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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-16

Indefinite Integration

ANSWER KEY OF CAPS-16

1.	(A)	**2.**	(A)	**3.**	(A)	**4.**	(C)	**5.**	(D)
6.	(C)	**7.**	(C)	**8.**	(A)	**9.**	(D)	**10.**	(B)
11.	(D)	**12.**	(10)	**13.**	(16)	**14.**	(12)	**15.**	(1)
16.	(8)	**17.**	(A) R ; (B) P ; (C) Q			**18.**	(A) R, (B) S, (C) Q		

SCQ (Single Correct Type) :

1. The value of $\int x \cdot \dfrac{\ell n\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx$ equals:

(A) $\sqrt{1+x^2}\ \ell n\left(x+\sqrt{1+x^2}\right) - x + C$

(B) $\dfrac{x}{2}\cdot \ell n^2\left(x+\sqrt{1+x^2}\right) - \dfrac{x}{\sqrt{1+x^2}} + C$

(C) $\dfrac{x}{2}\cdot \ell n^2\left(x+\sqrt{1+x^2}\right) + \dfrac{x}{\sqrt{1+x^2}} + C$

(D) $\sqrt{1+x^2}\ \ell n\left(x+\sqrt{1+x^2}\right) + x + C$

Ans. (A)

Sol. $\int x \cdot \dfrac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx$

$I = \int \ell n\,(x+\sqrt{1+x^2}) \cdot \dfrac{x}{\sqrt{1+x^2}}\,dx$

$= \sqrt{1+x^2}\ \ell n\,(x+\sqrt{1+x^2}) - \int \dfrac{1}{\sqrt{1+x^2}} \cdot \sqrt{1+x^2}\,dx$ (using integration by parts)

$= \sqrt{1+x^2}\ \ell n\,(x+\sqrt{1+x^2}) - x + C$

Alternate:

Let $x + \sqrt{1+x^2} = e^t$ then $\sqrt{1+x^2} - x = e^{-t}$

$\dfrac{dx}{\sqrt{1+x^2}} = dt$

$I = \int x\,t\,dt = \int \dfrac{(e^t - e^{-t})t\,dt}{2} = \dfrac{1}{2}\,t(e^t + e^{-t}) - \dfrac{1}{2}\,(e^t - e^{-t}) + C = \sqrt{1+x^2}\ \ell n\,(x+\sqrt{1+x^2}) - x + C.$

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2. The value of $2\int \sin x \,.\, \text{cosec}\,4x\, dx$ is equal to

(A) $\frac{1}{2\sqrt{2}}\ \ell n \left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| - \frac{1}{4}\ \ell n \left|\frac{1+\sin x}{1-\sin x}\right| + C$

(B) $\frac{1}{2\sqrt{2}}\ \ell n \left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + \frac{1}{4}\ \ell n \left|\frac{1+\sin x}{1-\sin x}\right| + C$

(C) $\frac{1}{2\sqrt{2}}\ \ell n \left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| - \frac{1}{4}\ \ell n \left|\frac{1+\sin x}{1-\sin x}\right| + C$

(D) none of these

Ans. **(A)**

Sol. $I = 2\int \sin x \,.\, \text{cosec}\,4x\ dx = 2\int \frac{\sin x\, dx}{4\sin x \cos x \cos 2x}$

$= \frac{1}{2}\int \frac{\cos x dx}{(1-\sin^2 x)(1-2\sin^2 x)}$ Put sinx = t, then cos x dx = dt

$\Rightarrow I = \frac{1}{2}\int \frac{dt}{(1-t^2)(1-2t^2)} = \frac{1}{2}\left[-\int \frac{1}{1-t^2}dt + \int \frac{2}{1-2t^2}dt\right]$ [By partial fraction]

$= \frac{1}{2}\left[-\frac{1}{2}\ell n\left|\frac{1+t}{1-t}\right| + \frac{2}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}t}{1-\sqrt{2}\,t}\right)\right|\right] = \frac{1}{2\sqrt{2}}\ \ell n\left|\left(\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right)\right| - \frac{1}{4}\ \ell n \left|\left(\frac{1+\sin x}{1-\sin x}\right)\right| + C$

3. The values of a nad b which satisfy $\int \frac{2\sin 2x - \cos x}{4-\cos^2 x - 4\sin x}dx = a\log|\sin x - 1| + b\log|\sin x - 3| + c$, are ______.

(A) $-\frac{3}{2},\frac{11}{2}$ (B) $\frac{3}{2},\frac{11}{2}$ (C) $\frac{3}{2},-\frac{11}{2}$ (D) $-\frac{3}{2},-\frac{11}{2}$

Ans. **(A)**

Sol. $\int \frac{2\sin 2x - \cos x}{\sin^2 x - 4\sin x + 3}dx = \int \frac{(4\sin x - 1)\cos x dx}{(\sin x - 3)(\sin x - 1)}$

let $\sin = t \Rightarrow \cos x dx = dt$

$I = \int \frac{(4t-1)dt}{(t-3)(t-1)} = \int \frac{11}{2(t-3)}dt - \frac{3}{2(t-1)}dt$

$I = \frac{11}{2}\ln|\sin x - 3| - \frac{3}{2}\ln|\sin x - 1| + c$

On comparing, $a = \frac{-3}{2},\ b = \frac{11}{2}$

4. Let $\int \frac{g(x)((1+x)\ln x - 5)}{x(\ln x)^6}dx = \frac{g(x)}{(\ln x)^5} + c, g(1) = e$ and $g:(0,e)\to(0,\infty)$. If $g^n(x)$ denotes the n^{th} derivative of g(x), then the value of $g^{2012}(1)$ is ______.

(A) 2011g(1) (B) 2012g(1) (C) 2013g(1) (D) 2014g(1)

Ans. **(C)**

Sol. $\int \frac{g(x)\left((1+x)\ln x-5\right)}{x(\ln x)^6}dx = \frac{g(x)}{(\ln x)^5}+c$

$\therefore \frac{g(x)\left((1+x)\ln x-5\right)}{x(\ln x)^6} = \frac{(x\ln x)g'(x)-5g(x)}{x(\ln x)^6}$

$\Rightarrow (1+x)g(x) = xg'(x) \Rightarrow \frac{g'(x)}{g(x)} = 1+\frac{1}{x}$ $\qquad \Rightarrow \ln g(x) = x+\ln x \Rightarrow g(x) = kxe^x$

$\therefore g^n(x) = ne^x + xe^x$

$\Rightarrow g^n(1) = (n+1)e = (n+1)g(1)$

5. $\int e^{\sin x}\left(\frac{x\cos^3 x-\sin x}{\cos^2 x}\right)dx =$

(A) $e^{\sin x}(x+\sec x)+c$

(B) $e^{\sin x}(1+\sec x)+c$

(C) $e^{\sin x}(1-\sec x)+c$

(D) $e^{\sin x}(x-\sec x)+c$

Ans. **(D)**

Sol. $\int e^{\sin x}\left(\frac{x\cos^3 x-\sin x}{\cos^2 x}\right)dx = \int e^{\sin x}(x\cos x-\sec x\tan x)dx$

$= \int e^{\sin x}\left((x\cos x+1)-(1+\sec x\tan x)\right)dx$

$= \int e^{\sin x}(x) - e^{\sin x}(\sec x)+c$

$= e^{\sin x}(x-\sec x)+c$

6. If $\int \frac{xdx}{\sqrt[2012]{(1+x^2)^{1012}(2+x^2)^{3012}}} = \frac{\alpha}{\beta}(1-f(x))^{\frac{\beta}{2\alpha}}+k$ then which is true

(A) $\alpha = 503; \beta = 500, f(\sqrt{2}) = \frac{1}{\beta-\alpha}$

(B) $\alpha = 503; \beta = 250, f(\sqrt{2}) = \frac{1}{\alpha-\beta}$

(C) $\alpha = 503; \beta = 500, f(1) = \frac{1}{\alpha-\beta}$

(D) $\alpha = 503; \beta = 225, f(\sqrt{3}) = \frac{1}{\alpha-\beta}$

Ans. **(C)**

Sol. $\int \frac{x}{(1+x^2)^2\left(\frac{2+x^2}{1+x^2}\right)^{\frac{3012}{2012}}}dx$

Let $\frac{2+x^2}{1+x^2} = t$ then $f(x) = \frac{1}{2+x^2}$

$f(1) = \frac{1}{3} = \frac{1}{\alpha-\beta}$

where α = 503, β = 500

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7. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx = \frac{P(x)}{\sin^{2010} x} + C$, then value of $P\left(\frac{\pi}{3}\right)$ is _____.

(A) 0 (B) $\frac{1}{\sqrt{3}}$ (C) $\sqrt{3}$ (D) None of these

Ans. **(C)**

Sol. $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx \quad = \int \sec^2 (\sin x)^{-2010}\ 2010 \int \frac{1}{(\sin x)^{2010}} dx - I_1 - I_2$

Applying, by parts on I_1, we get

$$I_1 = \frac{\tan x}{(\sin x)^{2010}} + 2010 \int \frac{\tan x \cos x}{(\sin x)^{2011}} dx = \frac{\tan x}{(\sin x)^{2010}} + 2010 \int \frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I = I_1 - I_2 = \frac{\tan x}{(\sin x)^{2010}} = \frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \sqrt{3}$$

8. If $g(x) = \frac{f(x)}{(x-a)(x-b)(x-c)}$ where f(x) is a polynomial of degree < 3, then ____.

(A) $\int g(x) dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + k$

(B) $\frac{d}{dx} g(x) = \begin{vmatrix} 1 & a & f(a)\log(x-a)^{-2} \\ 1 & b & f(b)\log(x-b)^{-2} \\ 1 & c & f(c)\log(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

(C) $\int g(x) dx = \begin{vmatrix} 1 & a & f(a)\log|x-a| \\ 1 & b & f(b)\log|x-b| \\ 1 & c & f(c)\log|x-c| \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} + k$

(D) None of these

Ans. **(A)**

Sol. $g(x) = \frac{1}{(a-b)(b-c)(c-a)} \left[\frac{f(a)(c-b)}{x-a} + \frac{f(b)(a-c)}{x-b} + \frac{f(c)(b-a)}{x-c} \right]$

$$= \begin{vmatrix} 1 & a & \frac{f(a)}{x-a} \\ 1 & b & \frac{f(b)}{x-b} \\ 1 & c & \frac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

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$$\therefore \int g(x)\,dx = \begin{vmatrix} 1 & a & \dfrac{f(a)}{x-a} \\ 1 & b & \dfrac{f(b)}{x-b} \\ 1 & c & \dfrac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & f(a)\ln|x-a| \\ 1 & b & f(b)\ln|x-b| \\ 1 & c & f(c)\ln|x-c| \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + k$$

9. If $y = 2\cot^{-1}\left(\dfrac{x}{2}\right) + x$, then $\displaystyle\int \frac{x^2 dx}{\left(x^2-4\right)\sin x + 4x\cos x}$

(A) $\log|\operatorname{cosec} 2y + \cot y| + c$ (B) $\log|\operatorname{cosec} 2y - \cot 2y| + c$

(C) $\log|\operatorname{cosec} y + \cot y| + c$ (D) $\log|\operatorname{cosec} y - \cot y| + c$

Ans. **(D)**

Sol. $y = 2\tan^{-1}\left(\dfrac{2}{x}\right) + x = \tan^{-1}\left(\dfrac{4x}{x^2-4}\right) + x \Rightarrow \tan(y-x) = \dfrac{4x}{x^2-4} \quad \Rightarrow \cos(y-x) = \dfrac{x^2-4}{x^2+4}$

Given integral $\displaystyle\int \frac{\dfrac{x^2}{x^2}-4}{\sin x + \dfrac{x^2}{x^2-4}\cos x}\,dx \int \frac{\dfrac{x^2}{x^2-4} \times \dfrac{x^2-4}{x^2+4}}{\sin y}\,dx$

But $\dfrac{dy}{dx} = \dfrac{2\times 1}{1+\dfrac{4}{x^2}} - \dfrac{2}{x^2} + 1 = \dfrac{x^2}{x^2+4} = \displaystyle\int \frac{dy}{\sin y} = \int \operatorname{cosec} y\, dy$

Comprehension Type Question:

Comprehension # 1

In calculating a number of integrals we had to use the method of integration by parts several times in succession. The result could be obtained more rapidly and in a more concise form by using the so-called generalized formula for integration by parts

$\int u(x)v(x)\,dx = u(x)\,v_1(x) - u'(x)\,v_2(x) + u''(x)\,v_3(x) + \ldots\ldots + (-1)^{n-1}\,u^{n-1}(x)\,v_n(x) - (-1)^{n-1}\int u^n(x)\,v_n(x)\,dx$

where

$v_1(x) = \int v(x)\,dx$, $v_2(x) = \int v_1(x)\,dx$ ...., $v_n(x) = \int v_{n-1}(x)\,dx$

Of course, we assume that all derivatives and integrals appearing in this formula exist. The use of the generalized formula for integration by parts is especially useful when calculating $\int P_n(x)Q(x)\,dx$, where $P_n(x)$ is polynomial of degree n and the factor Q(x) is such that it can be integrated successively n + 1 times.

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10. If $\int (x^3-2x^2+3x-1)\cos 2x\,dx = \frac{\sin 2x}{4}$ u(x) + $\frac{\cos 2x}{8}$ v(x) + C then

(A) u(x) = x^3 –4x^2 + 3x (B) u(x) = 2x^3 –4x^2 + 3x

(C) v(x) = 3x^2 –4x + 3 (D) v(x) = 6x^2 – 8x

Ans. **[B]**

Sol. The given integral is equal to (x^3 – 2x^2 + 3x – 1) $\frac{\sin 2x}{2}$ – (3x^2 – 4x + 3) $\left(-\frac{\cos 2x}{4}\right)$ + (6x – 4)

$\left(-\frac{\sin 2x}{8}\right)$ – 6 $\frac{\cos 2x}{16}$ + C

= $\frac{\sin 2x}{4}$ [2x^3 –4x^2 + 3x] + $\frac{\cos 2x}{8}$ [6x^2 –8x + 3) + C

11. If $\int (3x^2+x-2)\sin^2(3x+1)\,dx = -\frac{u(x)}{72}$ sin (6x + 2) + $\frac{v(x)}{72}$ cos (6x + 2) + $\frac{1}{2}x^3$ + $\frac{1}{4}x^2$ –x + C

(A) u(x) = 3x^2 + 6x –13 (B) u(x) = 18x^2 + 2x –13

(C) v(x) = 3x + 1 (D) v(x) = –(6x + 1)

Ans. **[D]**

Sol. First write the given integral as

$\frac{1}{2}\int (3x^2+x-2)(1-\cos(6x+2))$ .Now applying the formula in comprehension, the last integral can be written as

$\frac{1}{2}\left[x^3+\frac{x^2}{2}-2x\right]-\frac{1}{2}$

$\left[(3x^2+x-2)\frac{\sin(6x+2)}{6}+(6x+1)\frac{\cos(6x+2)}{36}-6\frac{\sin(6x+2)}{6+36}\right]$ + C= $\frac{1}{2}\left[x^3+\frac{x^2}{2}-2x\right]$ – $\frac{18x^2+6x-13}{72}$ sin

(6x+2) – $\frac{1}{72}$ (6x +1) cos (6x +2) + C

Numerical based Questions :

12. If $\int \frac{\sqrt{4+x^2}}{x^6}dx = \frac{\left(a+x^2\right)^{3/2}.\left(x^2-b\right)}{120x^5}$ + C then a + b equals to :

Ans. **10**

Sol. $I = \int \frac{\sqrt{4+x^2}}{x^6}dx$ Put x = 2 tan θ

$\Rightarrow$ dx = 2 $\sec^2\theta\, d\theta$

$\Rightarrow I = \int \frac{2\sec\theta.2\sec^2\theta\, d\theta}{2^6\tan^6\theta} = \frac{1}{2^4}\int \frac{\cos^3\theta\, d\theta}{\sin^6\theta}$

$= \frac{1}{2^4}\int\left(\frac{1-\sin^2\theta}{\sin^6\theta}\right)\cos\theta\, d\theta$ put sin θ = t

$\Rightarrow \cos\theta\, d\theta = dt$

$= \frac{1}{2^4}\int\frac{dt}{t^6} - \int\frac{1}{t^4}dt = \frac{1}{16}\left[\frac{1}{3t^3}-\frac{1}{5t^5}\right] + C = \frac{1}{16}\left(\frac{5\sin^2\theta-3}{15\sin^5\theta}\right) + C$

but $\tan\theta = \frac{x}{2}$ So $\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$$= \frac{1}{16}\left(\frac{\frac{5x^2}{x^2+4}-3}{\frac{15x^5}{\left(x^2+4\right)^{5/2}}}\right) + C = \frac{1}{16}\left(\frac{\left(2x^2-12\right)\left(x^2+4\right)^{3/2}}{15x^5}\right) + C = \frac{1}{120}\left(\frac{\left(x^2+4\right)^{3/2}\left(x^2-6\right)}{x^5}\right) + C$$

13. Let $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$ and $g(0) = 0$ then value of $32\,g\left(\frac{\pi}{2}\right)$ is.

Ans. **16**

Sol. $g(x) = \int \frac{1+2\cos x}{(\cos x+2)^2}dx$; $g(x) = \int \frac{\cos x(\cos x+2)+\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \int \cos x.\frac{1}{(\cos x+2)}dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$= \frac{1}{(\cos x+2)}.\sin x - \int -\frac{(-\sin x)}{(\cos x+2)^2}.\sin x dx + \int \frac{\sin^2 x}{(\cos x+2)^2}dx$

$g(x) = \frac{\sin x}{(\cos x+2)} + C$

$g(0) = 0$

$\Rightarrow C = 0$

$g(x) = \frac{\sin x}{\cos x+2}$

$g\left(\frac{\pi}{2}\right) = \frac{1}{2}$; $32g\left(\frac{\pi}{2}\right) = 16$

14. If $f(x) = \sqrt{x-1}$; $g(x) = e^x$ and $\int fog(x)dx = A fog(x) + B\tan^{-1}(fog(x)) + C$ then $A^3 + B^2$ equals

Ans. **12**

Sol. $fog(x) = \sqrt{e^x-1}$

$\int fog(x)dx = \int \sqrt{e^x-1}\,dx$ put $e^x - 1 = t^2$

$= \int \frac{2t^2}{1+t^2}dt = 2t - 2\tan^{-1}t + C = 2\sqrt{e^x-1} - 2\tan^{-1}\sqrt{e^x-1} + C$

$A^3 + B^2 = 2^3 + (-2)^2 = 12$

15. If $\int \frac{1+x\cos x}{x\left(1-x^2e^{2\sin x}\right)}dx = k\,\ell n\sqrt{\frac{x^2e^{2\sin x}}{1-x^2e^{2\sin x}}} + C$ then k is equal to :

Ans. **1**

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Sol. $\int \frac{1+x\cos x}{x\left(1-x^2 e^{2\sin x}\right)}\,dx$

Put $xe^{\sin x} = t \Rightarrow (xe^{\sin x}.\cos x + e^{\sin x})\,dx = dt \quad e^{\sin x}(x\cos x + 1)\,dx = dt$

$\Rightarrow \quad I = \int \frac{dt}{t(1-t^2)} = \int \frac{1}{t(1-t)(1+t)}dt$

Let $\frac{1}{t(1-t)(1+t)} = \frac{A}{t} + \frac{B}{(1-t)} + \frac{C}{1+t}$

$1 = A(1-t)(1+t) + B(t)(1+t) + C(t)(1-t)$

put $t = 0 \quad \Rightarrow \quad A = 1$

Put $t = 1 \quad \Rightarrow \quad B = 1/2$

Put $t = -1 \quad \Rightarrow \quad C = -1/2$

$\Rightarrow \quad I = \int\left\{\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)}\right\}dt = \ell n\,|t| - \frac{1}{2}\ell n\,|1-t| - \frac{1}{2}\ell n\,|1+t| + C$

$= \ell n\,|x\,e^{\sin x}| - \frac{1}{2}\log|1 - x^2 e^{2\sin x}| + C = \ell n\sqrt{\frac{x^2e^{2\sin x}}{1-x^2e^{2\sin x}}} + C$

16. Let $\int \frac{f'(x)g(x) - g'(x)f(x)}{\left(f(x)+g(x)\right)\sqrt{f(x)g(x) - g^2(x)}}dx = \sqrt{m}\tan^{-1}\left(\sqrt{\frac{f(x)-g(x)}{ng(x)}}\right) + C$,

where m, n $\in$ N and 'C' is constant of integration $(g(x) > 0)$. Find the value of $(m^2 + n^2)$.

Ans. 8

Sol. We have $I = \int \frac{\{f'(x)g(x) - g'(x)f(x)\}dx}{\{f(x)+g(x)\}\sqrt{g(x)(f(x) - g^2(x))}}$

$I = \int \frac{\left\{\frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}\right\}}{\left(\frac{f(x)}{g(x)} + 1\right)\sqrt{\frac{f(x)}{g(x)} - 1}}dx$

Let $\frac{f(x)}{g(x)} - 1 = t^2 \quad \Rightarrow \frac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}dx = 2t\,dt$

$\therefore\ I = \int \frac{2t\,dt}{(t^2+2)\cdot t} = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \sqrt{2}\tan^{-1}\left\{\sqrt{\frac{f(x)}{2g(x)} - \frac{1}{2}}\right\} + C$

$= \sqrt{2}\tan^{-1}\left\{\sqrt{\frac{f(x)-g(x)}{2g(x)}}\right\} + C$

Hence $m = 2$, $n = 2$

$\therefore \quad m^2 + n^2 = 8$ **Ans.**

Matrix Match Type :

17. Let $I = \int \frac{e^x}{e^{4x}+1}dx$ and $J = \int \frac{e^{-x}}{e^{-4x}+1}dx$

Then for any arbitrary constant C, match the following

Column-I		**Column-II**	
(A)	I	(P)	$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+C$
(B)	J + I	(Q)	$\frac{1}{2\sqrt{2}}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)+C$
(C)	J – I	(R)	$\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)-\frac{1}{2}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)\right)+C$
		(S)	$\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+\frac{1}{2}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)\right)+C$

Ans. **(A) R ; (B) P ; (C) Q]**

Sol. Given $I = \int \frac{e^x}{1+e^{4x}}dx$ and $J = \int \frac{e^{-x}}{1+e^{-4x}}dx$

$$J = \int \frac{e^{4x}\cdot e^{-x}}{1+e^{4x}}dx = \int \frac{e^{3x}\,dx}{1+e^{4x}}$$

$\therefore$ $I + J = \int \frac{e^x(1+e^{2x})}{1+e^{4x}}dx$

put $e^x = t \Rightarrow e^x dx = dt$

$\therefore$ $I + J = \int \frac{1+t^2}{1+t^4}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$

put $t-\frac{1}{t} = y \Rightarrow y = \left(t+\frac{1}{t^2}\right)dt = dy$

$$= \int \frac{dy}{y^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t^2-1}{\sqrt{2}\,t}$$

$J + I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+C$ ....(1) $\Rightarrow$ **(P)**

|||ly $J - I = \int \frac{e^x(-1+e^{2x})}{1+e^{4x}}dx = \int \frac{-1+t^2}{1+t^4}dt = +\int \frac{-1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = +\int \frac{-1+\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}dt \left(t+\frac{1}{t}=y\right)$

$\therefore$ $J - I = +\int \frac{dy}{y^2-2} = \frac{1}{2\sqrt{2}}\ln\frac{y-\sqrt{2}}{y+\sqrt{2}}$

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$$J - I = -\frac{1}{2\sqrt{2}}\ln\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}} = \frac{1}{2\sqrt{2}}\cdot\frac{t^2-\sqrt{2}\,t+1}{t^2+\sqrt{2}\,t+1} = \frac{1}{2\sqrt{2}}\cdot\frac{e^{2x}-\sqrt{2}\,e^x+1}{e^{2x}+\sqrt{2}\,e^x+1} \quad ....(2) \Rightarrow \textbf{(Q)}$$

(1) – (2) gives

$$2I = \frac{1}{\sqrt{2}}\tan^{-1}\frac{e^{2x}-1}{\sqrt{2}\,e^x} - \frac{1}{2\sqrt{2}}\ln\frac{e^{2x}-\sqrt{2}\,e^x+1}{e^{2x}+\sqrt{2}\,e^x+1}$$

$$I = \frac{1}{2\sqrt{2}}\left[\tan^{-1}\frac{e^{2x}-1}{\sqrt{2}\,e^x} - \frac{1}{2}\ln\frac{e^{2x}-\sqrt{2}\,e^x+1}{e^{2x}+\sqrt{2}\,e^x+1}\right] \qquad \Rightarrow \qquad \textbf{(R)}$$

18.

Column-I	**Column-II**
(A) $\int \frac{x^2(x^6+x^5-1)dx}{(2x^6+3x^5+2)^2}$	(P) $\frac{1}{6}\left(\frac{1}{x^{-3}}+\frac{3}{x^{-2}}\right)^{\frac{1}{2}}+C$
(B) $\int \frac{(x^5+x^4+x^2)dx}{\sqrt{4x^7+5x^6+10x^4}}$	(Q) $\frac{1}{2}(1 + x^{-2} + x^{-5})^{-2} + C$
(C) $\int \frac{(2x^{12}+5x^9)dx}{(x^5+x^3+1)^3}$	(R) $\frac{-1}{6}\ (2x^3 + 3x^2 + 2x^{-3})^{-1} + C$
(where C is the constant of integration.)	(S) $x\left(\frac{x^3}{25}+\frac{x^2}{20}+\frac{1}{10}\right)^{\frac{1}{2}}+C$

Ans. **(A) R, (B) S, (C) Q**

Sol. **(A)** Let $I_1 = \int \frac{x^2(x^6+x^5-1)dx}{(2x^6+3x^5+2)^2} = \int \frac{x^2(x^6+x^5-1)dx}{x^6\left(2x^3+3x^2+\frac{2}{x^3}\right)^2} = \int \frac{(x^2+x-x^{-4})dx}{(2x^3+3x^2+2x^{-3})^2}$

Putting $2x^3 + 3x^2 + 2x^{-3} = t \Rightarrow 6\left(x^2+x-\frac{1}{x^4}\right)dx = dt \quad \Rightarrow \frac{(x^6+x^5-1)}{x^4}\ dx = \frac{1}{6}dt$

Hence $I_1 = \frac{1}{6}\int\frac{dt}{t^2} = \frac{-1}{6t} + C = \frac{-1}{6(2x^3+3x^2+2x^{-3})} + C = \frac{-1}{6}\ (2x^3 + 3x^2 + 2x^{-3})^{-1} + C$ **Ans.**

(B) Let $I_2 = \int \frac{(x^5+x^4+x^2)dx}{\sqrt{4x^7+5x^6+10x^4}} = \int \frac{(x^5+x^4+x^2)dx}{x\sqrt{4x^5+5x^4+10x^2}} = \int \frac{(x^4+x^3+x)dx}{\sqrt{4x^5+5x^4+10x^2}}$

Putting $4x^5 + 5x^4 + 10x^2 = t \Rightarrow 20\ (x^4 + x^3 + x)\ dx = dt$

Hence $I_2 = \frac{1}{20}\int\frac{dt}{\sqrt{t}} = \frac{1}{10}\sqrt{t}+C = \frac{1}{10}\sqrt{4x^5+5x^4+10x^2} + C = \frac{1}{10}x\sqrt{4x^3+5x^2+10}+C$

$= x\left(\frac{x^3}{25}+\frac{x^2}{20}+\frac{1}{10}\right)^{\frac{1}{2}}+C$ **Ans.**

(C) Let $I_3 = \int \frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx = \int\frac{2x^{12}+5x^9}{x^{15}(1+x^{-2}+x^{-5})^3}dx = \int\frac{(2x^{-3}+5x^{-6})dx}{(1+x^{-2}+x^{-5})^3}dx$

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Putting $1 + x^{-2} + x^{-5} = t \Rightarrow (-2x^{-3} - 5x^{-6})\,dx = dt$

$\therefore I_3 = -\int \frac{dt}{t^3} = \frac{1}{2t^2} + C = \frac{1}{2}\left(1 + x^{-2} + x^{-5}\right)^{-2} + C$ **Ans.**]

Subjective Type Questions :

19. Evaluate : $\int \cos x \cdot e^x \cdot x^2 dx$

Ans. $\frac{1}{2}e^x\,[(x^2 - 1)\cos x + (x - 1)^2 \,.\, \sin x] + C$

Sol. $I = \int \cos x.\, e^x x^2\, dx$

$I = x^2 \int \cos x.\, e^x dx - \int 2x \left(\int \cos x . e^x dx\right) dx$

$\left(\because \quad \int \cos x . e^x \; dx = \frac{e^x}{2}(\cos x + \sin x) \;\&\; \int \sin x . e^x dx = \frac{e^x}{2}(\sin x - \cos x)\right)$

$= x^2 \left[\frac{e^x}{2}(\cos x + \sin x)\right] - \int x . e^x (\cos x + \sin x)\; dx$

$= \frac{x^2 e^x}{2}\;(\cos x + \sin x) - \int x e^x \cos x\, dx - \int x e^x \sin x\; dx$

$= \frac{x^2 e^x}{2}\;(\cos x + \sin x) - \left(x\frac{e^x}{2}(\cos x + \sin x) - \int \frac{e^x}{2}(\cos x + \sin x)\; dx\right)$

$- \left(x\frac{e^x}{2}(\sin x - \cos x) - \int \frac{e^x}{2}(\sin x - \cos x)\; dx\right)$

$= \frac{x^2 e^x}{2}\;(\cos x + \sin x) - xe^x \sin x + \int e^x \sin x\; dx$

$= \frac{x^2 e^x}{2}(\cos x + \sin x) - xe^x \sin x + \frac{e^x}{2}\;(\sin x - \cos x) + C$

$= \frac{e^x \cos x}{2}\;(x^2 - 1) + \frac{e^x \sin x}{2}\;(x^2 - 2x + 1) + C$

$= \frac{e^x}{2}\;((x^2 - 1)\cos x + (x - 1)^2 \sin x) + C$

20. Evaluate : $\tan^{-1} x.\ \ell n\,(1 + x^2)\, dx.$

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Ans. $x \tan^{-1} x.\ \ell n\,(1 + x^2) + (\tan^{-1} x)^2 - 2x \tan^{-1} x + \ell n\,(1 + x^2) - \left(\ell n\sqrt{1 + x^2}\right)^2 + C$

Sol. $\int \tan^{-1} x . \ell n(1 + x^2)\; dx$

$= x \tan^{-1} x \,.\, \ell n(1 + x^2) - \int \frac{x}{1 + x^2} \ell n(1 + x^2)\, dx \; - \int \frac{2x^2 \tan^{-1} x}{(1 + x^2)} dx$

$= x \tan^{-1} x\ \ell n\,(1 + x^2) - \frac{1}{4}\ \ell n^2\,(1 + x^2) - 2\int \tan^{-1} x\; dx + 2\int \frac{\tan^{-1} x}{1 + x^2} dx$

$= x \tan^{-1}\ \ell n\,(1 + x^2) - \frac{1}{4} \times 4\ \ell n^2 \left(\sqrt{1 + x^2}\right) - 2x \tan^{-1} x + \ell n\,(1 + x^2) + (\tan^{-1} x)^2 + C$

21. Evaluate : $\int e^x \dfrac{1+nx^{n-1}-x^{2n}}{\left(1-x^n\right)\sqrt{1-x^{2n}}}dx$

Ans. $e^x\sqrt{\dfrac{1+x^n}{1-x^n}}+C$

Sol. $I=\int e^x\left(\dfrac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx = \int e^x\left(\sqrt{\dfrac{1+x^n}{1-x^n}}+\dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx$

Let $f(x)=\sqrt{\dfrac{1+x^n}{1-x^n}} \Rightarrow f'(x)=\dfrac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}$

$\therefore \quad I = e^x.\sqrt{\dfrac{1+x^n}{1-x^n}}+C$

22. If $\int \dfrac{x\cos\alpha+1}{\left(x^2+2x\cos\alpha+1\right)^{3/2}}dx=\dfrac{f(x)}{\sqrt{g(x)}}+C$ then find f(x) and g(x).

Ans. x; $x^2+2x\cos\alpha+1$

Sol. $I=\int\dfrac{(x\cos\alpha+1).\,dx}{(x^2+2x\cos\alpha+1)^{3/2}} = \int\dfrac{x^3\left(x\dfrac{\cos\alpha}{x^3}+\dfrac{1}{x^3}\right)dx}{x^3\left(1+\dfrac{2\cos\alpha}{x}+\dfrac{1}{x^2}\right)^{3/2}}$

Put $1+\dfrac{2\cos\alpha}{x}+\dfrac{1}{x^2}=t$, then $\left(\dfrac{\cos x}{x^2}+\dfrac{1}{x^3}\right)dx=-\dfrac{dt}{2}$

$\Rightarrow \quad I=\int\dfrac{-dt/2}{t^{3/2}}=\dfrac{1}{t^{1/2}}+C=\dfrac{x}{\sqrt{x^2+2x\cos\alpha+1}}+C$

comparing : f(x) = x and, g(x) = $x^2+2x\cos\alpha+1$

23. $\int\dfrac{dx}{(\sin^2x+2\cos^2x)^2}dx$

Ans. $\dfrac{3}{8\sqrt2}\tan^{-1}\left(\dfrac{t^2x-2}{2\sqrt2\tan x}\right)-\dfrac{\tan x}{4(\tan^2x+2)}+C$

Sol. Let $I=\int\dfrac{dx}{(\sin^2x+2\cos^2x)^2}=\int\dfrac{\sec^4x\,dx}{(2+\tan^2x)^2}dx=\int\dfrac{1+\tan^2x}{(2+\tan^2x)^2}\sec^2x\,dx$

Putting tan x = t and $\sec^2 x\,dx = dt$, we have $I=\int\dfrac{t^2+1}{t^4+4t^2+4}dt$

Now, let $t^2+1=a(t^2+2)+b(t^2-2)$

Comparing the coefficient's we have $a=\dfrac{3}{4}$ and $b=\dfrac{1}{4}$

Hence, we have

$$I=\frac{3}{4}\int\frac{t^2+2}{t^4+4t^2+4}dt+\frac{1}{4}\int\frac{t^2-2}{t^4+4t^2+4}dt=\frac{3}{4}I_1+\frac{1}{4}I_2$$

Now, we have

$$I_1 = \int \frac{t^2+2}{t^4+4t^2+4}\,dt = \int \frac{1+\frac{2}{t^2}}{t^2\frac{4}{t^2}+4}\,dt = \int \frac{d\left(t-\frac{2}{t}\right)}{\left(t-\frac{2}{t}\right)+8}$$

$$= \int \frac{du}{u^2+8} \qquad \left[\text{putting } t-\frac{2}{t}=u\right]$$

$$= \frac{1}{\sqrt{8}}\tan^{-1}\left(\frac{u}{\sqrt{8}}\right) = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2-2}{2\sqrt{2}\,t}\right)$$

and $$I_2 = \int \frac{t^2-2}{t^4+4t^2+4}\,dt = \int \frac{1-\frac{2}{t^2}}{t^2+\frac{4}{t^2}+4}\,dt = \int \frac{d\left(t+\frac{2}{t}\right)}{\left(t+\frac{2}{t}\right)^2}$$

$$= \int \frac{dv}{v^2} \qquad \left[\text{putting } t+\frac{2}{t}=v\right]$$

$$= \frac{-1}{v} = \frac{-t}{t^2+2}$$

Hence, we have

$$I = \frac{3}{8\sqrt{2}}\tan^{-1}\left(\frac{t^2-2}{2\sqrt{2}t}\right) - \frac{t}{4(t^2+2)} + C = \frac{3}{8\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-2}{2\sqrt{2}\tan x}\right) - \frac{\tan x}{4(\tan^2 x+2)} + C.$$

24. Evaluate $\int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\frac{x}{2}}{x-\sin x}+3x\right\}dx$

Sol. $$I = \int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\frac{x}{2}}{x-\sin x}+3x\right\}dx = \int \frac{(x-\sin x)^{\frac{3}{2}}\left(6x^2\sin^2\frac{x}{2}+3x^2-3x\sin x\right)}{\sqrt{x}\,(x-\sin x)}dx$$

$$\int \frac{\sqrt{x-\sin x}}{\sqrt{x}}\left(6x^2-3x\left(\sin x+x\cos x\right)\right)dx$$

$$= 3\int \sqrt{x^2-x\sin x}\,\left(2x-\sin x-x\cos x\right)dx$$

Let $= x^2 - x\sin x \qquad \Rightarrow dt = \left(2x - x\cos x - \sin x\right)dx$

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$$\Rightarrow I = 3\int \sqrt{t}\,dt = 3\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = 2\left(x^2 - x\sin x\right)^{\frac{3}{2}} + c$$