\vec{d}\,\hat{}\,\vec{a})=\alpha$ and $(\vec{d}\,\hat{}\,\vec{b})=\beta$, if $(\vec{d}\,\hat{}\,\vec{c})=\cos^{-1}(m\cos\beta+n\cos\alpha)$ then m – n is :

Ans. **2**

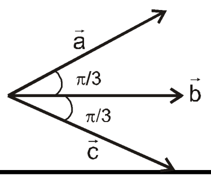
Sol. Angle between vector $\vec{a}$ & $\vec{b}$ remains same even if we presume them as unit vector. Here for sake of convinience let $\vec{a},\vec{b},\vec{c},\vec{d}$ are unit vectors.

$\vec{a}.\vec{b}=\cos\frac{\pi}{3}=\frac{1}{2}$ ....(1) ; $\vec{b}.\vec{c}=\frac{1}{2}$ ....(2)

$\vec{a}.\vec{d}=\cos\alpha$ ....(3) ; $\vec{b}.\vec{d}=\cos\beta$ ....(4)

also $\vec{b}=\lambda(\vec{a}+\vec{c})$

Since $\vec{b}$ is presumed as unit vector



$|\lambda(\vec{a}+\vec{c})|=1\quad\Rightarrow\lambda^2(\vec{a}^2+\vec{c}^2+2\vec{a}.\vec{c})=1$

or $\quad \lambda^2(1+1-1)=1 \;\Rightarrow \lambda = 1$

$\therefore\; \vec{b} = (\vec{a}+\vec{c}) \quad \Rightarrow \vec{c} = \vec{b}-\vec{a}$

again $\vec{d}.\vec{c} = \left|\vec{d}\right|\left|\vec{c}\right|\cos\theta = \vec{d}.(\vec{b}-\vec{a}) \;\Rightarrow\; \cos\theta = \cos\beta - \cos\alpha \;\Rightarrow\; \theta = \cos^{-1}(\cos\beta - \cos\alpha)$

18. Given $f^2(x) + g^2(x) + h^2(x) \le 9$ and $U(x) = 3f(x) + 4g(x) + 10h(x)$, where $f(x)$, $g(x)$ and $h(x)$ are continuous $\forall x \in R$. If maximum value of $U(x)$ is $\sqrt{N}$. Then the value of cube root of $(N-1000)$ is:

Ans. **5**

Sol. $\max\{U(x)\} = \max.\{(3\hat{i}+4\hat{j}+10\hat{k}) \cdot \left(f(x)\hat{i}+g(x)\hat{j}+h(x)\hat{k}\right)\}$

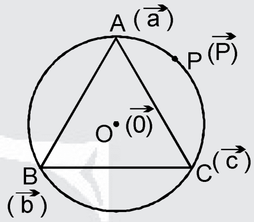
$= \max.\left\{\sqrt{125}\cdot\sqrt{f(x)^2+g(x)^2+f(x)^2}\cdot\cos\theta\right\} = \sqrt{1125}$

$\Rightarrow \quad N = 1125 \Rightarrow \quad N - 1000 = 125$

19. In an equilateral ΔABC find the value of $\dfrac{|\overrightarrow{PA}|^2+|\overrightarrow{PB}|^2+|\overrightarrow{PC}|^2}{R^2}$ where P is any arbitrary point lying on its circumcircle, is

Ans. **6**

Sol. Let O $(\vec{0})$ be the circumcentre of ΔABC



Given $(\vec{a}) = |\vec{b}| = |\vec{c}| = R \;\Rightarrow 0 = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3} \;\Rightarrow \vec{a}+\vec{b}+\vec{c} = 0$

$\dfrac{|\overrightarrow{PA}|^2+|\overrightarrow{PB}|^2+|\overrightarrow{PC}|^2}{R^2} = ? \Rightarrow \dfrac{|\vec{P}-\vec{a}|^2+|\vec{P}-\vec{b}|^2+|\vec{P}-\vec{c}|^2}{R^2}$

$\dfrac{3|\vec{P}|+|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2 - 2\vec{P}.(\vec{a}+\vec{b}+\vec{c})}{R^2};\; \dfrac{6R^2}{R^2} = 6$

20. If represents the position vector of point R in which the line AB cuts the plane CDE, where position vectors of points A, B, C, D, E are respectively $\vec{a} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+\hat{j}+2\hat{k}$, $\vec{c} = -4\hat{j}+4\hat{k}$, $\vec{d} = 2\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{e} = 4\hat{i}+\hat{j}+2\hat{k}$, then $\vec{r}^2$ is :

Ans. **18**

Sol. Equation of line AB is $\vec{r} = \vec{a}+\lambda(\vec{b}-\vec{a}) \;\Rightarrow \vec{r} = (\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ ....(1)

$\overrightarrow{CD} = 2\hat{i}+2\hat{j}-2\hat{k}$; $\quad \overrightarrow{CE} = 4\hat{i}+5\hat{j}-2\hat{k}$; $\quad \vec{n} = \overrightarrow{CD}\times\overrightarrow{CE} = 6\hat{i}-4\hat{j}+2\hat{k}$

So equation of plane CDE is $3x - 2y + z = 12$. Solve with line $\vec{r} = (\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$

$3(1+\lambda)-2(2-\lambda)+1+\lambda = 12$

$\Rightarrow \lambda = 2$

Hence R is $3\hat{i}+3\hat{k}$

21. Line L_1 is parallel to vector $\vec{\alpha} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and passes through a point A(7, 6, 2) and line L_2 is parallel to a vector $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ and passes through a point B(5, 3, 4). Now a line L_3 parallel to a vector $\vec{r} = 2\hat{i} - 2\hat{j} - \hat{k}$ intersects the lines L_1 and L_2 at points C and D respectively, then $|4\overrightarrow{CD}|$ is equal to :

Ans. **36**

Sol. Equation of line L_1 is $7\hat{i} + 6\hat{j} + 2\hat{k} + \lambda\ (-3\hat{i} + 2\hat{j} + 4\hat{k})$

Equation of line L_2 is $5\hat{i} + 3\hat{j} + 4\hat{k} + \mu\ (2\hat{i} + \hat{j} + 3\hat{k})$

$\overrightarrow{CD} = 2\hat{i} + 3\hat{j} - 2\hat{k} + \lambda\ (-3\hat{i} + 2\hat{j} + 4\hat{k})\ .\ \mu(2\hat{i} + \hat{j} + 3\hat{k})$. since it is parallel to $2\hat{i} - 2\hat{j} - \hat{k}$

$\therefore \quad \dfrac{2 - 3\lambda - 2\mu}{2} = \dfrac{3 + 2\lambda - \mu}{-2} = \dfrac{-2 + 4\lambda - 3\mu}{-1} \qquad \therefore \quad \lambda = 2, \quad \mu = 1$

$\therefore \quad \overrightarrow{CD} = -6\hat{i} + 6\hat{j} + 3\hat{k} \qquad \therefore \quad |4\overline{CD}| = 36$

Matrix Match Type :

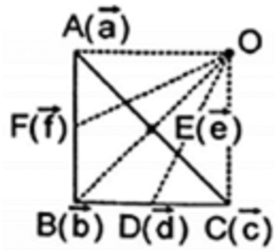
22. Match the following.

Column – I	Column – II
(A) If D, E and F are the mid points of the sides BC, CA and AB respectively of a triangle ABC and λ is a scalar such that $\overline{AD} + \frac{2}{3}\overline{BE} + \frac{1}{3}\overline{CF} - \lambda\overline{AC}$, then λ is equal to	(p) 0
(B) If $\bar{A}, \bar{B}$ and $\bar{C}$ are vectors such that $\|\bar{B}\| = \|\bar{C}\|$, then the value of $\left(\left(\left(\bar{A}+\bar{B}\right)\times\left(\bar{A}+\bar{C}\right)\right)\times\left(\bar{B}\times\bar{C}\right)\right)\cdot\left(\bar{B}+\bar{C}\right)$ is	(q) $\frac{1}{3}$
(C) In a ΔABC, points D, E and F are taken on the sides BC, CA and AB respectively such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2$. If area($\Delta$DEF) $=\lambda$ area(ΔABC) , then λ is equal to	(r) $\frac{1}{2}$
(D) The corner P of the square OPQR is folded up so that the plane OPQ is perpendicular to the plane OQR. If θ is angle between OP and QR, then $\|\cos\theta\|$ is equal to	(s) $\frac{3}{5}$
	(t) $\frac{3}{4}$

(A) A–p; B–r; C–q; D–s
(B) A–r; B–p; C–s; D–r
(C) A–s; B–q; C–p; D–r
(D) A–r; B–p; C–q; D–r

Ans. **(D)**

Sol. (A) Consider the point O as origin,



Here, $\overline{AD}+\frac{2}{3}\overline{BE}+\frac{1}{3}\overline{CF}$

$$=\left(\bar{d}-\bar{a}\right)+\frac{2}{3}\left(\bar{e}-\bar{b}\right)+\frac{1}{3}\left(\bar{f}-\bar{c}\right)$$

$$=\frac{1}{2}(\bar{c}-\bar{a})$$

$$=\frac{1}{2}\left(\overline{AC}\right)$$

(B) Let $\overline{R_1}=\bar{A}+\bar{B},\ \overline{R_2}=\bar{B}+\bar{C}$ and $\overline{R_3}=\bar{C}+\bar{A}$

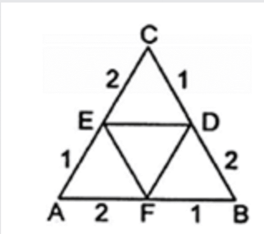
LHS $=\left(\left(\overline{R_1}\times\overline{R_2}\right)\times\left(\bar{B}\times\bar{C}\right)\right)\cdot\overline{R_3}$

$$=\left(\left(\overline{R_1}\cdot\left(\bar{B}\times\bar{C}\right)\right)\overline{R_2}-\left(\overline{R_2}\cdot\left(\bar{B}\times\bar{C}\right)\right)\overline{R_1}\right)\cdot\overline{R_3}$$

$$=\left[\bar{A}\ \bar{B}\ \bar{C}\right]\left(|\bar{C}|^2-|\bar{B}|^2\right)$$

$= 0$

(C)



Take A as the origin and let the position vectors of the points B and C be $\bar{b}$ and $\bar{c}$ respectively.

$\therefore$ The position vectors of D, E and F are

$\frac{2\bar{c}+\bar{b}}{3},\frac{\bar{c}}{3},\frac{2\bar{b}}{3}$ respectively

$\overline{FD}=\overline{AD}-\overline{AD}=\frac{2\bar{c}-\bar{b}}{3}$ and $\overline{EF}=\overline{AF}-\overline{AE}=\frac{2\bar{b}-\bar{c}}{3}$

Now area $(\Delta ABC)=\frac{1}{2}\left|\bar{b}\times\bar{c}\right|$

and the area of $\Delta ABC=\frac{1}{2}\left|\overline{FD}\times\overline{FE}\right|=\frac{1}{2(3)^2}\left|\left(2\bar{b}-c\right)\times\left(2\bar{c}-\bar{b}\right)\right|$

$\therefore$ area of $\Delta ABC=\frac{3}{(3)^2}$ area $(\Delta ABC)=\frac{1}{3}\left(\text{area}\left(\Delta ABC\right)\right)$

(D)

After folding OPQ, PS $\perp$ SR

Here SQ $\perp$ SR, SQ $\perp$ PS

Let $\overline{SR} = \frac{a}{\sqrt{2}}\hat{k},\ \overline{SQ} = \frac{a}{\sqrt{2}}\hat{i},\ \overline{SP} = \frac{a}{\sqrt{2}}\hat{j}$

$$\overline{OP} = -\overline{SO} + \overline{SP} = \frac{a}{\sqrt{2}}\hat{j} + \frac{a}{\sqrt{2}}\hat{i} + \frac{a}{\sqrt{2}}\left(\hat{i} + \hat{j}\right)$$

$$\overline{QR} = \frac{a}{\sqrt{2}}\left(\hat{k} + \hat{i}\right)$$

$\overline{OP} = a,\ \left|\overline{QR}\right| = a$

Cosine of angle between $\overline{OP}$ and $\overline{QR} = \dfrac{\overline{OP}\cdot\overline{QR}}{\left|\overline{OP}\right|\left|\overline{QR}\right|}$

$\cos\theta = -\frac{1}{2}$ $\qquad\qquad \therefore\ \theta = \frac{2\pi}{3}$

Subjective based Questions :

23. Let ABC be a triangle.Points M, N and P are taken on the sides AB, BC and CA respectively such that $\frac{AM}{AB} = \frac{BN}{BC} = \frac{CP}{CA} = \lambda$. Prove that the vectors $\overrightarrow{AN}$, $\overrightarrow{BP}$ and $\overrightarrow{CM}$ form a triangle. Also find λ for which the area of the triangle formed by these vectors is the least.

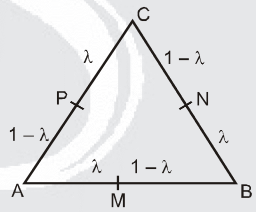
Ans. $\lambda = \frac{1}{2}$

Sol. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively.

$\overrightarrow{OM} = \lambda\vec{b} + (1-\lambda)\vec{a}$, $\overrightarrow{ON} = \lambda\vec{c} + (1-\lambda)\,\vec{b}$, $\overrightarrow{OP} = \lambda\vec{a} + (1-\lambda)\,\vec{c}$

$\therefore \qquad \overrightarrow{AN} = \lambda\vec{c} + (1-\lambda)\,\vec{b} - \vec{a}$

$\qquad \overrightarrow{BP} = \lambda\vec{a} + (1-\lambda)\,\vec{c} - \vec{b}$ and $\overrightarrow{CM} = \lambda\vec{b} + (1-\lambda)\,\vec{a} - \vec{c}$



Clearly $\overrightarrow{AN} + \overrightarrow{BP} + \overrightarrow{CM} = \vec{0}$

$\therefore \qquad \overrightarrow{AN}, \overrightarrow{BP}, \overrightarrow{CM}$ form a triangle

Area of triangle formed by the vectors $\overrightarrow{AN}, \overrightarrow{BP}, \overrightarrow{CM}$ is

$$= \frac{1}{2}\,|\,(\lambda\vec{c} + (1-\lambda)\vec{b} - \vec{a}) \times (\lambda\vec{a} + (1-\lambda)\,\vec{c} - \vec{b})\,|$$

$$= \frac{1}{2}\,|\lambda^2 - \lambda + 1|\ \ (\Delta\,ABC)$$

is least if $|\lambda^2 - \lambda + 1|$ is least

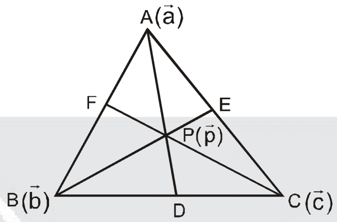
Which is when $\lambda = \frac{1}{2}$

24. Let P be an interior point of a triangle ABC and AP, BP, CP meet the sides BC, CA, AB is D, E, F respectively. Show that

$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$

Sol. Since A, B, C, P are co-planar, there exists 4 scalars x, y, z, w not all zero simultaneously such that

$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{p} = 0$



where x + y + z + w = 0

Also, $\frac{x\vec{a} + w\vec{p}}{x + w} = \frac{y\vec{b} + z\vec{c}}{y + z}$

Hence $\frac{AP}{PD} = -\frac{w}{x} - 1$

Also $\frac{x\vec{a} + y\vec{b}}{x + y} = \frac{z\vec{c} + w\vec{p}}{z + w}$ $\Rightarrow \frac{AF}{FB} = \frac{y}{x}$

Similarly $\frac{AE}{EC} = \frac{z}{x}$

Thus to show that $-\frac{w}{x} - 1 = \frac{y}{x} + \frac{z}{x}$ $\Rightarrow$ x + y + z + w = 0 which is true.

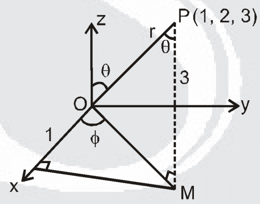
Hence proved.

25. Let PM be the perpendicular from the point P(1, 2, 3) to the x-y plane. If OP makes an angle θ with the positive direction of the z-axis and OM makes an angle ϕ with the positive direction of the x-axis, where O is the origin, then find θ and ϕ

Ans. $\theta = \cos^{-1}\frac{3}{\sqrt{14}}$ and $\phi = \cos^{-1}\frac{1}{\sqrt{5}}$

Sol. OP = $\sqrt{14}$

3 = OP cos θ



$\therefore$ $\cos\theta = \frac{3}{\sqrt{14}}$ $\Rightarrow$ $\theta = \cos^{-1}\frac{3}{\sqrt{14}}$

OM = OP sin θ = $\sqrt{5}$

1 = OM cos ϕ = $\sqrt{5}$ cos ϕ

$\therefore$ $\cos\phi = \frac{1}{\sqrt{5}}$ $\Rightarrow \phi = \cos^{-1}\frac{1}{\sqrt{5}}$