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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-17

Definite Integration

ANSWER KEY OF CAPS-17

1.	(D)	**2.**	(C)	**3.**	(D)	**4.**	(A)	**5.**	(ABCD)
6.	(AC)	**7.**	(ABCD)	**8.**	(C)	**9.**	(B)	**10.**	(D)
11.	(8)	**12.**	(6)	**13.**	(2)	**14.**	(61)	**15.**	(10)
16.	(2	**17.**	(4)	**18.**	(0)	**19.**	(2)	**20.**	(25)
21.	(6	**22.**	(11)	**23.**	(A)	**24.**	$\frac{1}{\sqrt{11}}\ ln\ \frac{\sqrt{11}+1}{\sqrt{11}-1}$	**25.**	$(\frac{19}{72})$

SCQ (Single Correct Type) :

1. The value of the definite integral $\int_2^3 \left[\sqrt{2x-\sqrt{5(4x-5)}}+\sqrt{2x+\sqrt{5(4x-5)}}\right]dx$ is equal to

(A) $4\sqrt{3}-\frac{2\sqrt{2}}{3}$ (B) $4\sqrt{2}$ (C) $4\sqrt{3}-\frac{4}{3}$ (D) $\frac{\sqrt{10}}{2}+\frac{7\sqrt{7}-5\sqrt{5}}{3\sqrt{2}}$

Ans. **(D)**

Sol. Let $5(4x-5)=t^2$ $\Rightarrow$ $20x-25=t^2$ $\Rightarrow$ $x=\frac{t^2+25}{20}$

Also $20\,dx=2t\,dt$ or $dx=\frac{t}{10}dt$

$$\therefore\quad I=\int_{\sqrt{15}}^{\sqrt{35}}\left(\sqrt{\frac{t^2+25}{10}-t}+\sqrt{\frac{t^2+25}{10}+t}\right)\frac{t}{10}\,dt=\int_{\sqrt{15}}^{\sqrt{35}}\left(\frac{|t-5|}{\sqrt{10}}+\frac{|t+5|}{\sqrt{10}}\right)\frac{t}{10}\,dt$$

$$=\int_{\sqrt{15}}^{5}\left(\frac{-t+5}{\sqrt{10}}+\frac{t+5}{\sqrt{10}}\right)\frac{t}{10}dt+\int_{5}^{\sqrt{35}}\left(\frac{t-5}{\sqrt{10}}+\frac{t+5}{\sqrt{10}}\right)dt=\frac{t}{10}\left[\frac{t^2}{2}\right]_{\sqrt{15}}^{5}+\frac{1}{5\sqrt{10}}\int_5^{\sqrt{35}}t^2\,dt$$

$$=\frac{1}{\sqrt{10}}\left[\frac{25}{2}-\frac{15}{2}\right]+\frac{1}{15\sqrt{10}}\left[(\sqrt{35})^3-125\right]=\frac{\sqrt{10}}{2}+\frac{7\sqrt{7}-5\sqrt{5}}{3\sqrt{2}}$$

2. If $\int_{\ell n2}^{x}\frac{dx}{\sqrt{e^x-1}}=\frac{\pi}{6}$, then x is equal to

(A) 4 (B) ℓn 8 (C) ℓn 4 (D) none of these

Ans. **(C)**

Sol. $\because$ $I = \int \frac{dx}{\sqrt{e^x - 1}} = \int \frac{e^{x/2}\, dx}{e^{x/2}\sqrt{(e^{x/2})^2 - 1}}$

Let $e^{x/2} = z \Rightarrow$ $I = 2\int \frac{dz}{|z|\sqrt{z^2 - 1}} = 2\sec^{-1}z + c = 2\sec^{-1}(e^{x/2}) + c$

Given $\frac{\pi}{6} = \int_{\ell n 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = 2\,[\sec^{-1}(e^{x/2}) - \sec^{-1}\sqrt{2}\,]$

$\therefore$ $\sec^{-1}(e^{x/2}) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$

$\therefore$ $e^{x/2} = 2 \Rightarrow \frac{x}{2} = \ell n\, 2 \Rightarrow x = \ell n\, 4$

3. The tangent to the graph of the function y = f(x) at the point with abscissa x =1 form an angle of and at the point x = 2, an angle of $\pi/3$ and at the point x = 3, an angle of $\pi/4$. The value of $\int_1^3 f'(x)f''(x)dx + \int_2^3 f''(x)dx$ (f''(x) is supposed to be continuous) is :

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$ (C) $\frac{4-\sqrt{3}}{3}$ (D) None of these

Ans. **(D)**

Sol. Given, $f'(1) = \left(\frac{dy}{dx}\right)_{x=1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$f'(2) = \left(\frac{dy}{dx}\right)_{x=2} = \tan\frac{\pi}{3} = \sqrt{3}$

$f'(3) = \left(\frac{dy}{dx}\right)_{x=3} = \tan\frac{\pi}{4} = 1$

Let , $I = \int_1^3 f'(x).f''(x)dx + \int_2^3 f''(x)dx = I_1 + I_2$

$\therefore$ $I_1 = \int_1^3 f'(x).f''(x)dx$

$I_1 = f'(x).f'(x)\Big|_1^3 - \int_1^3 f''(x)f'(x)\,dx$

$2I_1 = \{f'(3)\}^2 - \{f'(1)\}^2$

$2I_1 = 1 - \frac{1}{3}$

$I_1 = \frac{1}{3}$ and, $I_2 = \int_2^3 f''(x)dx = f'(x)\Big|_2^3 = f'(3) - f'(2) = 1 - \sqrt{3}$

$\therefore$ $I = I_1 + I_2 = \frac{1}{3} + 1 - \sqrt{3} = \frac{4}{3} - \sqrt{3}$

4. $\displaystyle\int_{5/2}^{5} \frac{\sqrt{\left(25-x^2\right)^3}}{x^4}\,dx$ equals to :

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) none

Ans. **(A)**

Sol. Let x = 5 sinθ ⇒ dx = 5 cos θ dθ

$$\therefore \quad I = \int_{\pi/6}^{\pi/2} \frac{\sqrt{(25-25\sin^2\theta)^3}}{5^4\sin^4\theta}\,.\,5\cos\theta\,d\theta = \int_{\pi/6}^{\pi/2}\frac{5^3\cos^3\theta.5\cos\theta}{5^4\sin^4\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}\cot^2\theta(\text{cosec}^2\theta-1)\,d\theta = \int_{\pi/6}^{\pi/2}\cot^2\theta\,\text{cosec}^2\theta\,d\theta - \int_{\pi/6}^{\pi/2}\cot^2\theta\,d\theta$$

$$= \int_{\pi/6}^{\pi/2}\cot^2\theta\,\text{cosec}^2\theta\,d\theta - \int_{\pi/6}^{\pi/2}(\text{cosec}^2\theta-1)\,d\theta$$

Let $I_1 = \displaystyle\int_{\pi/6}^{\pi/2}\cot^2\theta\,\text{cosec}^2\theta\,d\theta$ put cot θ = t

so $-\text{cosec}^2\theta\,d\theta = dt$

i.e. $I_1 = \displaystyle\int_{\sqrt{3}}^{0} t^2\,dt = \sqrt{3}$

Let $I_2 = \displaystyle\int_{\pi/6}^{\pi/2}(\text{cosec}^2\theta-1)\,d\theta = \sqrt{3} - \frac{\pi}{3}$

so $I_1 - I_2 = \frac{\pi}{3}$

MCQ (One or more than one correct) :

5. If $\displaystyle\int_{-\alpha}^{\alpha} e^x + \cos x\ln\left(x\sqrt{1+x^2}\right)dx > \frac{3}{2}$, then α can be _____.

(A) 1 (B) 2 (C) 3 (D) 4

Ans. **(ABCD)**

Sol. Let $\displaystyle\int_{-\alpha}^{\alpha} e^x + \cos x\ln\left(x+\sqrt{1+x^2}\right)dx$

$$= \int_{-\alpha}^{\alpha} e^x\,dx + \int_{-\alpha}^{\alpha}\cos x\,dx\,\ln\left(x+\sqrt{1+x^2}\right)dx$$

Since cos x is an even function and ln $\left(x+\sqrt{1+x^2}\right)$ is an odd function,

so cos ln $\left(\sqrt{1+x^2}\right)$ is an odd function

Hence $\displaystyle\int_{-\alpha}^{\alpha}\cos x\ln\left(x+\sqrt{1+x^2}\right)dx = 0$

Thus, $I=\int_{-\alpha}^{\alpha} e^x dx = e^{\alpha}-e^{-\alpha}$

Now since $I>\frac{3}{2} \Rightarrow e^{\alpha}-e^{-\alpha}>\frac{3}{2}$

$\Rightarrow 2e^{2\alpha}-3e^{\alpha}-2>0 \Rightarrow e^{\alpha}<-\frac{1}{2}$ or $e^{\alpha}>2$

But $e^{\alpha}>0 \forall \alpha \in \mathbb{R} \Rightarrow e^{\alpha}>2$

$\Rightarrow \alpha \in (\log_e 2, \infty)$

6. The value of the definite integral $\int_0^1 \frac{dx}{\sqrt{(x+1)^3(3x+1)}}$ equals ______.

(A) $\sqrt{2}-1$ (B) $\tan\frac{\pi}{2}$ (C) $\tan\frac{\pi}{8}$ (D) $\tan\frac{5\pi}{12}$

Ans. **(AC)**

Sol. $I=\int_0^1 \frac{dx}{\sqrt{(x+1)^3(3x+1)}}=\int_0^1 \frac{dx}{\sqrt{(x+1)(3(x+1)-2)}}$

Put $x+1=\frac{1}{t} \Leftrightarrow dx=-\frac{1}{t^2}dt$

$\therefore I=\int_{\frac{1}{2}}^{1} \frac{dt}{\sqrt{3-2t}}=\sqrt{2}-1$

7. If $f(x) = 2^{\{x\}}$, where $\{x\}$ denotes the fractional part of x. Then which of the following is true ?

(A) f is periodic

(B) $\int_0^1 2^{\{x\}}dx=\frac{1}{\ell n2}$

(C) $\int_0^1 2^{\{x\}}\, dx=\log_2 e$

(D) $\int_0^{100} 2^{\{x\}}\, dx=100\log_2 e$

Ans. **(ABCD)**

Sol. $f(x) = 2^{\{x\}}$

Clearly f(x) is periodic with period 1.

Now $\int_0^1 2^{\{x\}}dx = \int_0^1 2^x dx = \left[\frac{2^x}{\ell n2}\right]_0^1 = \frac{1}{\ell n2} = \log_2 e$

Also $\int_0^{100} 2^{\{x\}}dx = 100\int_0^1 2^{\{x\}}dx = 100 \log_2 e$ $\left[\text{Using} \int_0^{na} f(x)dx = n\int_0^a f(x)dx \text{ if a is the period of } f(x)\right]$

Comprehension Type Question:

Comprehension # 1

Consider the integral $I = \int_0^{10\pi} \frac{\cos 6x \cos 7x \cos 8x \cos 9x}{1+e^{2\sin^3 4x}} dx$

8. If $k \int_0^{\pi/2} \cos 6x \cos 7x \cos 8x \cos 9x \, dx$ then k is _____.

(A) 1 (B) 5 (C) 10 (D) 20

Ans. **(C)**

Sol. Let $f(x) = \frac{\cos 6x \cos 7x \cos 8x \cos 9x}{1+e^{2\sin^3 4x}}$

As $f(\pi + x) = f(x)$

$$\therefore \quad I = 10\int_0^{\pi} \frac{\cos 6x \cos 7x \cos 8x \cos 9x}{1+e^{2\sin^3 4x}} dx \quad .....(1)$$

$$I = 10\int_0^{\pi} \frac{\cos 6x \cos 7x \cos 8x \cos 9x \left(e^{2\sin^3 4x}\right)}{1+e^{2\sin^3 4x}} dx \quad .....(2)$$

$$2I = 10\int_0^{\pi} \cos 6x \cos 7x \cos 8x \cos 9x \, dx$$

$$I = 5\int_0^{\frac{\pi}{2}} \cos 6x \cos 7x \cos 8x \cos 9x \, dx \quad = 10\int_0^{\frac{\pi}{2}} \cos 6x \cos 7x \cos 8x \cos 9x dx$$

$\therefore$ k = 10

9. If $I = \lambda \int_0^{\pi/4} \cos 6x \cos 8x \cos 2x \, dx$ then λ is _____.

(A) 5 (B) 10 (C) 20 (D) $\frac{5}{2}$

Ans. **(B)**

Sol. $I = 5\int_0^{\frac{\pi}{2}} \cos 6x \cos 8x \left(\cos 16x \cos 2x\right) dx$

$$I = 5\int_0^{\frac{\pi}{2}} \left(\cos 6x \cos 8x \cos 2x\right) dx + 5\int_0^{\frac{\pi}{2}} \cos 6x \cos 8x \cos 16x \, dx$$

$$= 10\int_0^{\frac{\pi}{4}} \left(\cos 6x \cos 8x \cos 2x\right) dx + 0$$

$\therefore \lambda$ = 10

10. The value of I is _____.

(A) $\frac{5\pi}{4}$ (B) $\frac{5\pi}{16}$ (C) $\frac{5\pi}{32}$ (D) $\frac{5\pi}{8}$

Ans. **(D)**

Sol. $I = 5\int_0^{\frac{\pi}{4}} (\cos 14x + \cos 2x)\cos 2x \, dx$ $\quad = 5\int_0^{\frac{\pi}{4}} \frac{(\cos 16x + \cos 12x)\cos 4x + 1}{2} dx$

$= \frac{5}{2}\left[x + \frac{\sin 4x}{4} + \frac{\sin 12x}{12} + \frac{\sin 16x}{16}\right]_0^{\frac{\pi}{4}}$ $\quad = \frac{5\pi}{8}$

Numerical based Questions :

11. Given $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\int_0^1 \frac{\ln x}{1-x^2} dx = -\frac{\pi^2}{\lambda}$ then value of λ is _____.

Ans. **8**

Sol. $\int_0^1 \frac{\ln x}{1-x^2} dx \left[\ln x \times \frac{1}{2}\ln\frac{1+x}{1-x}\right]_0^1 - \frac{1}{2}\int_0^1 \ln\left(\frac{1+x}{1-x}\right) \times \frac{1}{x} dx$

$= 0 - \frac{1}{2}\int_0^1 \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - ....\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \right)}{x} dx$

$= -\int_0^1 \left(1 + \frac{x^2}{3} + \frac{x^4}{5} + ...\right) dx = -\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right)$

As $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6}$

or, $\left(\frac{1}{1^2} + \frac{1}{3^2} + ...\right) + \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{6}$

$\therefore \int_0^1 \frac{\ln x}{1-x^2} dx = -\frac{3}{4} \times \frac{\pi^2}{6} = -\frac{\pi^2}{8}$

12. $\int_1^e \left(\frac{1}{\sqrt{x \ln x}} + \sqrt{\frac{\ln x}{x}}\right) dx = L$ then the value of $\frac{3L}{\sqrt{e}}$ is _____.

Ans. **6**

Sol. $I = \int_1^e \frac{1 + \ln x}{\sqrt{x \ln x}} dx$

Putting $(x \ln x) = t^2$, $(\ln x + 1)dx = 2t dt$, we get

$I = \int_0^{\sqrt{e}} \frac{2t dt}{t} = 2\sqrt{e}$