13. If $\int_0^{\frac{\pi}{2}} \sqrt[3]{\sin 2x}\, \sin x dx = k\int_0^{\frac{\pi}{4}} \sqrt[3]{\cos 2x}\, \cos x\, dx$ then the value of k^2 is_______.

Ans. **2**

Sol. $I_1 = \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin 2x}\, \sin x dx$ ...(i)

$\because \int_0^b f(x) dx = \int_0^b f(a+b-x) dx$

$\therefore I_1 = \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin 2x} \cdot \cos x dx$ .....(ii)

(i) + (ii)

$\therefore 2I_1 = \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin 2x}\left[\sin x + \cos x\right] dx$

$\int_0^{2a} f(x) dx = 2\int_0^a f(x) dx$ if $f(x) = f(2a - x)$

$I_1 = \int_0^{\frac{\pi}{4}} \sqrt[3]{\sin 2x}\left[\sin x + \cos x\right] dx$

$I_1 = \int_0^{\frac{\pi}{4}} \sqrt[3]{\sin 2x} \cdot \sin\left(\frac{\pi}{4} + x\right) \cdot \sqrt{2} dx$

$\therefore I_1 = \int_0^{\frac{\pi}{4}} \sqrt[3]{\cos 2x} \cdot \cos x \sqrt{2} dx$

$\Rightarrow k = \sqrt{2}$ or $k^2 = 2$

14. $\int_2^4 \frac{3x^2+1}{(x^2-1)^3} dx = \frac{\lambda}{n^2}$ where λ, $n \in N$ and gcd(λ, n) = 1, then find the value of $\lambda + n$

Ans. **61**

Sol. Note that $(x + 1)^3 - (x - 1)^3 = 2(3x^2 + 1)$

Hence $I = \frac{1}{2}\left[\int_2^4 \frac{(x+1)^3 - (x-1)^3}{(x+1)^3(x-1)^3} dx\right] = \frac{1}{2}\left[\int_2^4 \frac{dx}{(x-1)^3} - \int_2^4 \frac{dx}{(x+1)^3}\right]$

$= \frac{1}{2}\left[\frac{1}{2(x+1)^2} - \frac{1}{2(x-1)^2}\right]_2^4 = \left[\left(\frac{1}{25} - \frac{1}{9}\right) - \left(\frac{1}{9} - 1\right)\right] = \frac{46}{225}$

15. Let f(x) be a function satisfying $f(x) = f\left(\frac{100}{x}\right) \forall x > 0$. If $\int_1^{10} \frac{f(x)}{x} dx = 5$ then find the value of $\int_1^{100} \frac{f(x)}{x} dx$

Ans. **10**

Sol. $\int_1^{100} \frac{f(x)}{x} dx = \int_1^{10} \frac{f(x)}{x} dx + \int_{10}^{100} \frac{f(x)}{x} dx = 5 + \int_{10}^{1} \frac{f\left(\frac{100}{t}\right)}{100} . t\left(\frac{-100dt}{t^2}\right)$ (substituting $x = \frac{100}{t}$)

$= 5 + \int_t^{10} f\left(\frac{100}{t}\right) . \frac{dt}{t} = 5 + \int_1^{10} \frac{f(t)}{t} dt = 10$

16. Evaluate

$$\frac{2005 \int_0^{1002} \frac{dx}{\sqrt{1002^2 - x^2} + \sqrt{1003^2 - x^2}} + \int_{1002}^{1003} \sqrt{1003^2 - x^2}\, dx}{\int_0^1 \sqrt{1 - x^2}\, dx} = k,$$ then find the sum of squares of digits of natural number k.

Ans. **29**

Sol. $$\frac{2005 \int_0^{1002} \frac{\sqrt{1003^2 - x^2} - \sqrt{1002^2 - x^2}}{2005} + \int_{1002}^{1003} \sqrt{1003^2 - x^2}\, dx}{\int_0^1 \sqrt{1 - x^2}\, dx} = \frac{\int_0^{1003} \sqrt{1003^2 - x^2}\, dx - \int_0^{1002} \sqrt{1002^2 - x^2}\, dx}{\int_0^1 \sqrt{1 - x^2}\, dx}$$

$$= \frac{(1003^2 - 1002^2)\left(\frac{\pi}{4}\right)}{\left(\frac{\pi}{4}\right)} = 2005$$

Hence $2^2 + 5^2 = 29$

17. Let $I_1 = \int_0^{\pi/4} (1 + \tan x)^2 dx$, $I_2 = \int_0^1 \frac{dx}{(1+x)^2(1+x^2)}$ then find the value of $\frac{I_1}{I_2}$

Ans. **4**

Sol. Substitute x = tanθ in I_2.

$$I_2 = \int_0^{\pi/4} \frac{d\theta}{(1 + \tan\theta)^2} = \int_0^{\pi/4} \frac{d\theta}{\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right)^2} \qquad \left[\int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right]$$

$$= \int_0^{\pi/4} \frac{(1 + \tan\theta)^2}{4} d\theta \qquad \Rightarrow \qquad I_2 = \frac{I_1}{4}$$

18. If f, g, h be continuous functions on [0, a] such that f (a – x) = f (x), g (a – x) = – g (x) and 3 h (x) – 4 h (a – x) = 5, then find the value of $\int_0^a$ f (x) g (x) h (x) dx

Ans. 0

Sol. $I = \int_0^a f(x)\,g(x)\,h(x)\,dx$

$$I = \int_0^a f(a-x)\,g(a-x)\,h(a-x)\,dx = \int_0^a f(x)\,.\,(-g(x))\,\frac{3h(x)-5}{4}\,dx$$

$$I = -\frac{3}{4}\int_0^a f(x)\,g(x)\,h(x) + \frac{5}{4}\int_0^a f(x)\,g(x)\,dx$$

$$I = -\frac{3}{4}\,I + 0\,\frac{7}{4}\,I = 0$$

$\frac{7}{4}\,I = 0 \qquad \Rightarrow \qquad I = 0 \qquad \because \qquad f(x)\,g(x)\,dx = 0$

$I_1 = \int_0^a f(x)\,g(x)\,dx \qquad$ ......(i)

$I_1 = \int_0^a f(a-x)\,g(a-x)\,dx$

$I_1 = \int_0^a f(x)\,(-g(x))\,dx \qquad$ ......(ii)

(i) + (ii) $2I_1 = 0 \Rightarrow \qquad I_1 = 0$

19. If $f(x) = \frac{\sin x}{x}$ $\forall$ $x \in (0, \pi]$, If $\frac{\pi}{k}\int_0^{\pi/2} f(x)\,f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)\,dx$ then find the value of k.

Ans. 2

Sol. $I = \int_0^{\pi} f(x)dx = \int_0^{\pi}\frac{\sin x}{x}\,dx \qquad \ldots$ (i)

$I = \int_0^{\pi} f(\pi-x)dx = \int_0^{\pi}\frac{\sin(\pi-x)}{\pi-x}dx = \int_0^{\pi}\frac{\sin x}{\pi-x}\,dx \qquad \ldots$ (ii)

(i) + (ii)

$\Rightarrow \qquad 2I = \int_0^{\pi}\left\{\frac{\sin x}{x}+\frac{\sin x}{\pi-x}\right\}dx \quad \Rightarrow I = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{x(\pi-x)}dx \qquad \ldots$ (iii)

Now $\frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(x)f\left(\frac{\pi}{2}-x\right)dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}\,x\,\frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x}dx$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}\cdot\frac{\cos x}{\frac{\pi}{2}-x}\,dx \quad = \frac{\pi}{4}\int_0^{\pi/2}\frac{\sin 2x}{x\left(\frac{\pi}{2}-x\right)}\,dx\,\cdot$$

$$= \frac{\pi}{8}\int_0^{\pi}\frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)}dt\ ,\quad \text{where } t = 2x \qquad \ldots (iv)$$

(iii) + (iv)

$$\Rightarrow \frac{\pi}{2}\int_0^{\pi} f(x)f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)dx$$

20. Let a be a real number in the interval [0, 314] such that $\int_{-\pi+a}^{3\pi+a} |x-a-\pi|\sin\left(\frac{x}{2}\right)dx = -16$, then determine number of such values of k.

Ans. **25**

Sol. Let $I = \int_{-\pi+a}^{3\pi+a} |x-a-\pi|\sin\left(\frac{x}{2}\right)dx$ Let $x - a - \pi = t$

$$\Rightarrow I = \int_{-2\pi}^{2\pi} |t|\sin\left(\frac{t+a}{2}+\frac{\pi}{2}\right)dt = \int_{-2\pi}^{2\pi} |t|\cos\left(\frac{a+t}{2}\right)dt = \int_{-2\pi}^{2\pi} t\left\{\cos\left(\frac{a+t}{2}\right)+\cos\left(\frac{a-t}{2}\right)\right\}dt$$

$$= \int_{-2\pi}^{2\pi} t\left\{2\cos\frac{a}{2}\cos\frac{t}{2}\right\}dt = 2\cos\left(\frac{a}{2}\right)\int_0^{2\pi}\cos\left(\frac{t}{2}\right)dt \qquad \text{Let } \frac{t}{2} = y$$

$$8\cos\left(\frac{a}{2}\right)\int_0^{\pi} y\cos y\,dy = 8\cos\left(\frac{a}{2}\right)\{y\sin y+\cos y\}_0^{\pi} = -16\cos\left(\frac{a}{2}\right)$$

Now $I = -16 \quad \Rightarrow \quad \cos\left(\frac{a}{2}\right) = 1 \quad \Rightarrow \quad a = 4k\pi,\ k \in I$

Hence number of value of 'a' is 25.

21. If $f(x) = x + \int_0^1 t(x+t)f(t)dt$, then the value of the definite integral $\int_0^1 f(x)dx$ can be expressed in the form of rational as $\frac{p}{q}$ (where p and q are coprime). Find (p + q).

Ans. **65**

Sol. $f(x) = x + \int_0^1 t(x+t)\,f(t)dt$; $f(x) = x + x\int_0^1 tf(t)dt + \int_0^1 t^2 f(t)dt$

Let $\int_0^1 tf(t)dt = a$ & $\int_0^1 t^2 f(t)dt = b$ Hence $f(x) = (a + 1)x + b$

so $a = \int_0^1 t\{(a+1)t+b\}dt = \frac{a+1}{3}+\frac{b}{2}$

$\Rightarrow \quad 4a - 3b = 2 \quad$ ....(1)

& $\quad b = \int_0^1 t^2\{(a+1)t+b\}dt = \frac{a+1}{4}+\frac{b}{3}$

$\Rightarrow \quad 8b - 3a = 3 \quad$ ....(2)

from (1) & (2) $a = \frac{25}{23}$ & $b = \frac{18}{23}$

so $\int_0^1 f(t)dt = \frac{6}{23}\int_0^1 (8t+3)dt = \frac{6}{23}.7 = \frac{42}{23} \quad \Rightarrow \quad p + q = 65$

22. If the minimum of the following function f(x) defined at $0 < x < \frac{\pi}{2}$.

$f(x) = \int\limits_0^x \frac{d\theta}{\cos\theta} + \int\limits_x^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ is equal to $\ell n(a+\sqrt{b})$ where a, b $\in$ N and b is not a perfect square then find the value of (a + b)

Ans. **11**

Sol. $f(x) = \int\limits_0^x \frac{d\theta}{\cos\theta} + \int\limits_x^{\pi/2} \frac{d\theta}{\sin\theta}$

$\Rightarrow f'(x) = \frac{1}{\cos x} - \frac{1}{\sin x} = \frac{\sin x - \cos x}{\sin x - \cos x}$

$f'(x) = 0 \qquad \Rightarrow \qquad x = \frac{\pi}{4}$

$\Rightarrow$ f'(x) changes sign from (–) to (+) hence it is a point of minima.

Now $f\left(\frac{\pi}{4}\right) = \int\limits_0^{\pi/4} \sec\theta\, d\theta + \int\limits_{\pi/4}^{\pi/2} \text{cosec}\, d\theta = 2\int\limits_0^{\pi/4} \sec\theta\, d\theta = 2\ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = 2\ell n(\sqrt{2} + 1) = \ell n(3 + \sqrt{8})$

Hence a + b = 11

Matrix Match Type :

23. Match the following:

[x] $\rightarrow$ greatest integer less than or equal to x

Column-I	Column-II
(A) $25\int\limits_0^{\frac{\pi}{4}}\left(\tan^6(x-[x]) + \tan^4(x-[x])\right)dx =$	(p) 4
(B) $I_1 = \int\limits_{-2}^{2} \frac{x^6 + 3x^5 + 7x^4}{x^4 + 2}dx$ $I_2 = \int\limits_{-3}^{1} \frac{2(x+1)^2 + 11(x+1) + 14}{(x+1)^4 + 2}dx$ then $I_1 + I_2 =$	(q) 2
(C) $\frac{2}{\pi}\int\limits_{-1}^{3}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx =$	(r) $\frac{100}{3}$
(D) $f(x) = \frac{2-x}{\pi}\cos\pi(x+3) + \frac{1}{x^2}\sin\pi(x+3)$ where $0 < x < 4$. The number of points of local extrema are	(s) 5
	(t) $\frac{102}{3}$

(A) A $\rightarrow$ s; B $\rightarrow$ r; C $\rightarrow$ q; D $\rightarrow$ q
(B) A $\rightarrow$ s; B $\rightarrow$ s; C $\rightarrow$ q; D $\rightarrow$ r
(C) A $\rightarrow$ s; B $\rightarrow$ r; C $\rightarrow$ s; D $\rightarrow$ q
(D) A $\rightarrow$ s; B $\rightarrow$ r; C $\rightarrow$ r; D $\rightarrow$ q

Ans. **(A)**

Sol. (A) $I = 25\int_0^{\pi}\left(\tan^6 x + \tan^4 x\right)dx = 25\int_0^{\frac{\pi}{4}}\left(\tan^4 x + \sec^2 x\right)dx$

(B) Let x + 1 = t in I_2 then $I_1 + I_2 = \int_{-2}^{2}\left(x^2 + 7\right)dx = \frac{100}{3}$

(C) Let $I = \int_{-1}^{0} -\frac{\pi}{2}dx + \int_0^3 \frac{\pi}{2}dx = \pi$

(D) $f'(x) = 0$ at x = 1, 2, 3 but x = 2 is point of inflexion

Subjective Based Questions :

24. Evaluate $\int_0^1 \frac{1}{\left(5+2x-2x^2\right)\left(1+e^{(2-4x)}\right)}dx$

Ans. $\frac{1}{\sqrt{11}}\, ln\, \frac{\sqrt{11}+1}{\sqrt{11}-1}$

Sol.
$$I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})} \quad ...(1)$$

$$= \int_0^1 \frac{dx}{[5+2(1-x)-2(1-x)^2]\ [1+e^{2-4(1-x)}]} \qquad \text{By } \int_a^b f(x)\ dx = \int_a^b f(a+b-x)\,dx$$

$$= \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{-2+4x})}$$

$$= \int_0^1 \frac{e^{2-4x}\ \ dx}{(5+2x-2x^2)(1+e^{2-4x})} \quad ...(2)$$

(1) + (2)

$$\Rightarrow 2I = \int_0^1 \frac{e^{2-4x}+1\ \ dx}{(5+2x-2x^2)(1+e^{2-4x})} = \int_0^1 \frac{dx}{(5+2x-2x^2)}$$

$$= -\frac{1}{2}\int_0^1 \frac{dx}{\left(x^2-x-\frac{5}{2}\right)} = -\frac{1}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{11}}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{11}}\left[\ell n\left(\frac{\frac{\sqrt{11}}{2}+x-\frac{1}{2}}{\frac{\sqrt{11}}{2}-x+\frac{1}{2}}\right)\right]_0^1 = \frac{1}{\sqrt{11}}\ \ell n\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)$$

25. Evaluate $\int_0^1 \left(\{2x\}-1\right)\left(\{3x\}-1\right)dx$, where { } denotes fractional part of x.

Ans. $\frac{19}{72}$

Sol. $\int_0^1 (\{2x\} - 1)(\{3x\} - 1)\, dx$

$$= \int_0^{1/3} (2x-1)\ (3x-1)\, dx + \int_{1/3}^{1/2} (2x-1)\ (3x-2)\, dx + \int_{1/2}^{2/3} (2x-2)\ (3x-2)\, dx + \int_{2/3}^{1} (2x-2)\ (3x-3)\, dx$$

$$= \int_0^{1/3} (6x^2 - 5x + 1)\, dx + \int_{1/3}^{1/2} (6x^2 - 7x + 2)\, dx + \int_{1/2}^{1/3} (6x^2 - 10x + 4)\, dx + \int_{2/3}^{1} (6x^2 - 12x + 6)\, dx = \frac{19}{72}.$$