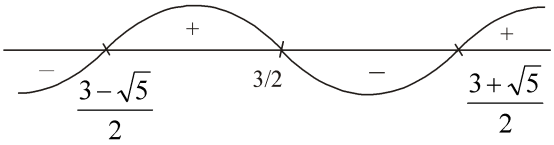
(i) $f(x) = (x^2 - 3x + 1)^2$

$\Rightarrow f'(x) = 2(x^2 - 3x + 1)(2x - 3) = 0 \Rightarrow x = \dfrac{3-\sqrt{5}}{2}, \dfrac{3}{2}$ and $\dfrac{3+\sqrt{5}}{2}$

sign scheme for f '(x) will be



Clearly f(x) has local maxima at $x = \dfrac{3}{2}$ and local minima at $x = \dfrac{3 \pm \sqrt{5}}{2}$

$\therefore$ f(x) has exactly one local maxima and two local minima. $\Rightarrow$ **(C)**

(ii) We have $g(x) = x^3 - 6x^2 + 11x + 6$

$g'(x) = 3x^2 - 12x + 11 = 3(x-2)^2 - 1 = 3\left[(x-2)^2 - \dfrac{1}{3}\right]$

$\therefore\ g'(x) > 0 \Rightarrow x \in \left(-\infty, 2 - \dfrac{1}{\sqrt{3}}\right) \cup \left(2 + \dfrac{1}{\sqrt{3}}, \infty\right)$ and $g'(x) < 0 \Rightarrow x \in \left(2 - \dfrac{1}{\sqrt{3}}, 2 + \dfrac{1}{\sqrt{3}}\right)$

$\therefore$ g(x) monotonically increases for $x \in \left(-\infty, 2 - \dfrac{1}{\sqrt{3}}\right) \cup \left(2 + \dfrac{1}{\sqrt{3}}, \infty\right)$ and monotonically decreases for

$x \in \left(2 - \dfrac{1}{\sqrt{3}}, 2 + \dfrac{1}{\sqrt{3}}\right)$

For $x \in [1, 3]$

$g(x) = (x-1)(x-2)(x-3) + 12$

$\Rightarrow g(1) = 12$ and $g(3) = 12$

$\therefore$ By Rolle's theorem in [1, 3] we have, $g'(c) = 0$

$\Rightarrow c = 2 \pm \dfrac{1}{\sqrt{3}}$ (both $\in (1, 3)$)

$\therefore$ There exists two distinct tangents to the curve y = g(x) which are parallel to the chord joining (1, g(1)) and (3, g(3))

For $x \in [0, 4]$

$g(0) = 6$ and $g(4) = 18$

$\therefore$ By LMVT

$g'(c) = \dfrac{18-6}{4-0} \Rightarrow 3c^2 - 12c + 11 = 3 \Rightarrow 3c^2 - 12c + 8 = 0 \Rightarrow c = 2 \pm \dfrac{2}{\sqrt{3}}$ (both $\in (0, 4)$)

$\therefore$ There exists exactly two distinct Lagrange's mean value in (0, 4) for y = g(x). $\Rightarrow$ **(D)**

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(iii) We have $h(x) = x^2 - 3x + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

The curve y = h(x) is an upward parabola, intersecting x axis at two distinct points.

$\therefore$ h(x) has exactly one critical point (i.e. the vertex) and no any point of inflection.

Also h(x) = 0 $\Rightarrow$ x = $\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ (both $\in$ (0, 3))

$\therefore$ h(x) = 0 has exactly two distinct real zeroes in (0, 3). $\Rightarrow$ **(C)**

Every quadratic function has exactly one tangent on x-axis or parallel to x-axis.]

Comprehension # 2

Consider f(x) = $\begin{cases} |-x^2 - x| + \lambda; & x \le 0 \\ \lim\limits_{n\to\infty} \dfrac{x^n - x^{-n}}{x^n + x^{-n}} + k; & 0 < x < 1 \quad n \in N, k \in R \\ b \quad ; & x = 1 \\ \text{sgn}(\ell n(e^x + e^{-x} + 1)); & x > 1 \end{cases}$

20. The value of k + b + λ so that f(x) is continuous in R is

(A) 3 (B) 2 (C) 4 (D) 1

Ans. [C]

Sol. $\lim\limits_{n\to\infty} \dfrac{x^n - x^{-n}}{x^n + x^{-n}} = -1$

$e^x + e^{-x} + 1 > 3 \;\; \forall x \in R$

$\therefore \ell n\,(e^x + e^{-x} + 1) > 0$

$\text{sgn}\,(\ell n\,(e^x + e^{-x} + 1)) = 1$

Now f(x) = $\begin{cases} x^2 + x + \lambda; & x < -1 \\ -x^2 - x + \lambda; & -1 \le x \le 0 \\ k - 1; & 0 < x < 1 \\ b; & x = 1 \\ 1; & x > 1 \end{cases}$

f(x) to be continuous in R

b = 1, – 1 + k = 1 $\Rightarrow$ k = 2

λ = – 1 + k $\Rightarrow$ λ = 1

k + b + λ = 2 + 1 + 1 = 4

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21. Number of point(s) where continuous function f(x) is non differentiable, is

(A) 0 (B) 1 (C) 2 (D) 3

Ans. [C]

Sol. $f(x) = \begin{cases} x^2 + x + 1; & x < -1 \\ -x^2 - x + 1; & -1 \le x \le 0 \\ 1; & x > 0 \end{cases}$

$f'(x) = \begin{cases} 2x + 1; & x < -1 \\ -2x - 1; & -1 < x < 0 \\ 0; & x > 0 \end{cases}$

$f'(x)$ does not exist at $x = -1, 0$

22. If f(x) is continuous then set of values of x for which $f'(x)$ is decreasing, is

(A) $(-\infty, -1)$ (B) $(-1, 0)$ (C) $(0, 1)$ (D) $(-1, 1)$

Ans. **[B]**

Sol. $f''(x) = \begin{cases} 2; & x < -1 \\ -2; & -1 < x < 0 \\ 0; & x > 0 \end{cases}$

$f''(x) < 0$ in $(-1, 0)$

$\therefore f'(x)$ is decreasing in this interval.

Comprehension # 3

A function f(x) having the following properties;

(i) f(x) is continuous except at x = 3

(ii) f(x) is differentiable except at x = – 2 and x = 3

(iii) $f(0) = 0$, $\lim\limits_{x\to 3} f(x) \to -\infty$, $\lim\limits_{x\to -\infty} f(x) = 3$, $\lim\limits_{x\to \infty} f(x) = 0$

(iv) $f'(x) > 0 \ \forall\ x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \le 0 \ \forall\ x \in (-2, 3)$

(v) $f''(x) > 0 \ \forall\ x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0 \ \forall\ x \in (0, 3) \cup (3, \infty)$

then answer the following questions

23. Maximum possible number of solutions of $f(x) = |x|$ is

(A) 2 (B) 1 (C) 3 (D) 4

Ans. **(C)**

24. Graph of function $y = f(-|x|)$ is

(A) differentiable for all x, if $f'(0) = 0$

(B) continuous but not differentiable at two points, if $f'(0) = 0$

(C) continuous but not differentiable at one points, if $f'(0) = 0$

(D) discontinuous at two points, if $f'(0) = 0$

Ans. **(B)**

25. $f(x) + 3x = 0$ has five solutions if

(A) $f(-2) > 6$ (B) $f'(0) < -3$ and $f(-2) > 6$

(C) $f'(0) > -3$ (D) $f'(0) > -3$ and $f(-2) > 6$

Ans. **(D)**

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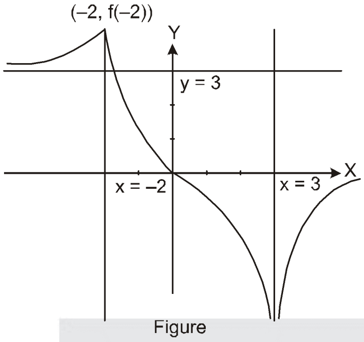
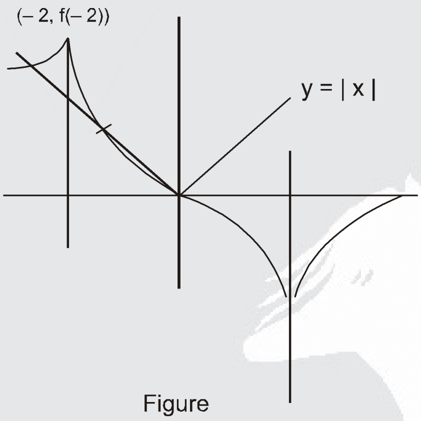
Sol. **(23 to 25)**

Graph of y = f(x)



Figure

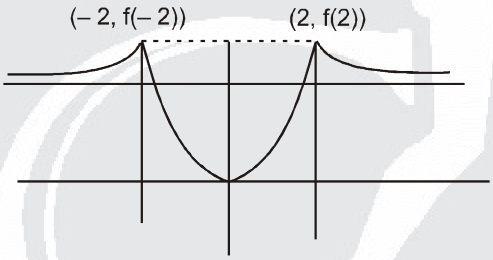
23.



Figure

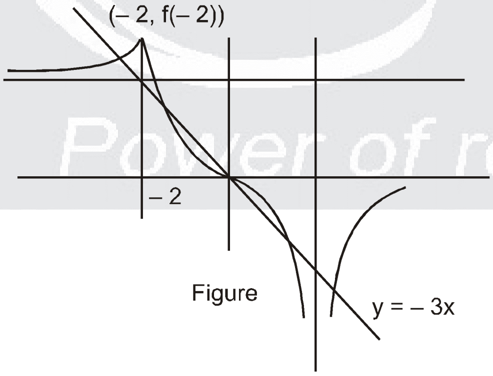
Three points of intersection. Three solutions

24.



Figure

25.



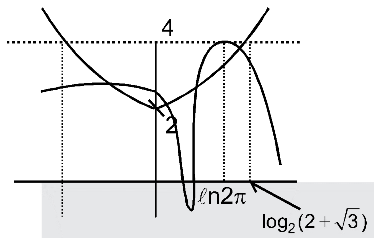
Figure

Numerical based Questions :

26. If $\ln 2\pi < \log_2\left(2+\sqrt{3}\right) < \ell n 3\pi$, then number of roots of the equation $4\cos(e^x) = 2^x + 2^{-x}$, is

Ans. **4**

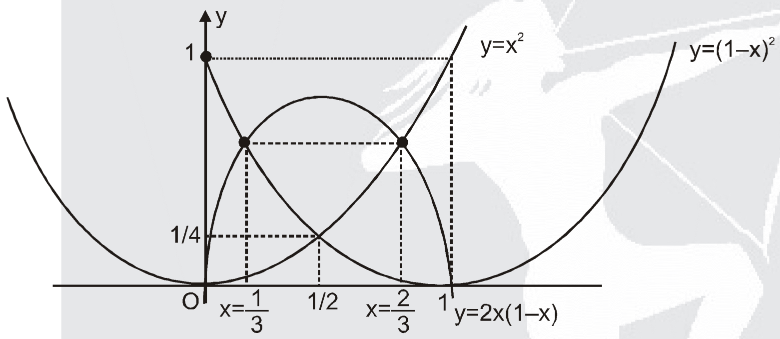
Sol. Using graph of expressions on both the sides, we get only two roots.



27. Let $f(x) = \text{Max.}\{x^2, (1-x)^2, 2x(1-x)\}$ where $x \in [0, 1]$ If Rolle's theorem is applicable for f(x) on largest possible interval [a, b] then the value of 2(a + b + c) when $c \in (a, b)$ such that $f'(c) = 0$, is

Ans. **3**

Sol.



From figure we can see Rolle's theorem is applicable for $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $f'(c) = 0 = 2 - 4c$

$\Rightarrow c = \frac{1}{2}$

$a + b + c = \frac{1}{3} + \frac{2}{3} + \frac{1}{2} = \frac{3}{2}$

28. $f(x) = \begin{cases} \left(\sqrt{2} + \sin\frac{1}{x}\right)e^{\frac{-1}{|x|}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Number of points where f (x) has local extrema when $x \neq 0$ be n_1 . n_2 be the value of global minimum of f (x) then $n_1 + n_2 =$

Ans. **(0)**

Sol. Local extremum does not occur at any value of $x \neq 0$

But global minimum = f (0) = 0

$\therefore n_1 = 0, n_2 = 0$ then $n_1 + n_2 = 0$

29. f (x) is a polynomial of 6th degree and $f(x) = f(2 - x)\ \forall\ x \in R$. If f (x) = 0 has 4 distinct real roots and two real and equal roots then sum of roots of f (x) = 0

Ans. **(6)**

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Sol. $f(\alpha) = f(2-\alpha) = 0$

Sum of roots = 4

When $\alpha \neq 2-\alpha$

Where $\alpha = 2-\alpha$ i.e., $\alpha = 1$

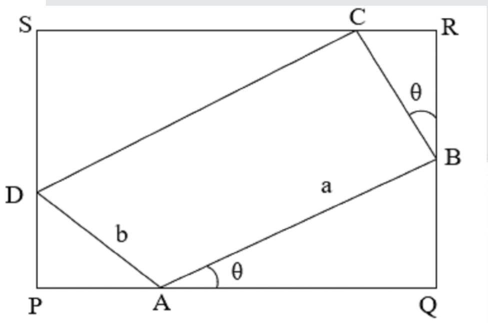
Sum of roots = 2

$\therefore$ Total sum = 6

30. ABCD and PQRS are two variable rectangles, such that A,B,C and D lie on PQ,QR,RS and SP respectively and perimeter ' x ' of ABCD is constant. If the maximum area of PQRS is 32, then $\frac{x}{4} =$

Ans. **(4)**

Sol.



$2(a+b) = x$ (a constant)

Area of PQRS = $(b\sin\theta + a\cos\theta)(a\sin\theta + b\cos\theta)$

$$= ab + \frac{a^2+b^2}{2}.\sin 2\theta \leq \frac{(a+b)^2}{2} = \frac{x^2}{8}$$

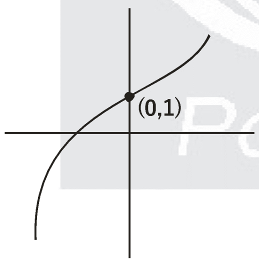
$$\therefore \frac{x^2}{8} = 32 \Rightarrow x = 16$$

31. Find number of distinct read roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$

Ans. 2

Sol. $f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f'(0) = 1$



$f''(x) = 12x^2 - 24x + 24$

$= 12\left(x^2 - 2x + 2\right)$

$> 0 \forall x \in R$

$f'(x) = 0$

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Matrix Match Type :

32. Match the following:

Column-I	**Column-II**
(A) If $x^2 + y^2 = 1$, then minimum value of x + y is	(p)
(B) If maximum value of y = $a\cos x - \frac{1}{3}\cos 3x$ occurs at $x = \frac{\pi}{6}$, then value of 'a' is	(q) $-\sqrt{2}$
(C) If f(x) = x –2 sin x, $0 \le x \le 2\pi$ is increasing in the interval $(a\pi, b\pi)$ then a + b is	(r) 3
(D) If equation of tangent to the curve $y = -e^{-x/2}$ where it crosses the y-axsi is $\frac{x}{p} + \frac{y}{q} = 1$, then p–q is	(s) 2
	(t) –2

(A) A–p; B–r; C–q; D–s
(B) A–q; B–s; C–s; D–r
(C) A–s; B–q; C–p; D–r
(D) A–r; B–p; C–q; D–r

Ans. (B)

Sol. (A) Let x = sinθ and y = cosθ

⇒ x + y = sin cosθ + cosθ

$$= \sqrt{\sin\left(\theta + \frac{\pi}{4}\right)}$$

⇒ minimum value = $-\sqrt{2}$

(B) $y = a\cos x - \frac{1}{3}\cos 3x$

$$y'\left(\frac{\pi}{6}\right) = 0$$

$$\Rightarrow -a\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{2}\right) = 0$$

⇒ a = 2

(C) $f'(x) = 1 - 2\cos x$

$$f'(x) > 0 \Rightarrow \cos x < \frac{1}{2}$$

$$\Rightarrow x \in \left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$$

$$\Rightarrow a + b = 2$$

(D) At x = 0, y = −1 also $y'(0) = \frac{1}{2}$

Equation of tangent $y + 1 = \frac{1}{2}(x - 0)$

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$$\Rightarrow \frac{x}{2} + \frac{y}{-1} = 1$$

$$\Rightarrow p - q = 3$$

Subjective based Questions :

33. Find the possible values of a such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution

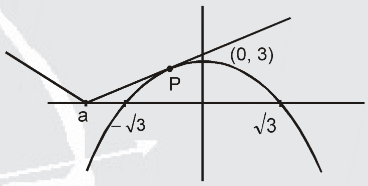
Ans. $a \in \left(-\frac{13}{4}, 3\right)$

Sol. $3 - x^2 > |\,x - a\,|$

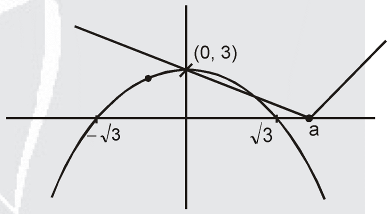
Case (i) $a < 0$ and $y = x - a$ is tangent of $y = 3 - x^2$ (see figure)

$-2x = 1 \Rightarrow x = -\frac{1}{2}$

$P\left(-\frac{1}{2}, \frac{11}{4}\right)$



Since $y = x - a$ passes through $\left(-\frac{1}{2}, \frac{11}{4}\right) \Rightarrow a = x - y$



$= -\left(\frac{11}{4} + \frac{1}{2}\right) = -\frac{13}{4}$ (minimum value of a)

Case (ii) $a > 0$ and $y = -x + a$ passes through (0, 3),

then $a = 3$ (maximum value of a) (see figure)

$\Rightarrow a \in \left(-\frac{13}{4}, 3\right)$

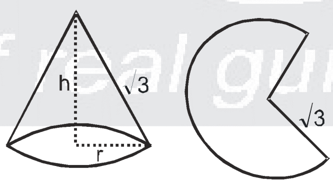
34. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and keeping the cut edges of the remaining piece together. Then find the maximum volume attainable for the cone

Ans. **$2\pi/3$**

Sol. $3 = h^2 + r^2$

$\Rightarrow r^2 = 3 - h^2$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3 - h^2)h$



Figure

$\frac{dV}{dh} = \frac{1}{3}\pi(3 - 3h^2)$

$\frac{dV}{dh} = 0$ at $h = 1$

$\frac{d^2V}{dh^2} < 0$ at $h = 1$

$\Rightarrow V_{max} = \frac{2\pi}{3}$

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35. f (x) and g (x) are differentiable functions for $0 \le x \le 2$ such that f (0) = 5, g (0) = 0, f (2) = 8, g (2) = 1.

Show that there exists a number c satisfying 0 < c < 2 and f ' (c) = 3 g' (c).

Sol. f (0) = 5 ; g(0) ; f (2) = 8, g (2) = 1

F (x) = f (x) – 3g (x) continuous and differentiable

F(0) = f(0) – 3g(0)

= 5 – 0 = 5

F(2) = f(2) – 3g(2) } Hence Rolles is applicable on [0, 2]

= 8 – 3 = 5

$\exists$ some c $\in$ (0, 2) where f ′ (c) = 0

$\Rightarrow$ f ′ (c) – 3g′(c) $\Rightarrow$ f ′ (c) = 3g′ (c)

36. Find maximum value of function $g(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$ $(0 \le x \le \pi e)$.

Ans. *ln* π

Sol. Since increases on [0, ∞) so it is enough to consider $f(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$

$$f'(x) = \frac{\log(e+x) \times \dfrac{1}{\pi + x} - \log(\pi + x)\dfrac{1}{e+x}}{(\log(e+x))^2} = \frac{\log(e+x)\times(e+x) - (\pi + x)\log(\pi + x)}{(\pi + x)(e+x)\,(\log(e+x))^2}$$

Since log function is an increasing function and $e < \pi$, $\log(e + x) < \log(\pi + x)$.

Thus $(e + x) \log (e + x) < (e + x) \log (\pi + x) < (\pi + x) \log (\pi + x)$ for all $x > 0$.

Thus, $f'(x) < 0$ for $\forall x > 0 \Rightarrow f(x)$ decreases on $(0, \infty)$

Hence maximum value of g(x) = g(0).

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