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# MATHEMATICS

TARGET : JEE- Advanced 2021

# CAPS-2

## Quadratic Equation

**ANSWERKEY (CAPS-2)**

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1.** | (D) | **2.** | (A) | **3.** | (A) | **4.** | (B) | **5.** | (A) | **6.** | (A) | **7.** | (B) |
| **8.** | (ABD) | **9.** | (AC) | **10.** | (AC) | **11.** | (C) | **12.** | (B) | **13.** | (2.00) | **14.** | (0.00) |
| **15.** | (8) | **16.** | (1) | **17.** | (1) | **18.** | (9) | **19.** | [1, 2) | **20.** | (±2) | **21.** | (0) |
| **22.** | (A) → q, (B) → (p,r,s), (C) → (r), (D)→(s) | | | | | | | | | | | | |

**SCQ (Single Correct Type) :**

**1.** If $\alpha, \beta$ be the roots of $x^2 + x + 2 = 0$ and $\gamma, \delta$ be the roots of $x^2 + 3x + 4 = 0$, then $(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)$ is equal to

(A) – 18 (B) 18 (C) 24 (D) 44

**Ans.** **(D)**

**Sol.** We have $\alpha + \beta = -1;\ \alpha\beta = 2,\ \gamma + \delta = -3,\ \gamma\delta = 4$

Now $(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta) = (\alpha^2 - 3\alpha + 4)(\beta^2 - 3\beta + 4)$....(1)

But $\alpha^2 + \alpha + 2 = 0$ and $\beta^2 + \beta + 2 = 0$

$\therefore\quad (\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta) = (-\alpha - 2 - 3\alpha + 4)(-\beta - 2 - 3\beta + 4) = (2 - 4\alpha)(2 - 4\beta)$

$= 4 - 8(\alpha + \beta) + 16\alpha\beta = 4 + 8 + 32 = 44$ **Ans.**

**2.** The number of value of k for which $(x^2 - (k - 2)x - 2k)(x^2 + kx + 2k - 4)$ is a perfect square is

(A) 1 (B) 2 (C) 2 (D) 0

**Ans.** **(A)**

**Sol.** Let the expression is of the form $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

It can be a perfect square if either both expression are simultaneously perfect squares of both roots common, i.e., $(k - 2)^2 + 8k = 0$ and $k^2 - 4(2k - 4) = 0$ or $\frac{2-k}{k} = \frac{-2k}{2k-4}$ Solving we get k can take only value

**3.** The value of 'a' for which the quadratic expression $ax^2 + |2a - 3|x - 6$ is positive for exactly three integral values of x is

(A) $\left(-\frac{3}{5}, -\frac{1}{2}\right]$ (B) $\left[-\frac{3}{5}, -\frac{1}{2}\right]$ (C) $\left[-\frac{3}{5}, -\frac{1}{6}\right]$ (D) None of these

**Ans.** **(A)**

**Sol.** Since expression is positive only three integral values of x, therefore parabola $y = ax^2 + |2a - 3|x - 6$ will open downward

$\Rightarrow\quad a < 0$ and $y = ax^2 + (3 - 2a)x - 6$

or $\quad y = (ax + 3)(x - 2)$

$\Rightarrow\quad 5 < -\frac{3}{a} \le 6 \quad \Rightarrow \quad -\frac{3}{5} < a \le -\frac{1}{2}$

**4.** Consider the quadratic equation $ax^2 - bx + c = 0$, a, b, $c \in N$. If the given equation has two distinct real roots belonging to (1,2) then

(A) $1 < a < 5$ (B) $a \geq 5$ (C) $a = 4$ (D) $a = 3$

**Ans.** **(B)**

**Sol.** $\alpha + \beta = \frac{b}{a}, \alpha\beta = \frac{c}{a}, \alpha, \beta \in (1, 2)$

$\Rightarrow$ a – 1, $\beta - 1$, $2 - \alpha$, $2 - \beta \in (0, 1)$

Apply AM $\geq$ GM

$$\frac{(\alpha-1)+(2-\alpha)}{2} > \sqrt{(\alpha-1)(2-\alpha)}$$

And $\frac{(\beta-1)+(2-\beta)}{2} > \sqrt{(\beta-1)(2-\beta)}$

$\Rightarrow$ $(a - 1)(2 - \alpha) < \frac{1}{4}$ and $(\beta - 1)(2 - \beta) < \frac{1}{4}$

$(\alpha - 1)(\beta - 1)(2 - \alpha)(2 - \beta) < \frac{1}{16}$

$\therefore$ $(\alpha - 1), (2 - \alpha), (\beta - 1), (2 - \beta) > 0$

$\Rightarrow$ $0 < (\alpha - 1)(2 - \alpha)(\beta - 1)(2 - \beta) < \frac{1}{16}$

$\Rightarrow$ $0 < \left(\frac{c}{a} - \frac{b}{a} + 1\right)\left(4 - \frac{2b}{a} + \frac{c}{a}\right) < \frac{1}{16}$

$(a - b + c)(4a - 2b + c) \geq 1$ $(\because a, b, c \in N)$

From (1), $\frac{a^2}{16} > 1$ $\Rightarrow a \geq 5$

**5.** The set of values of 'a ' for which in-equation $(a - 1)x^2 - (a + 1)x + (a - 1) \geq 0$ is true for all $x \geq 2$ is

(A) $\left(\frac{7}{3}, \infty\right)$ (B) $\left(1, \frac{7}{3}\right]$ (C) $(-\infty, 1)$ (D) $(-3, -2)$

**Ans.** **(A)**

**Sol.** $(a-1)x^2 - (a+1)x + (a-1) \geq 0$

$a(x^2 - x + 1) - (x^2 + x + 1) \geq 0$

$$a \geq \frac{x^2+x+1}{x^2-x+1} = 1 + \frac{2x}{x^2-x+1} = 1 + \frac{2}{x + \frac{1}{x} - 1} \Rightarrow a \in \left(1, \frac{7}{3}\right)$$

**6.** Let $a_m$ (m = 1, 2, 3, ...p) be the possible integral values of 'a' for which the graphs of $f(x) = ax^2 + 2bx + b$ and $g(x) = 5x^2 - 3bx - a$ meets at some point for all real values of b.

Let $T_r = \prod_{m=1}^{p}(r - a_m)$ and $S_n = \sum_{r=1}^{n} T_r (n \in N)$

sum of all the possible values of 'n' for which $T_n$ vanishes, is ____

(A) 10 (B) 15 (C) 21 (D) 20

**Ans.** **(A)**

**Sol.** $ax^2 + 2bx + b = 5x^2 - 3bx - a$

$\Rightarrow \quad (a-5)x^2 + 5bx + b + a = 0$

Since, this equation has real solutions $\forall b \in R$

Hence, $a - 5 \neq 0 \Rightarrow a \neq 5$

and $\quad D \geq 0 \quad \forall b \in R$

$\Rightarrow \quad 25b^2 - 4(a-5)(b+a) \geq 0 \quad \forall b \in R$

$\Rightarrow \quad 25b^2 - 4(a-5)b - 4a(a-5) \geq 0 \quad \forall b \in R$

$\Rightarrow$ Its discriminant $\leq 0$

$16(a-5)^2 + 4(25) \times 4a(a-5) \leq 0$

$\Rightarrow \quad 16(a-5)[a - 5 + 25a] \leq 0$

$\Rightarrow \quad (a-5)(26a-5) \leq 0$

$\Rightarrow \quad a \in \left[\frac{5}{25}, 5\right) \quad (\because a \neq 5)$

So, integral values of a are 1, 2, 3, 4.

Now, $T_r = \prod_{m=1}^{p} (r - a_m)$

$\Rightarrow \quad T_r = (r-1)(r-2)(r-3)(r-4)$

$S_n = \sum_{r=1}^{n} T_r$

$S_n = \sum_{r=1}^{n} (r-1)(r-2)(r-3)(r-4)$

Clearly $S_n$ vanishes for n =1, 2, 3, 4

Hence, required sum =1+ 2 + 3 + 4 =10

**7.** If p, q, r, s $\in$ R, then equaton $(x^2 + px + 3q)(-x^2 + rx + q)(-x^2 + sx - 2q) = 0$ has

(A) 6 real roots
(B) at least two real roots
(C) 2 real and 4 imaginary roots
(D) 4 real and 2 imaginary roots

**Ans.** **(B)**

**Sol.** Dis. of $x^2 + px + 3q$ is $p^2 - 12q \equiv D_1$

Dis. of $-x^2 + rx + q$ is $r^2 + 4q \equiv D_2$

Dis. of $-x^2 + sx - 2q$ is $s^2 - 8q \equiv D_3$

**Case 1** : If $q < 0$, then $D_1 > 0$, $D_3 > 0$ and $D_2$ may or may not be positive

**Case 2** : If $q > 0$, then $D_2 > 0$ and $D_1$, $D_3$ may or may not be positive

**Case 3** : If $q = 0$, then $D_1 \geq 0$, $D_2 \geq 0$ and $D_3 \geq 0$

from **Case 1**, **Case 2** and **Case 3** we can say that the given equation has at least two real roots.

**MCQ (One or more than one correct) :**

**8.** If $|ax^2 + bx + 1| \leq 1$ for all x in [0, 1] then

(A) $|a| \leq 8$
(B) $|b| > 8$
(C) $b > 0$
(D) $|a| + |b| \leq 16$

**Ans.** **(ABD)**

**Sol.** Putting x = 1, and ½

$-2 \leq a + b \leq 0$ and $-8 \leq a + 2b \leq 0$

$-4 \leq 2a + 2b \leq 0$ and $0 \leq -a - 2b \leq 8$

By solving we get

$-4 \leq a \leq 8$ and $-8 \leq b \leq 2$

**9.** The set 'S' of all real 'x' for which $(x^2 - x + 1)^{x-1} < 1$ contains

(A) (–5, –1) (B) (–1, 1) (C) (–1, 0) (D) (–3, 1)

**Ans.** **(AC)**

**Sol.** $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \quad \forall x \in R$

$\left(x^2 - x + 1\right)^{x-1} < 1 \qquad \Rightarrow (x - 1) \log(x^2 - x + 1) < 0$

**Case 1 :** $x \in (0, 1)$

**Case 2 :** $x - 1 < 0 \qquad \Rightarrow \log(x^2 - x + 1) > 0$

$x \in (-\infty, 1) \cap x \in (-\infty, 0) \cup (1, \infty)$

$x \in (-\infty, 0)$

**10.** Let $\Delta^2$ be the discriminant and $\alpha, \beta$ be the roots of the equation $ax^2 + bx + c = 0$. Then, $2a\alpha + \Delta$ and $2a\beta - \Delta$ can be the roots of the equation

(A) $x^2 + 2bx + b^2 = 0$ (B) $x^2 - 2bx + b^2 = 0$

(C) $x^2 + 2bx - 3b^2 + 16ac = 0$ (D) $x^2 - 2bx - 3b^2 + 16ac = 0$

**Ans.** **(AC)**

**Sol.** Since $\alpha, \beta = \frac{-b \pm \Delta}{2a}$ and $\Delta^2 = b^2 - 4ac$

either $2a\alpha + \Delta = -b$ and $2a\beta - \Delta = -b$

or $2a\alpha + \Delta = -b + 2\Delta$ and $2a\beta - \Delta = -b - 2\Delta$

$\therefore$ Sum of roots of required equation = $-2b$

And product of roots = $b^2$ or $b^2 - 4\Delta^2 = -3b^2 + 16ac$

$\therefore$ Required equation is

either $x^2 + 2bx + b^2 = 0$ or $x^2 + 2bx - 3b^2 + 16ac = 0$

**Comprehension Type Question:**

**Comprehension # 1**

If $\alpha, \beta, \gamma$ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$. To obtain the equation whose roots are $f(\alpha), f(\beta), f(\gamma)$, where f is a function, we put $y = f(\alpha)$ and simplify it to obtain $\alpha = g(y)$ (some function of y). Now $\alpha$ is a root of the equation $ax^3 + cx + d = 0$, then we obtain the desired equation which is $a\{g(y)\}^3 + b\{g(y)\}^2 + c\{g(y)\} + d = 0$

For example, if $\alpha, \beta, \gamma$ are the roots of $ax^3 + bx^2 + cx + d = 0$. To find equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ we put $y = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{\gamma}$

As $\alpha$ is a root of $ax^3 + bx^2 + cx + d = 0$

we get $\frac{a}{y^3} + \frac{b}{y^2} + \frac{c}{y} + d = 0 \quad \Rightarrow dy^3 + cy^2 + by + a = 0$

This is desired equation.

On the basis of above information, answer the following questions :

**11.** If $\alpha, \beta$ are the roots of the equation $ax^2 + bx + c = 0$, then the roots of the equation $a(2x + 1)^2 + b(2x + 1)(x - 1) + c(x - 1)^2 = 0$ are-

(A) $\frac{2\alpha+1}{\alpha-1}, \frac{2\beta+1}{\beta-1}$ (B) $\frac{2\alpha-1}{\alpha+1}, \frac{2\beta-1}{\beta+1}$ (C) $\frac{\alpha+1}{\alpha-2}, \frac{\beta+1}{\beta-2}$ (D) $\frac{2\alpha+3}{\alpha-1}, \frac{2\beta+3}{\beta-1}$

**Ans.** **(C)**

**Sol.** $a\left(\frac{2x+1}{x-1}\right)^2 + b\left(\frac{2x+1}{x-1}\right) + c = 0$

Let $\frac{2x+1}{x-1} = a$

$\Rightarrow \quad x\alpha - \alpha = 2x + 1$

$\Rightarrow \quad x(\alpha - 2) = 1 + \alpha \quad \Rightarrow \quad x = \frac{1+\alpha}{\alpha - 2}$

Similarly other root can be find out.

**12.** If $\alpha, \beta$ are the roots of the equation $2x^2 + 4x - 5 = 0$, the equation whose roots are the reciprocals of $2\alpha - 3$ and $2\beta - 3$ is -

(A) $x^2 + 10x - 11 = 0$
(B) $11x^2 + 10x + 1 = 0$
(C) $x^2 + 10x + 11 = 0$
(D) $11x^2 - 10x + 1 = 0$

**Ans.** **(B)**

**Sol.** $\frac{1}{2\alpha - 3} = x \quad \Rightarrow 2\alpha - 3 = \frac{1}{x} \quad \Rightarrow \alpha = \frac{1}{2}\left(\frac{1}{x} + 3\right)$

$\Rightarrow \quad 2\left(\frac{1+3x}{2x}\right)^2 + 4\left(\frac{1+3x}{2x}\right) - 5 = 0$

$\Rightarrow \quad 11x^2 + 10x + 1 = 0.$

**Numerical based Questions :**

**13.** If $X^2 - 2P_sX + s = 0$, $s = 1, 2, 3$ are three equations of which each pair has exactly one root common and no root is common to all three equations, then the number of solutions of the triplet $(p_1, p_2, p_3)$ is

**Ans.** **(2.00)**

**Sol.** $\alpha + \beta = 2p_1$ ; $\alpha\beta = 1$

$\beta + \gamma = 2p_2$ ; $\beta\gamma = 2$

$\gamma + \alpha = 2p_3$ ; $\gamma\alpha = 3$

From here we get two sets of $\alpha, \beta, \gamma$

i.e. $(P_1, P_2, P_3)$

**14.** If $\alpha$ and $\beta$ are the roots of the equation $x^2 - a(x + 1) - b = 0$, where $a, b \in R-\{0\}$ and $(a + b) \neq 0$, then the value of $\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} - \frac{2}{a+b}$ is equal to

**Ans.** **(0.00)**

**Sol.** As $\alpha^2 - a\alpha - a - b = 0$

$\Rightarrow \quad \frac{1}{\alpha^2 - a\alpha} = \frac{1}{a+b} = \frac{1}{\beta^2 - a\beta}$

**15.** Find product of all real values of x satisfying $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$

**Ans.** **(8)**

**Sol.** $\left(5+2\sqrt{6}\right)^{x^2-3} + \dfrac{1}{\left(5+2\sqrt{6}\right)^{x^2-3}} = 10 \quad \Rightarrow \quad t + \dfrac{1}{t} = 10 \quad \Rightarrow \quad t^2 - 10t + 1 = 0$

$$t = \frac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6} \quad \Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6}) \text{ or } \frac{1}{5+2\sqrt{6}}$$

$\Rightarrow \quad x^2 - 3 = 1 \quad \text{or} \quad x^2 - 3 = -1$

$\Rightarrow \quad x = 2 \text{ or } -2 \quad \text{or} \quad -\sqrt{2} \text{ or } \sqrt{2}$

Product 8

**16.** If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$. Then find the value of $(a - c)(b - c)(a + d)(b + d)/(q^2 - p^2)$.

**Ans.** **(1)**

**Sol.** $x^2 + px + 1 = 0 \begin{matrix} a \\ b \end{matrix} \quad a + b = -p,\ ab = 1 \quad ;\ x^2 + qx + 1 = 0 \begin{matrix} c \\ d \end{matrix} \quad c + d = -q,\ cd = 1$

$a + b = -p,\ ab = 1 \Rightarrow c + d = -q,\ cd = 1$

$\text{RHS} = (a - c)(b - c)(a + d)(b + d) = (ab - ac - bc + c^2)(ab + ad + bd + d^2)$

$= (1 - ac - bc + c^2)(1 + ad + bd + d^2)$

$= 1 + ad + bd + d^2 - ac - a^2cd - abcd - acd^2 - bc - abcd - b^2cd$

$\qquad - bcd^2 + c^2 + adc^2 + bdc^2 + c^2d^2$

$= 1 + ad + bd + d^2 - ac - a^2 - 1 - ad - bc - 1 - b^2 - bd + c^2 + ac + bc + 1 \quad [\because\ ab = cd = 1]$

$= c^2 + d^2 - a^2 - b^2 = (c + d)^2 - 2cd - (a + b)^2 + 2ab = q^2 - 2 - p^2 + 2 = q^2 - p^2 = \text{LHS}.$

**Proved.**

**Aliter :**

$\text{RHS} = (ab - c(a + b) + c^2)(ab + d(ab + d(a + b) + d^2) = (c^2 + pc + 1)(1 - pd + d^2) \quad ...(1)$

Since c & d are the roots of the equation $x^2 + qx + 1 = 0$

$\therefore\ c^2 + qc + 1 = 0 \Rightarrow c^2 + 1 = -qc \ \&\ d^2 + qd + 1 = 0 \Rightarrow d^2 + 1 = -qd.$

$\therefore$ (i) Becomes $= (pc - qc)(-pd - qd) = c(p - q)(-d)(p + q) = -cd(p^2 - q^2)$

$= cd(q^2 - p^2) = q^2 - p^2 = \text{LHS}.$ **Proved.**

**17.** If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$, where K, L & M are real numbers, then the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is – n. Find the value of n.

**Ans.** **(1)**

**Sol.** $x^4 - Kx^3 + Kx^2 + Lx + M = 0 \begin{matrix} \alpha \\ \beta \\ \gamma \\ \delta \end{matrix} \quad \Rightarrow \quad \sum \alpha = K,\ \sum \alpha\beta = K,\ \sum \alpha\beta\gamma = -L$

$\alpha\beta\gamma\delta = M \quad \Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2\sum \alpha\beta$

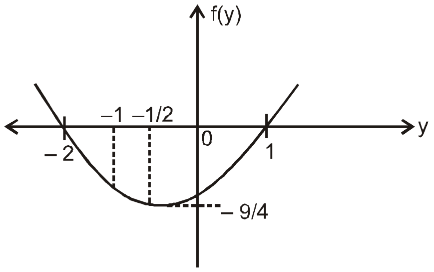
$K^2 - 2K = (K - 1)^2 - 1 \Rightarrow \quad (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)_{min} = -1$

**18.** Consider $y = \dfrac{2x}{1 + x^2}$, where x is real , then the range of expression $y^2 + y - 2$ is [a, b].

Find b – 4a.

**Ans.** **(9)**

**Sol.** $y = \dfrac{2x}{1+x^2} \quad \Rightarrow \quad x^2y - 2x + y = 0 \quad \forall \ x \in R$




$D \geq 0$

$4 - 4y^2 \geq 0 \quad \Rightarrow \quad y \in [-1, 1]$

Now $\quad f(y) = y^2 + y - 2$

$\Rightarrow \quad f(y) \in \left[-\dfrac{9}{4}, 0\right] \ \Rightarrow a = \dfrac{-9}{4}$ , $b = 0 \Rightarrow b - 4a = 0 - 4\left(\dfrac{-9}{4}\right) = 9$ . **Ans.**

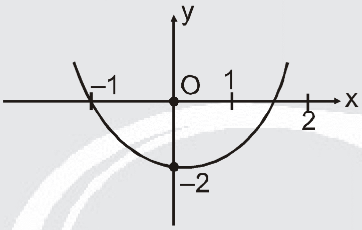
**19.** Find the values of a, for which the quadratic expression $ax^2 + (a - 2)x - 2$ is negative for exactly two integral values of x.

**Ans.** **[1, 2)**

**Sol.** Let $\quad f(x) = ax^2 + (a - 2)x - 2$

$\because \quad f(0) = -2$

and $\quad f(-1) = 0$




Since the quadratic expression is negative for exactly two integral values

$\Rightarrow \quad f(1) < 0 \quad$ and $\quad f(2) \geq 0$

$\Rightarrow \quad a + a - 2 - 2 < 0 \quad$ and $\quad 4a + 2a - 4 - 2 \geq 0$

$\Rightarrow \quad a < 2 \quad$ and $\quad a \geq 1$

$\therefore \quad a \in [1, 2)$

**20.** If $\alpha$, $\beta$ are roots of the equation $x^2 - 34x + 1 = 0$, evaluate $\sqrt[4]{\alpha} - \sqrt[4]{\beta}$

**Ans.** **(±2)**

**Sol.** $\because \quad \alpha, \beta$ are the roots of $x^2 - 34x + 1 = 0$

$\Rightarrow \quad \alpha + \beta = 34$ and $\alpha\beta = 1$

$\because \quad \left(\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}\right)^2 = \sqrt{\alpha} + \sqrt{\beta} - 2(\alpha\beta)^{1/4} \quad \because \quad \alpha\beta = 1$

$\therefore \quad \left(\alpha^{\frac{1}{4}} - \beta^{\frac{1}{4}}\right)^2 = \sqrt{\alpha} + \sqrt{\beta} - 2 \quad$ ........(1)

$\because \quad \left(\sqrt{\alpha} + \sqrt{\beta}\right)^2 = \alpha + \beta + 2\sqrt{\alpha\beta} \quad \because \quad \alpha + \beta = 34$ and $\alpha\beta = 1$

$\therefore \quad \left(\sqrt{\alpha}+\sqrt{\beta}\right)^2 = 36 \qquad \because$ we consider the principal value

$\therefore \quad \sqrt{\alpha}+\sqrt{\beta} = 6$ put in (1), we get.

$\left(\alpha^{\frac{1}{4}}-\beta^{\frac{1}{4}}\right)^2 = 4 \qquad \therefore \quad \alpha^{\frac{1}{4}}-\beta^{\frac{1}{4}} = \pm 2$ **Ans.**

**21.** Find the number of real roots of $\left(x+\frac{1}{x}\right)^3 + \left(x+\frac{1}{x}\right) = 0$

**Ans.** **(0)**

**Sol.** $\left(x+\frac{1}{x}\right)^3 + \left(x+\frac{1}{x}\right) = 0$

$\Rightarrow \quad \left(x+\frac{1}{x}\right)\left(\left(x+\frac{1}{x}\right)^2+1\right) = 0$

$\Rightarrow \quad \left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}+3\right) = 0$

$\Rightarrow$ No real value of x.

$\therefore$ Number of real roots = 0

**Matrix Match Type :**

**22.** Consider the equation $x^2 + 2(a - 1)x + a + 5 = 0$, where 'a' is parameter. Match of the real values of 'a' so that the given equation has

| | **Column-I** | | **Column-II** |
|---|---|---|---|
| A | imaginary roots | p | $\left(-\infty, -\frac{8}{7}\right)$ |
| B | one root smaller than 3 and other root greater than 3 | q | (–1, 4) |
| C | exactly one root in the interval (1, 3) & 1 and 3 are not be root of the equation | r | $\left(-\frac{4}{3}, -\frac{8}{7}\right)$ |
| D | one root smaller than 1 and other root greater than 3 | s | $\left(-\infty, -\frac{4}{3}\right)$ |



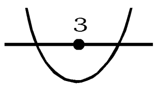
**Ans.** **(A) → q, (B) → (p,r,s), (C) → (r), (D)→(s)**

**Sol.** $x^2 + 2(a - 1)x + a + 5 = 0$

$D = 4(a - 1)^2 - 4(a + 5)$

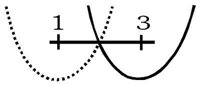
$= (a^2 - 3a - 4) = 4(a - 4)(a + 1)$

(A) $D < 0 \Rightarrow a \in (-1, 4)$

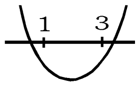
(B) 

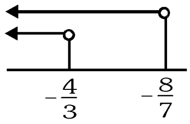
$\Rightarrow \quad f(3) < 0$

$\Rightarrow 7a + 8 < 0 \Rightarrow \quad a \in \left(-\infty, -\frac{8}{7}\right)$

(C)  $\Rightarrow f(1)\,.\,f(3) < 0$

$(3a + 4)(7a + 8) < 0 \quad a \in \left(-\frac{4}{3}, -\frac{8}{7}\right)$

(D)  $\Rightarrow f(1) < 0$ & $f(3) < 0$




$\Rightarrow a \in \left(-\infty, -\frac{4}{3}\right)$

**Subjective Type Questions :**

**23.** Let $\alpha_1$, $\alpha_2$ and $\beta_1$, $\beta_2$ be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has non-trivial solution, prove that

$$\frac{b^2}{q^2} = \frac{ac}{pr}$$

**Sol.** Given $\alpha_1$, $\alpha_2$ are the roots of the equation $ax^2 + bx + c = 0$

$$\alpha_1 + \alpha_2 = -\frac{b}{a}, \alpha_1\alpha_2 = \frac{c}{a} \qquad ....(1)$$

Also, $\beta_1$, $\beta_2$ are the roots of $px^2 + qx + r = 0$, then

$$\beta_1 + \beta_2 = -\frac{q}{p}, \beta_1\beta_2 = \frac{r}{p} \qquad .....(2)$$

The equation $\alpha_1 y + \alpha_2 z = 0$ will have non-zero solution if

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \Rightarrow \alpha_1\beta_2 - \alpha_2\beta_1 = 0$$

$$\Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} \Rightarrow \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$$

$$\Rightarrow \quad \frac{\left(-\frac{b}{a}\right)}{\left(-\frac{q}{p}\right)} = \sqrt{\frac{\frac{c}{a}}{\frac{r}{p}}} \Rightarrow \frac{pb}{qa} = \sqrt{\frac{pc}{ar}}$$

$$\Rightarrow \quad \frac{p^2b^2}{q^2a^2} = \frac{pc}{ar} \Rightarrow \frac{b^2}{q^2} = \frac{ac}{pr}.$$



**24.** The equation $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ have a common root and the difference of their other roots is one. Then show that $|ac| = |a^2 - c^2|$. Hence or otherwise find the maximum and minimum value of $\left|\frac{a}{c}\right|$.

**Sol.** Roots of both the equations are recipocal of each other and they have a common root, which can be 1 or –1.

$\Rightarrow |b| = |a + c|$

Let $\pm$ 1, $\beta$ be roots of the equation $ax^2 + bx + c = 0$

and $\pm$ 1, $\beta'$ be the roots of the equation $cx^2 + bx + a = 0$

Now, $\pm\ 1 + \beta = -\frac{b}{a}$

$\pm\ 1 + \beta' = -\frac{b}{c}$

$\Rightarrow \beta - \beta' = b\left(\frac{a-c}{ac}\right) \quad \Rightarrow \left|\frac{b(a-c)}{ac}\right| = 1 \Rightarrow |ac| = |a^2 - c^2|$

$\Rightarrow \quad \left|\frac{a}{c}\right| = \left|\frac{a^2}{c^2} - 1\right|$

Now, $\left|\frac{a^2}{c^2} - 1\right| \geq \left|\left|\frac{a^2}{c^2}\right| - 1\right| \quad \Rightarrow \left|\frac{a}{c}\right| \geq \left|\left|\frac{a^2}{c^2}\right| - 1\right| \quad \Rightarrow \left|\frac{a}{c}\right| \leq \left|\frac{a^2}{c^2}\right| - 1 \leq \left|\frac{a}{c}\right|$

Solving, we get

$\Rightarrow \quad \left|\frac{a}{c}\right| \in \left[\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right].$

Hence minimum value = $\frac{-1+\sqrt{5}}{2}$

And maximum value = $\frac{1+\sqrt{5}}{2}$.

**25.** If the roots of the equation $ax^2 - 2bx + c = 0$ are imaginary, find the number of real roots of $4e^x + (a + c)^2 (x^3 + x) = 4b^2x$.

**Sol.** $f(x) = 4e^x + (a + c)^2 (x^3 + x) = 4b^2x$

$f'(x) = 4e^x + (a + c)^2 - 4b^2 + 3x^2 (a + c)^2$

Given $ax^2 - 2bx + c = 0$ has imaginary roots

$\Rightarrow \quad b^2 - ac < 0 \quad \Rightarrow \quad -4b^2 + 4ac > 0$

$\Rightarrow \quad -4b^2 + (a - c)^2 + 4ac > 0 \quad \Rightarrow \quad -4b^2 + (a + c)^2 > 0$

So, $f'(x) > 0$

$\Rightarrow$ f(x) is an monotonically increasing function.

Also $f(0) = 4$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Hence f(x) has one and only one real root.

**26.** Prove that roots of $a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$ are not real, if $a + b > c$ and $|a - b| < c$. (where a, b, c are positive real numbers)

**Sol.** $a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$ ......(1)

$\because \quad a + b > c \Rightarrow a + b - c > 0$ ......(2)

and $|a - b| < c \Rightarrow \quad a - b - c < 0$ .......(3)

and $\quad a - b + c > 0$ .......(4)

Discriminant of equation (1)

i.e. $D = (b^2 + a^2 - c^2)^2 - 4a^2b^2$

$= (b^2 + a^2 - c^2 - 2ab)(b^2 + a^2 - c^2 + 2ab)$

$= \{(a - b)^2 - c^2\}\{(a + b)^2 - c^2\}$

$= (a - b + c)(a - b - c)(a + b + c)(a + b - c) < 0$ (using (2), (3), (4))

$D < 0 \quad \therefore \quad$ roots are not real.