competishun	CAPS-18
MATHEMATICS	**Area Under the Curves**
TARGET : JEE- Advanced 2021	

ANSWER KEY OF CAPS-18

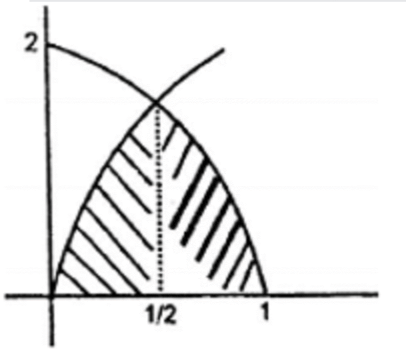
1.	(D)	**2.**	(A)	**3.**	(B)	**4.**	(C)	**5.**	(C)
6.	(A)	**7.**	(A)	**8.**	(A)	**9.**	(ABD)	**10.**	(AC)
11.	(AC)	**12.**	(D)	**13.**	(A)	**14.**	(C)	**15.**	(D)
16.	(D)	**17.**	(C)	**18.**	(A)	**19.**	(A)	**20.**	(B)
21.	(B)	**22.**	(8)	**23.**	(800)	**24.**	(8)	**25.**	(4)
26.	(2)	**27.**	(C)	**28.**	(c)				

SCQ (Single Correct Type) :

1. Let f (x) = $2\sqrt{x}$ and g(x) = $2\sqrt{1-x}$ be two functions then the area bounded by y = f(x), y = g(x) and x axis is

(A) $\frac{1}{3\sqrt{2}}$ sq unit (B) $\frac{2}{3\sqrt{2}}$ sq unit (C) $\frac{1}{\sqrt{2}}$ sq unit (D) $\frac{4}{3\sqrt{2}}$ sq unit

Ans. **(D)**



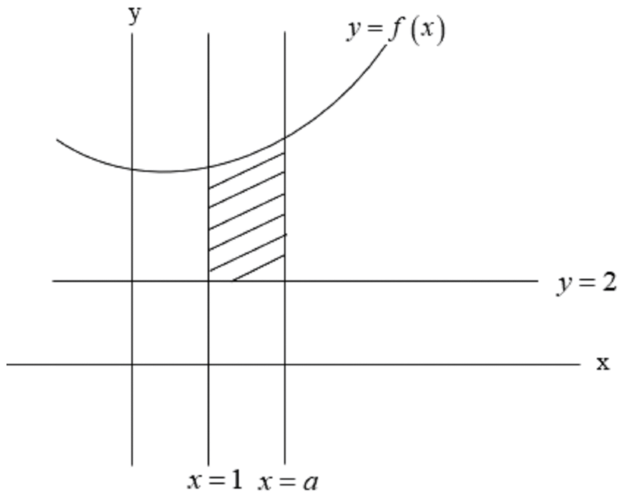
Sol.

$$\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{1} 2\sqrt{1-x}\,dx = \frac{4}{3\sqrt{2}}$$

2. The area bounded by the lines y = 2, x = 1, x = a and the curve y = f(x), which cuts the last two lines above the first line for all $a \geq 1$, is equal to $\frac{2}{3}((2a))^{3/2} - 3a + 3 - 2\sqrt{2}$ then f(x) =

(A) $2\sqrt{2x};\ x \geq 1$ (B) $\sqrt{2x};\ x \geq 1$ (C) $2\sqrt{x};\ x \geq 1$ (D) None of these

Ans. **(A)**



Sol.

$$\int_1^a (f(x)-2)\,dx = \frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right]$$

$$\int_1^a f(x)\,dx = \frac{2}{3}\left[(2a)^{3/2} - 3a + 3 - 2\sqrt{2}\right] + 2a$$

$$f(a) = \frac{2}{3}\left[3(2a)^{1/2} - 3\right] + 2$$

$$f(a) = 2\sqrt{2a}$$

3. A variable circle C touches y = 0 and passes through the point (0,1). Let the locus of the centre of C is the curve P. The area enclosed by the curve P and the line x + y = 2 is

(A) $\frac{14}{3}$ (B) $\frac{16}{3}$ (C) $\frac{15}{3}$ (D) $\frac{11}{3}$

Ans. **(B)**

Sol. Let A(h, k) is the centre of circle C

$$\Rightarrow \sqrt{h^2 + (k-1)^2} = |k|$$

$$\Rightarrow k = \frac{1+h^2}{2}$$

Hence curve P is $y = \frac{1+x^2}{2}$

Point of intersection of $y = \frac{1+x^2}{2}$ and x + y = 2 is (1, 1) and (–3, 5)

Area $\int_{-3}^{1}\left(2 - x - \frac{1+x^2}{2}\right)dx$

$$= \frac{1}{2}\int_{-3}^{1}\left(3 - 2x - x^2\right)dx = \frac{16}{3}$$

4. Let n S_n be the area of the figure enclosed by a curve $y = x^3(1-x^2)^n, 0 \le x \le 1$ and the x-axis. The value of $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} S_k$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Ans. **(C)**

Sol. $S_k = \int\limits_0^1 x^3\left(1-x^2\right)^k dx$

Let $x^2 = t$

$\Rightarrow x\,dx = \frac{1}{2}dx$

$\Rightarrow S_k = \frac{1}{2}\int\limits_0^1 t(1-t)^k dt = \frac{1}{2}\left[\frac{1}{k+1} - \frac{1}{k+2}\right]$

$\lim\limits_{n\to\infty}\frac{1}{2}\sum\limits_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{4}$

5. Find the area of the region enclosed between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of the points where the curve has local minima

(A) $\frac{7}{60}$ (B) $\frac{7}{80}$ (C) $\frac{7}{120}$ (D) $\frac{7}{160}$

Ans. **(C)**

Sol. $y = 2x^4 - x^2$

$y' = 8x^3 - 2x = 0$

$\Rightarrow x = 0, \frac{1}{2}, -\frac{1}{2}$

$y'' = 24x^2 - 2$

Curve has local minima at $-\frac{1}{2}$ and $\frac{1}{2}$

$\therefore$ area = $-\int\limits_{1/2}^{1/2}\left(2x^4 - x^2\right)dx = -2\int\limits_0^{1/2}\left(2x^4 - x^2\right)dx = \frac{7}{120}$

6. Let $f(x) = \begin{cases} Ln(x-[x]); & x \notin I \\ 0\ ; & x \in I \end{cases}$ Area enclosed by curves $y = f(x), y = f(|x|), -5 \le x \le 5$ and $y = 0$ is

(A) 5 (1 – 1n2) (B) 0 (C) 51n2 (D) none of these

Ans. **(A)**

Sol. $\Delta = 5\times 2\int\limits_{-1/2}^{0} \ln\left(x-[x]\right)dx = 10\int\limits_{-1/2}^{0}\ln\left(x+1\right) = 5\left(1-\ln 2\right)$

7. The area bounded by the curves y = 2 − |x −1| , y = sin x; x = 0 and x = 2 is _____.

(A) $1+ 2\cos^2 1$ (B) $2 + \sin^2 1$ (c) $\frac{\pi}{2}$ (d) 1+ log2

Ans. **(A)**

Sol. Area $= \int_0^1 (1 + x - \sin x)dx + \int_1^2 (3 - x - \sin x)dx$

$= \frac{3}{2} + (\cos 1 - 1) + 3 - \frac{3}{2} + (\cos 2 - \cos 1) \quad = 2 + \cos 2$

8. The area bound by the curve y = f (x) , the x–axis and the line y =1, where $f(x) = 1 + \frac{1}{x}\int_1^x f(t)dt, (x \neq 0)$ is _____.

(A) $2\left(1 - \frac{1}{e}\right)$ (B) $2\left(1 + \frac{1}{e}\right)$ (C) $\left(1 + \frac{1}{e}\right)$ (D) $\left(1 - \frac{1}{e}\right)$

Ans. **(A)**

Sol. $f(x) = 1 + \ln|x|$

Area = $2\int_0^1 e^{y-1}dy = 2\left(1 - \frac{1}{e}\right)$

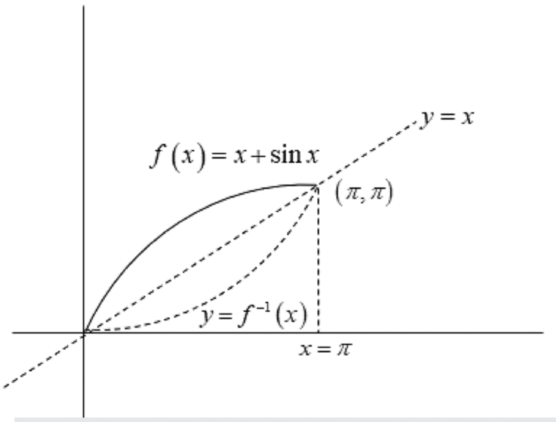
MCQ (One or more than one correct) :

9. The area bounded by the curve $y = f^{-1}(x)$, x-axis, x = 0 and x = π, where f(x) is given by f(x) = x + sin x is $\frac{\pi^2 - a}{b}$ then the correct statement are :

(A) a = 2b (B) a + 2b = 8 (C) 3a =b (d) $\left(\frac{2\pi}{b}, \frac{a\pi}{4}\right)$ lies on f(x)

(A) - (B) - (C) - (D) -

Ans. **(ABD)**



Sol.

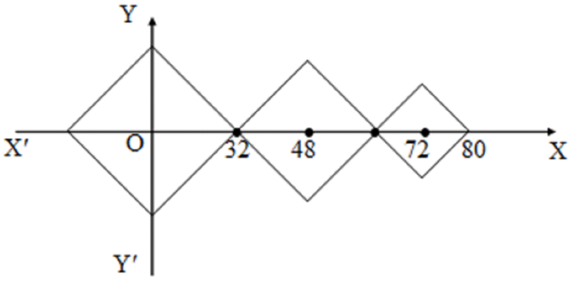
$\int_0^{\pi} f^{-1}(x)dx = \frac{1}{2} \times \pi \times \pi - 2$

$= \frac{\pi^2 - 4}{2}$ $\quad \therefore$ a = 4 & b = 2

10. If A_i is the area bounded by $|x - a_i| + |y| = b_i, i \in \mathbb{N}$ where 1 $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}, a_1 = 0, b_1 = 32$, then _____.

(A) A_3 =128 (B) A_3 = 256 (C) $\lim_{n\to\infty}\sum_{i=1}^{n} A_i = \frac{8}{3}(32)^2$ (D) $\lim_{n\to\infty}\sum_{i=1}^{n} A_i = \frac{4}{3}(16)^2$

Ans. **(AC)**



Sol.

$$a_1 = 0,\ b_1 = 32,\ a_2 = a_1 + \frac{3}{2}b_1 = 48,\ b_2 = \frac{b_1}{2} = 16$$

$$a_3 = 48 = \frac{3}{2} \times 16 = 72,\ b_3 = \frac{16}{2} = 8$$

So the three lops form i = 1 to i = 3 are alike

Now area of i^{th} loop (square) = $\frac{1}{2}$ (diagonal)^2

$$A_i = \frac{1}{2}(2b_i)^2 = 2(b_i)^2$$

So, $\frac{A_{i+1}}{A_i} = \frac{2(b_{i+1})}{2(b_i)^2} = \frac{1}{4}$

So the area form a GP series

So, the sum of the GP upto infinite terms

$$= A_i \frac{1}{1-r} = 2(32)^2 \times \frac{1}{1-\frac{1}{4}} = 2(32)^2 \times \left(\frac{4}{3}\right) = \frac{8}{3} \times (32)^2 \text{ square units}$$

11. Let T be the triangle with vertices (0, 0), (0, c^2) and (c, c^2) and let R be the region between y = cx and y = x^2 where c > 0 then

(A) Area (R) = $\frac{c^3}{6}$

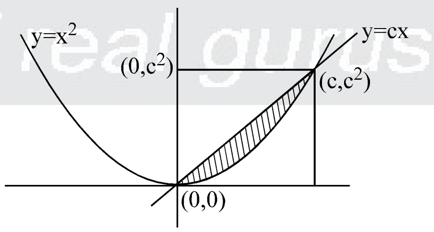
(B) Area of R = $\frac{c^3}{3}$

(C) $\lim\limits_{c \to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = 3$

(D) $\lim\limits_{c \to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = \frac{3}{2}$

Ans. **(AC)**

Sol. Area (T) = $\frac{c \cdot c^2}{2} = \frac{c^3}{2}$



Area (R) = $\frac{c^3}{2} - \int_0^c x^2\,dx$

$$= \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\therefore \quad \lim_{c \to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = \lim_{c \to 0^+} \frac{c^3}{2} \cdot \frac{6}{c^3} = 3]$$

Comprehension Type Question:

Comprehension # 1

Given f (x) = $x^3 + x - 2$, f :R $\to$ R, area of region bounded by $y = f(x), y = f^{-1}(x)$ and xy = 0 in 1^{st} quadrant is A, where $A = 3t - \frac{5}{2}$, where $t = a^{1/b}$ and $3(a+b) = n;\ a, b, n \in N$ and a is prime.

12. Difference of last two digits of $a^{a^{11}-36}$ is

(A) 0 (B) 1 (C) 2 (D) 3

Ans. **(D)**

Sol. f(x) and f^{-1} intersect at $f(x) = f^{-1}(x) = x$

So at $x = 2^{1/3}$

So area bounded is $\left(2^{1/3}\right)^2 - 2\int_1^{2^{1/3}} f(x)dx = 3.2^{1/3} - \frac{5}{2}$

So t = $2^{1/3}$

So a = 2,b = 3,n =15

Last two digits of 2^{2012} are 96

$\Rightarrow$ difference = 3

13. Number of ways of putting n alike apples and n alike oranges in six baskets if each basket can hold atmost five fruits is

(A) ${}^{20}C_5 - 6\ {}^{14}C_5 + 15\ {}^{8}C_5$

(B) $({}^{20}C_5 - 6\ {}^{14}C_5 + 15\ {}^{8}C_5)^2$

(C) $({}^{20}C_5 - 6\ {}^{14}C_5 + 15\ {}^{8}C_5)^2 - 5^6$

(D) none of these

Ans. **(A)**

Sol. Given, f(x) = $x^3 + x - 2$

y = f (x) and y = f^{-1} (x) intersect at $f(x) = f^{-1}(x) = x$

So at $x = 2^{1/3}$

So area of region bounded is $2 \times \left[\frac{1}{2} \times 2^{1/3} \times 2^{1/3} - \int_1^{2^{1/3}} \left(x^3 + x - 2\right)dx\right]$

$= 2\left[\frac{3}{4} \times 2^{1/3} - \frac{5}{4}\right]$

$A = 3 \times 2^{1/3} - \frac{5}{2}$

So t = $2^{1/3}$

So a = 2, b = 3,n =15

Number of ways of distribution of apples is coefficient of x^2 in $\left(1 + x + x^2 + ... + x^5\right)^6$

Number of ways of distribution of oranges is 1

Comprehension # 2

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function and bijective, defined such that $f(\alpha) = 0(\alpha \neq 0)$. The area bounded by $y = f(x)$, $x = \alpha$, $x = \alpha - t$ is equal to area bounded by $y = f(x)$, $x = \alpha$, $x = \alpha + t$, $\forall t \in \mathbb{R}$ then

14. y = f (x) is symmetrical about the point

(A) (0,0) (B) $(0, \alpha)$ (C) $(\alpha, 0)$ (D) (α, α)

Ans. **(C)**

Sol. Since is continuous and bijective, so curve is

symmetrical about the point $(\alpha, 0)$

$\Rightarrow f(\alpha + t) = -f(\alpha - t)$

Put $t = \alpha$, $\Rightarrow f(2\alpha) = -f(0)$

Now $f(\alpha + t) = y$

$\alpha + t = f^{-1}(y)$ ..... (1)

and $f(\alpha - t) = -y$

$\alpha - t) = f^{-1}(-y)$ ....(2)

add (1) and (2)

$2\alpha = f^{-1}(y) + f^{-1}(-y)$

Now $\int_{-\beta}^{\beta} f^{-1}(t)\,dt = \int_{0}^{\beta}\left(f^{-1}(t) + f^{-1}(-t)\right)dt = \int_{0}^{\beta} 2\alpha\,dt = 2\alpha\beta$

15. $f(2\alpha)$ equal to ______.

(A) $f(\alpha)$ (B) $-f(\alpha)$ (c) f(0) (d) − f(0)

Ans. **(D)**

Sol. Since f(x) is continous and bijective, so curve is symmetrical about the point $(\alpha, 0)$

$\Rightarrow f(\alpha + t) = -f(\alpha - t)$

Put $t = \alpha$, $\Rightarrow f(2\alpha) = -f(0)$

Now $f(\alpha + t) = y$

$\alpha + t = f^{-1}(y)$ ..... (1)

and $f(\alpha - t) = -y$

$\alpha - t) = f^{-1}(-y)$ ....(2)

add (1) and (2)

$2\alpha = f^{-1}(y) + f^{-1}(-y)$

Now $\int_{-\beta}^{\beta} f^{-1}(t)\,dt = \int_{0}^{\beta}\left(f^{-1}(t) + f^{-1}(-t)\right)dt = \int_{0}^{\beta} 2\alpha\,dt = 2\alpha\beta$