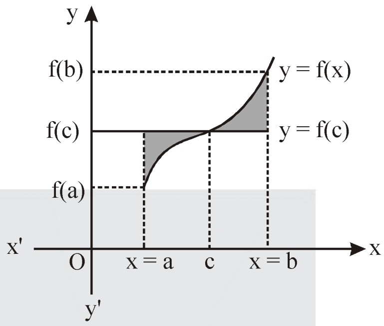
Comprehension # 3

If $y = f(x)$ is a monotonic function in (a, b), then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$ (where $c \in (a, b)$) is minimum when $c = \frac{a+b}{2}$.

Proof : $A = \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx$



$= f(c)(c-a) - \int_a^c (f(x))\,dx + \int_c^b (f(x))\,dx - f(c)(b-c)$

$\Rightarrow$ $A = [2c - (a + b) f(c) + \int_c^b (f(x))\,dx - \int_a^c (f(x))\,dx$

Differetiating w.r.t. c,

$\frac{dA}{dc} = \left[2c - (a+b)\right]$ f '(x) + 2 f(c) + 0 – f(c) – (f(c))

for maxima and minima $\frac{dA}{dc} = 0$

$\Rightarrow$ f '(x)[2c – (a + b)] = 0 (as f '(c) $\neq$ 0)

hence $c = \frac{a+b}{2}$

Also $c < \frac{a+b}{2}$, $\frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}$, $\frac{dA}{dc} > 0$.

Hence A is minimum when $c = \frac{a+b}{2}$.

16. If the area bounded by f(x) = and the straight lines x = 0, x = 2 and the x-axis is minimum, then the value of a is

(A) $\frac{1}{3}$ (B) 2 (C) 1 (D) $\frac{2}{3}$

Ans. (D)

17. The valueof the parameter a for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$ and the straight lines, which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least, is

(A) 2 (B) 0 (C) – 1 (D) 1

Ans. (C)

18. If the area enclosed by f(x) = sin x + cos x, y = a between the two consecutive points of extremum is minimum, then the value of a is

(A) 0 (B) – 1 (C) 1 (D) 2

Ans. (A)

Sol. **(i)** f(x) = $\frac{x^3}{3} - x^2 + a$

f '(x) = $x^2 - 2x = x(x - 2) < 0$ (Note that f(x) is monotonic in (0, 2)).

Hence for the minimum and f(x) must cross the x-axis at $\frac{0+2}{2}$ = 1.

Hence, $f(1) = \frac{1}{3} - 1 + a = 0$

$\Rightarrow \quad a = \frac{2}{3}$.

(ii) $f(x) = x^3 + 3x^2 + x + a$

$f'(x) = 3x^2 + 6x + 1 = 0$

$\Rightarrow \quad x = -1 \pm \frac{\sqrt{6}}{3}$.

Hence, f(x) cuts the x-axis at $\frac{1}{2}\left[\left(-1+\frac{\sqrt{6}}{3}\right)+\left(-1-\frac{\sqrt{6}}{3}\right)\right] = -1$.

$f(-1) = -1 + 3 - 1 + a = 0$

$a = -1$.

(iii) f(x) = sin x + cos x

$\Rightarrow \quad \frac{df(x)}{dx}$ = cos x – sin x

If $\frac{df(x)}{dx}$ = 0, then cos x = sin x $\Rightarrow x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ (considering any two of consecutive points of extremum). For minimum area bounded by y = f(x) and y = a, between $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, graphs of g(x) must cut y = a at $c = \frac{\frac{\pi}{4}+\frac{5\pi}{4}}{2} = \frac{3\pi}{4}$

$a = f\left(\frac{3\pi}{4}\right) \Rightarrow a = \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = 0.$

Comprehension # 4

Consider the areas S_0, S_1, S_2 ..... bounded by the x-axis and half-waves of the curve $y = e^{-x} \sin x$, where $x \geq 0$.

19. The value of S_0 is

(A) $\frac{1}{2}\left(1+e^{\pi}\right)$ sq. units

(B) $\frac{1}{2}\left(1+e^{-\pi}\right)$ sq. units

(C) $\frac{1}{2}\left(1-e^{-\pi}\right)$ sq. units

(D) $\frac{1}{2}\left(e^{\pi}-1\right)$ sq. units.

Ans. **(A)**

20. The sequence S_0, S_1, S_2, ..., forms a G.P. with common ratio

(A) $\frac{e^{\pi}}{2}$ (B) $e^{-\pi}$ (C) e^{π} (D) $\frac{e^{-\pi}}{2}$

Ans. **(B)**

21. $\sum_{n=0}^{\infty} S_n$ is equal to

(A) $\frac{1+e^{\pi}}{1-e^{-\pi}}$ (B) $\frac{\frac{1}{2}(1+e^{\pi})}{1-e^{-\pi}}$ (C) $\frac{1}{2(1-e^{-\pi})}$ (D) None of these

Ans. **(B)**

Sol. Since $-1 \le \sin x \le 1$, the curve $y = e^{-x} \sin x$ is bounded by the curves $y = e^{-x}$ and $y = -e^{-x}$.

Also, the curve $y = e^{-x} \sin x$ intersects the positive semi axis OX at the points where $\sin x = 0$, where $x_n = n\pi$, $n \in Z$. Also $| y_n | =$ | y coordinate in the half wave S_n|

$= (-1)^n e^{-x} \sin x$, and in S_n, $n\pi \le x \le (n + 1)\pi$

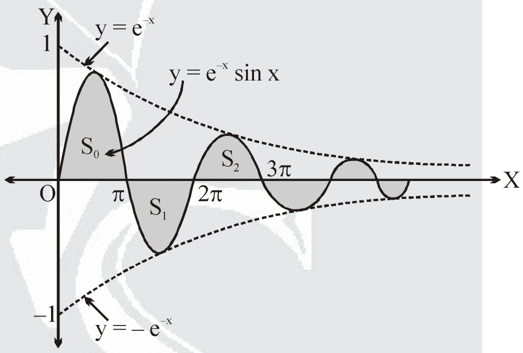
$$\therefore \quad S_n = (-1)^n \int_{n\pi}^{(n+1)\pi} e^{-x} \sin x \, dx$$

$$= \frac{(-1)^{n+1}}{2}\left[e^{-x}(-\sin x + \cos x)\right]_{n\pi}^{(n+1)\pi}$$

$$= \frac{(-1)^{n+1}}{2}\left[e^{-(n+1)\pi}(-1)^{n+1} - e^{n\pi}\beta(-1)^n\right]$$

$$= \frac{e^{-n\pi}}{2}\left(1+e^{\pi}\right)$$

$$= \frac{S_{n+1}}{S_n} = e^{-\pi} \text{ and } S_0 = \frac{1}{2}\left(1+e^{\pi}\right).$$



$\therefore$ the sequence S_0, S_1, S_2,......... forms an infinite G.P. with common ratio $e^{-\pi}$.

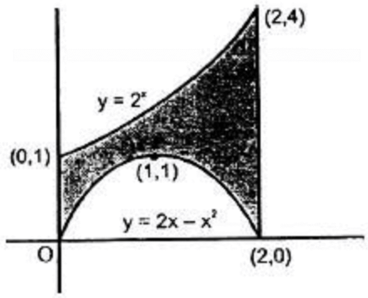
$$\therefore \sum_{n=0}^{\infty} S_n = \frac{\frac{1}{2}\left(1+e^{\pi}\right)}{1-e^{-\pi}}.$$

Numerical based Questions :

22. If A is the area of the figure bounded by the straight line x = 0 and x = 2 and the curves $y = 2^x$ and $y = 2x - x^2$, then the value of $6\left(\frac{3}{\ell n 2} - A\right)$ is

Ans. **(8)**

Sol. Finding area of shaded region $\int_0^2 \left[2^x - \left(2x - x^2\right)\right]dx$



$$= \left.\frac{2x}{\ell n2}\right]_0^2 - \left.\left(x^2 - \frac{x^3}{3}\right)\right]_0^2 \qquad = \frac{3}{\ell n2} - \frac{4}{2} = A$$

23. The graph of $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. The area of the bounded region is

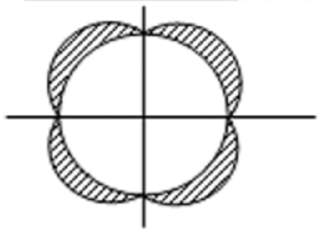
Ans. **(**800 sq. units**)**

Sol. $y^2 + 2xy + 40|x| - 400 = 0$ can be factorized as $(y+20)(y+2x-20) = 0$ for $x \geq 0$ and $(y-20)(y+2x-20) = 0$ for $x < 0$

The bounded region is parallelogram of area 20 × 40 = 800 sq. units

24. Area enclosed by the curve $4 \leq x^2 + y^2 \leq 2\left(|x| + |y|\right)$ is (in sq. units)

Ans. **(8)**



Sol.

Required area $= 4\left[\dfrac{\pi\left(\sqrt{2}\right)^2}{2} - \left(\pi - 2\right)\right] = 8$ sq. units

25. The area bounded by the curves $y = -\sqrt{4 - x^2}$, $x^2 = -y\sqrt{2}$ and x = y can be expressed in the form of $\pi + \frac{\lambda}{\mu}$, ($\lambda$, μ being relatively prime positive integers) then $\lambda + \mu$ equals to ______.

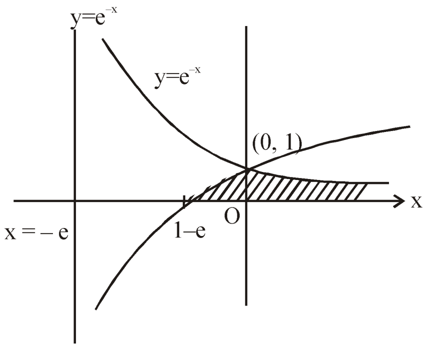
Ans. **(4)**

Sol. Area $= \dfrac{\pi}{2} + \left|\displaystyle\int_{-\sqrt{2}}^{0}\left(-\frac{x^2}{\sqrt{2}} + \sqrt{4 - x^2}\right)dx\right| = \dfrac{\pi}{2} + \dfrac{\pi}{2} + \dfrac{1}{3} = \pi + \dfrac{1}{3}$

$\therefore \lambda + \mu = 1 + 3 = 4$

26. Find the area enclosed between the curves : $y = \log_e(x + e)$, $x = \log_e(1/y)$ & the x–axis.

Ans. **2 sq. units**



Sol.

$y = \log_e(x + e)$, $x = \log_e(1/y)$

$x + e = e^y$ $\qquad \frac{1}{y} = e^x \Rightarrow y = e^{-x}$

$x = e^y - e$

$\log_e(x + e) = 0$

$x + e = 1$

$x = 1 - e$

$$A = \int_0^1 \left[(-\ln y) - (e^y - e)\right] dy$$

$$= \int_0^1 \left(e - \ln y - e^y\right) dy$$

Matrix Match Type :

27. Match the following

Column-I	Column-II
(a) Let $f(K) = \frac{K}{2009}$ and $g(K) = \frac{(f(K))^4}{(1-f(K))^4 + (f(K))^4}$ and $S = \sum_{K=0}^{2009} g(K)$ then sum of the diagonal in S =	(p) 0
(b) $f : D \to \mathbb{R}$ such that $f(k) = \sqrt{\sin(\cos x)} + \log_e\left(-2\cos^2 x + 3\cos x + 1\right)$ then $\int_{x_1}^{x_2} \left[\cos x - \frac{1}{2}\right] dx$ is equal to, $x_1, x_2 \in D$ and [.] denotes greatest integer function.	(q) 1
(c) The area of region, represented by $\|x\| + \|y\| \le 2$ and $x^2 \le y$ is S square units then 3S = _____.	(r) 7
(d) $\lim_{y\to\infty} \frac{\int_1^y \left[\tan^{-1} x\right] dx}{\int_1^y \left[1 + \frac{1}{x}\right] dx} =$ _____. ([.]) denotes greatest integer function)	(s) 6

(A) A→ s; B → p; C → s; D → q

(B) A → p; B → s; C → r; D → q

(C) A → s; B → p; C → r; D → q

(D) A → s; B → p; C → r; D → s

Ans. **(C)**

Sol. (a) Let (K) + f(2009–K)=1 and g(K)+g (2009–K) = 1 $\Rightarrow$ S = 1005

$\therefore$ Sum of the digits is 6

(b) f(x) is defined when

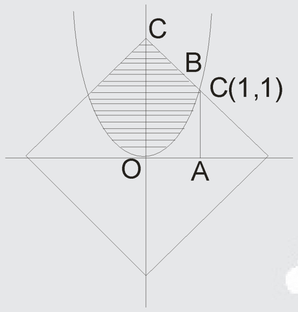
$-2\cos^2 x + 3\cos x - 1 > 0$

$2\cos^2 x - 3\cos x + 1 < 0$

$$\Rightarrow \frac{1}{2} < \cos x < 1$$

$$0 < \cos x - \frac{1}{2} < \frac{1}{2}$$

$$\Rightarrow \left[\cos x - \frac{1}{2}\right] = 0$$



(c)

$$\text{Required are } = 2\left\{\text{area OABC} - \int_0^1 x^2 dx\right.$$

$$= 2\left\{\frac{3}{2} - \frac{1}{3}\right\} = \frac{7}{3}$$

(d) $$\lim_{y\to\infty} \frac{\int_1^{\tan 1} 0dx + \int_{\tan 1}^{y} 1dx}{} = \lim_{y\to\infty} \frac{y - \tan 1}{y - 1} = 1$$

Matrix Matching : 3

28. **Column - I** **Column-II**

(A) The area bounded y the curve y = x | x | , x-axis and the ordinates x = 1, x = – 1 (P) $\frac{10}{3}$ sq. units

(B) The area of the region lying between the line x – y + 2 =0 and the curve x = $\sqrt{y}$ (Q) $\frac{64}{3}$ sq. units

(C) The area enclosed between the curves y^2 = x and y = | x | (R) $\frac{2}{3}$ sq. units

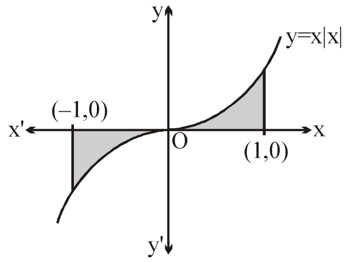
(D) The area bounded by parabola y^2 = x, straight line y = 4 and y-axis (S) $\frac{1}{6}$ sq. units

(A) A$\to$Q; B$\to$S; C$\to$P; D$\to$R **(B)** A$\to$S; B$\to$R; C$\to$Q; D$\to$P

(C) A$\to$R; B$\to$P; C$\to$S; D$\to$Q **(D)** A$\to$Q; B$\to$R; C$\to$P; D$\to$S

Ans. **(C)**

Sol. (A) Required area $= \left| \int_{-1}^{1} x\,|\,x\,|\,dx \right|$

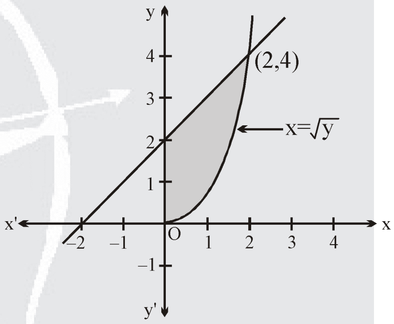


$= \left| \int_{-1}^{0} x\,|\,x\,|\,dx \right| + \int_{0}^{1} x\,|\,x\,|\,dx$

$= \left| \int_{-1}^{0} -x^2\,dx \right| + \int_{0}^{1} x^2\,dx$

$= \left| \left[\frac{x^3}{3}\right]_{-1}^{0} \right| - \left[\frac{x^3}{3}\right]_{0}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ sq. units.

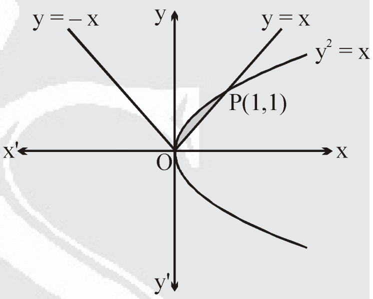
(B) Required area $= \int_{0}^{2} (y_1 - y_2)\,dx$



$= \int_{0}^{2} \left[(x+2) - (x^2)\right] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{0}^{2}$

$= 2 + 4 - \frac{8}{3} = \frac{10}{3}$ sq. units.

(C) Required area $= \int_{0}^{1} \left(\sqrt{x} - x\right) dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_{0}^{1}$

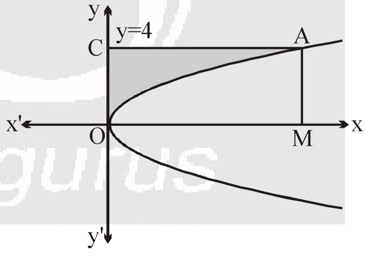


$= \left(\frac{1}{3/2} - \frac{1}{2}\right)$

$= \frac{2}{3} - \frac{1}{2}\ \frac{1}{6}$ sq. units.

(D) y = 4 meets the parabola $y^2 = x$ at A is (16, 4)

Required area = Area of rectangle OMAC – Area OMA



$= 4 \times 16 - \int_{0}^{16} \sqrt{x}\,dx = 64 - \left| \frac{x^{3/2}}{3/2} \right|_{0}^{16}$

$= 64 - \frac{2}{3}(4)^3 = 64 - \frac{128}{3} = \frac{64}{3}$ sq. units.