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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-13

AOD-2

ANSWER KEY OF CAPS-13

1.	(A)	**2.**	(A)	**3.**	(A)	**4.**	(A)	**5.**	(B)
6.	(C)	**7.**	(C)	**8.**	(B)	**9.**	(C)	**10.**	(A)
11.	(B)	**12.**	(BCD)	**13.**	(ABCD)	**14.**	(AC)	**15.**	(ABCD)
16.	(ACD)	**17.**	(C)	**18.**	(D)	**19.**	(C)	**20.**	(C)
21.	(C)	**22.**	(B)	**23.**	(C)	**24.**	(B)	**25.**	(D)
26.	(4)	**27.**	(3)	**28.**	(0)	**29.**	(6)	**30.**	(4)
31.	(2)	**32.**	(B)	**33.**	$a \in \left(-\frac{13}{4}, 3\right)$	**34.**	($2\pi/3$)	**36.**	($ln\ \pi$)

SCQ (Single Correct Type) :

1. f : [0, 4] → R is a differentiable function. Then for some a, b ∈ (0, 4) , $f^2(4) - f^2(0) =$

(A) 8f′(a) . f(b) (B) 4f′(b) f(a) (C) 2f′ (b) f(a) (D) f′(b) f(a)

Ans. (A)

Sol. Here f is a differentiable function then f is continuous function

So by L.M.V. theorem for any a ∈ (0, 4)

$$f'(a) = \frac{f(4) - f(0)}{4 - 0} \quad ...(1)$$

Again from mean value for any b ∈ (0, 4)

$$f(b) = \frac{f(4) + f(0)}{2} \quad ...(2)$$

Now multiplying (1) and (2), we get

$$\frac{f^2(4) - f^2(0)}{8} = f'(a) \cdot f(b) \quad \Rightarrow \quad f^2(4) - f^2(0) = 8f'(a) \cdot f(b)$$

2. The values of the parameter 'k' for which the equation $x^4 + 4x^3 - 8x^2 + k = 0$ has all roots real is given by

(A) k ∈ (0,3) (B) k ∈ (0, 128) (C) k ∈ (3, 128) (D) k ∈ (128, ∞)

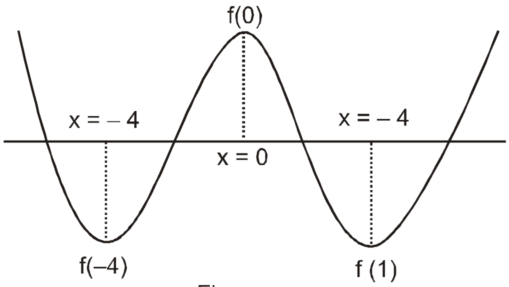
Ans. (A)

Sol. Let $f(x) = x^4 + 4x^3 - 8x^2 + k$

$f'(x) = 4x^3 + 12x^2| - 16x = 4x(x^2 + 3x - 4) = 4x(x + 4)(x - 1)$

$\Rightarrow f'(x) = 0 \Rightarrow x = -4, 0, 1$

$f''(x) = 12x^2 + 24x - 16 = 4(3x^2 + 6x - 4)$



Figure

$f''(-4) = 20 > 0$

$f''(0) = -16 < 0$

$f''(1) = 20 > 0$

$\Rightarrow$ x = – 4 and x = 1 are points of local minima whereas

x = 0 is point of local maxima

for f(x) = 0 to have 4 real roots

$f(-4) < 0 \Rightarrow \quad k < 128$

$f(0) > 0 \quad \Rightarrow \quad k > 0$

$f(1) < 0 \quad \Rightarrow \quad k < 3$

$\Rightarrow k \in (0, 3)$

3. A composite function $(f_1 o\ f_2 o\ f_3 o.....of_{21})(X)$ is an increasing function. If number of increasing functions in the set $\{f_1, f_2, ...f_{21}\}$ is r and remaining are decreasing functions, then maximum value of r(21−r) is

(A) 110 (B) $\frac{441}{4}$ (c) 105 (d) None of these

Ans. (A)

Sol. Number decreasing functions = (21−r)

Since the composite function is increasing, therefore, 21− r must be even

$\Rightarrow r \in \{1,3,5,....,21\}$

Now, $21r - r^2 = \frac{441}{4} - \left(r - \frac{21}{2}\right)^2$ is maximum at r = 11

Maximum of $21r - r^2 = \frac{441}{4} - \frac{1}{4} = 110$

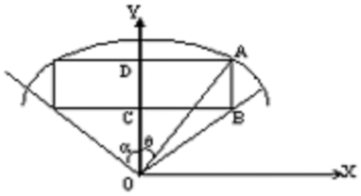
4. A sector subtends an angle 2α at the centre then the greatest area of the rectangle inscribed in the sector is (R is radius of the circle)

(A) $R^2 \tan\frac{\alpha}{2}$ (B) $\frac{R^2}{2}\tan\frac{\alpha}{2}$ (C) $R^2 \tan\alpha$ (D) $\frac{R^2}{2}\tan\alpha$

Ans. (A)

Sol. Let A be any point on the arc such that $\angle YOA = \theta$

Where $0 \le \theta \le \alpha$



DA = CB = R sinθ, OD = R cosθ

$\Rightarrow$ CO = CB cot α = R sin θ cot α

Now, CD = OD −OC = Rcosθ − Rsinθ cotα

= R(cosθ −sinθ cotα)

Area of rectangle ABCD, S = 2.CD.CB

$= 2R(\cos\theta - \sin\theta\cot\alpha)R\sin\theta$

$= 2R^2(\sin\theta\cos\theta - \sin^2\theta\cot\alpha)$

$R^2(\sin 2\theta - (1-\cos 2\theta)\cot\alpha) = \dfrac{R^2}{\sin\alpha}[\cos(2\theta-\alpha) - \cos\alpha]$

$S_{max} = \dfrac{R^2}{\sin\alpha}(1-\cos\alpha)\left(\text{for } \theta = \dfrac{\alpha}{2}\right)$

Hence, greatest area of the rectangle = $R^2\tan\dfrac{\alpha}{2}$

5. If the graphs of the functions y = ln x & y = ax intersect at exactly two points, then

(A) $a \in (0,e)$ (B) $a \in \left(0,\dfrac{1}{e}\right)$ (C) $a \in (-e,1)$ (d) $a \in (1,e)$

Ans. (B)

Sol. ln x = ax has exactly two solutions

$\dfrac{\ln x}{x}$ = a has exactly two solutions

Let $f(x) = \dfrac{\ln x}{x}$

Range of $y \in \left(-\infty, \dfrac{1}{e}\right)$

6. Let f (x) be a polynomial of degree 3 satisfying f(3) = 5, f (−1) = 9, f (x) has minimum at x = 0 and f′(x) has maximum at x =1. The distance between local maximum and local minimum of f (x) is

(A) $3\sqrt{2}$ (B) $\sqrt{15}$ (C) $2\sqrt{5}$ (D) $4\sqrt{3}$

Ans. (C)

Sol. $f(x) = -x^3 + 3x^2 + 5$

7. Statement 1: In ΔABC, sinA + sinB sin C $\leq \dfrac{3\sqrt{3}}{2}$

Statement 2: Let y = f (x) be a twice differentiable function such that f "(x) < 0 in [a,b] then $\dfrac{f(a_1)+f(a_2)+f(a_3)}{3} \geq f\left(\dfrac{a_1+a_2+a_3}{3}\right)$ for $a_1, a_2, a_3 \in [a,b]$

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct Explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Ans. (C)

Sol. Let f (x) = sin x $\Rightarrow$ f‘(x) = cos x $\Rightarrow$ f”(x) = −sin x < 0 $\forall$ x $\in [0,\pi]$

It is concave down and (A,sin A) (B,sin B) (C,sinC) are three points and

$\dfrac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\dfrac{A+B+C}{3}\right) \Rightarrow \sin A + \sin B + \sin C \leq \dfrac{3\sqrt{3}}{2}$

8. Let P(x) be a fourth degree polynomial with derivative P'(x) . Such that P(1) = P(2) = P(3) = P'(7) = 0 . Let k is the real number k ≠1,2,3 such that P(k) = 0, then k is equal to

(A) $\frac{317}{37}$ (B) $\frac{319}{37}$ (C) $\frac{321}{37}$ (D) $\frac{15}{37}$

Ans. (B)

Sol. $P(x) = a(x-1)(x-2)(x-3)(x-b)$

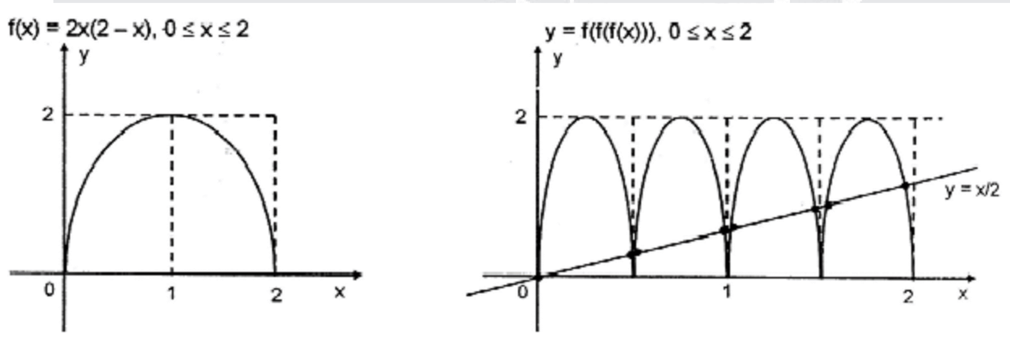
$\because P'(7) = 0$

$\Rightarrow a[20(7-b) + 24(7-b) + 30(7-b) + 120] = 0 \quad \Rightarrow b = \frac{319}{37}$

9. Let f (x) = 2x(2− x), 0 ≤ x ≤ 2. The number of solution of $f\left(f\left(f\left(x\right)\right)\right) = \frac{x}{2}$ is

(A) 2 (B) 4 (C) 8 (D) 12

Ans. (C)

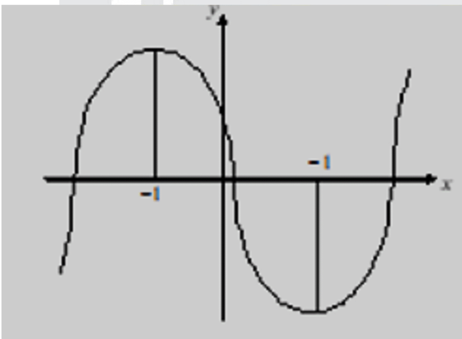


Sol.

10. Set of values of a for which one negative and two positive real roots of the equation $x^3 - 3x + a = 0$ are possible, is _____.

(A) (0, 2) (B) (0, 4) (C) (2, 4) (D) (0, 10)

Ans. (A)



Sol.

Let $f(x) = x^3 - 3x - a \quad \Rightarrow f'(x) = 3x^2 - 3 = 0 \quad \Rightarrow x = \pm 1$

$f(1)\,f(-1) < 0$

$(a+2)(a-2) < 0$

$a \in (-2, 2)$

From graph f(0) f(1) < 0

$a(a-2) < 0$

$a \in (0, 2)$

$\therefore a \in (0, 2)$

11. $f(x)=\begin{cases} e^x-2-e^{-2}, x<-2 \\ x^2-x+\lambda, -2\le x\le 2 \\ -\mu \ell nx, x>2 \end{cases}$

If y = f(x) has local maxima at x = –2, then range of λ is

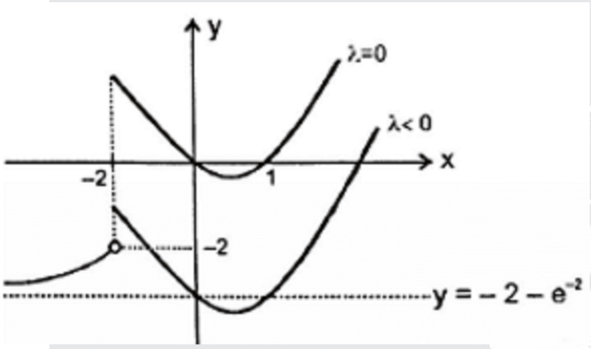
(A) $(-\infty, 8]$ (B) $[-8, \infty)$ (C) [–8, 8] (D) $(-\infty, -8] \cup [8, \infty)$

Ans. **(B)**

Sol. $y = e^x - 2 - e^{-2}$

at x = –2 maxima

$4+2+\lambda \ge -2 \quad \Rightarrow \lambda \ge -8 \quad \Rightarrow \lambda \in [-8, \infty)$



MCQ (One or more than one correct) :

12. Consider the function f : R $\to$ R defined as f (x) = x + sin x. Which of the following is/are the correct statement(s)?

(A) The function is strictly increasing at every point on R except at 'x' equal to an odd integral multiple of π where the derivative of f (x) is zero and where the function f is not strictly increasing.

(B) The function is bounded in every bounded interval but unbounded on whole real line.

(C) The graph of the function y = f (x) lies in the first and third quadrants only.

(D) The graph of the function y = f (x) cuts the line y = x at infinitely many points.

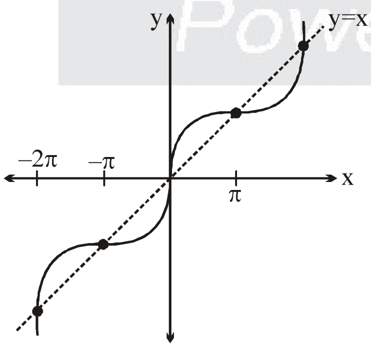
Ans. **(BCD)**

Sol. We have f ' (x) = 1 + cos x $\Rightarrow$ f is strictly increasing and has inflection point at x = nπ

Also there is no x for which f (x) is not increasing.

$\Rightarrow$ (A) is not correct

As f is continuous $\forall$ x $\in$ R hence bounded in every closed interval.



As f is odd hence symmetric w.r.t origin and its graphs lies in 1st and 3rd quadrant and y = x cut the graph at infinitely many points.

$\Rightarrow$ **B, C, D** are correct]

13. Let f (x) be a non constant twice derivable function defined on R such that f (2 + x) = f (2 – x) and $f'\left(\frac{1}{2}\right) = 0 = f'(1)$. Then which of the following alternative(s) is/are correct?

(A) f (– 4) = f (8).

(B) Minimum number of roots of the equation f '' (x) = 0 in (0, 4) are 4.

(C) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(2+x)\sin x\,dx = 0$.

(D) $\int_0^2 f(t)5^{\cos\pi t}\,dt = \int_2^4 f(4-t)5^{\cos\pi t}\,dt$.

Ans. (ABCD)

Sol. We have f (2 – x) = f (2 + x)

Replacing x by 2 – x, we get

f (x) = f (4 – x) ....(1)

Put x = – 4 in (1), we get

f (– 4) = f (8) $\Rightarrow$ **(A) is correct**

On differentiating (1) w.r.t. x, we get f ' (x) = – f ' (4 – x) ....(2)

Put $x = \frac{1}{2}$, 1, 2 in (2), we get $f'\left(\frac{1}{2}\right) = 0 = f'(1) = f'(2) = f'\left(\frac{7}{2}\right) = f'(3)$

Now, consider a function y = f ' (x)

As f ' (x) satisfy Rolle's theorem in $\left[\frac{1}{2}, 1\right]$, [1, 2], $\left[2, \frac{7}{2}\right]$, $\left[\frac{7}{2}, 3\right]$ respectively.

So, by Rolle's theorem, the equation f '' (x) = 0 has minimum 4 roots in (0, 4). $\Rightarrow$ **(B) is correct**

Now, consider $I_1 = \int_{-\pi/4}^{\pi/4} f(2+x)\sin x\,dx$ ....(3)

Applying king property, we get

$I_1 = \int_{-\pi/4}^{\pi/4} f(2-x)\sin(-x)\,dx = -\int_{-\pi/4}^{\pi/4} f(2+x)\sin(x)\,dx$

$\therefore$ $I_1 = -I_1$

Hence $I_1 = 0$ $\Rightarrow$ **(C) is correct**

Again, consider $I_2 = \int_0^2 f(t)\,5^{\cos\pi t}\,dt$

Put 4 – t = y $\Rightarrow$ dt = – dy

So, $I_2 = \int_4^2 f(4-y)\,5^{\cos\pi(4-y)}(-dy) = \int_2^4 f(4-y)\,5^{\cos\pi y}\,dy = \int_2^4 f(4-t)\,5^{\cos\pi t}\,dt$ $\Rightarrow$ **(D) is correct]**

14. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} [f(x)]$ (a $\in$ R), where [·] denotes greatest integer function and f (x) is a non constant continuous function, then

(A) $\lim\limits_{x \to a} f(x)$ is an integer.

(B) $\lim\limits_{x \to a} f(x)$ is non integer.

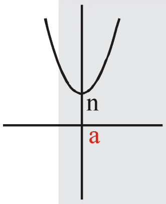
(C) f(x) has a local minimum at x = a.

(D) f(x) has a local maximum at x = a.

Ans. **(AC)**

Sol. Clearly $\lim\limits_{x \to a} [f(x)]$ is an integer and LHL and RHL should be same for existence of $\lim\limits_{x \to a} f(x)$

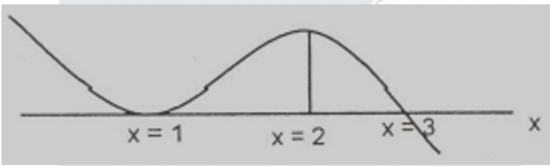
Let $\lim\limits_{x \to a} [f(x)]$ = n (n $\in$ I)



Clearly LHL and RHL both should be just greater than n as f(x) is cont. at x = a

$\therefore$ f(x) has local minimum at x = a]

15. If graph of y = f '(x) is



then which of the following can be true for y = f(x)

(A) point of inflection at x = 1 and x = 2

(B) concave down in $(-\infty,1) \cup (2,\infty)$

(C) point of local maxima at x = 3

(D) decreasing in interval $(3,\infty)$

Ans. **(ABCD)**

16. Let $g(x) = f(\tan x) + f(\cot x) \forall x \in \left(\frac{\pi}{2}, \pi\right)$. If $f''(x) < 0 \forall x \in \left(\frac{\pi}{2}, \pi\right)$, then

(A) g(x) is increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

(B) g(x) is increasing in $\left(\frac{3\pi}{4}, \pi\right)$

(C) g(x) is decreasing in $\left(\frac{3\pi}{4}, \pi\right)$

(D) g(x) has local maximum at $x = \frac{3\pi}{4}$

Ans. **(ACD)**

Sol. $g'(x) = f'(\tan x)\sec^2 x - f'(\cot x)\operatorname{cosec}^2 x$

$\because f''(x) < 0 \Rightarrow f'(x)$ is decreasing

$\tan x < \cot x \forall x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

$\therefore f'(\tan x) > f'(\cot x)$ & $\sec^2 x > \operatorname{cosec}^2 x \forall x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ $\Rightarrow$ g(x) is increasing in $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$

g(x) is decresing in $\left(\frac{3\pi}{4}, \pi\right)$ and g(x) has local maximum at $x = \frac{3\pi}{4}$

Comprehension Type Question:

Comprehension # 1

Consider f, g and h be three real valued differentiable functions defined on R. Let $g(x) = x^3 + g''(1) x^2 + (3g'(1) - g''(1) - 1)x + 3g'(1)$, $f(x) = x\,g(x) - 12x + 1$ and $f(x) = (h(x))^2$ where $h(0) = 1$.

17. The function y = f(x) has

(A) Exactly one local minima and no local maxima

(B) Exactly one local maxima and no local minima

(C) Exactly one local maxima and two local minima

(D) Exactly two local maxima and one local minima

Ans. (C)

18. Which of the following is/are true for the function y = g(x)?

(A) g(x) monotonically decreases in $\left(-\infty, 2-\frac{1}{\sqrt{3}}\right) \cup \left(2+\frac{1}{\sqrt{3}}, \infty\right)$

(B) g(x) monotonically increases in $\left(2-\frac{1}{\sqrt{3}}, 2+\frac{1}{\sqrt{3}}\right)$

(C) There exists exactly one tangent to y = g(x) which is parallel to the chord joining the points (1, g(1)) and (3, g(3))

(D) There exists exactly two distinct Lagrange's mean value in (0, 4) for the function y = g(x).

Ans. (D)

19. Which one of the following does not hold good for y = h(x)?

(A) Exactly one critical point

(B) No point of inflection

(C) Exactly one real zero in (0, 3)

(D) Exactly one tangent parallel to x-axis

Ans. (C)

Sol. We have $g(x) = x^3 + g''(1)x^2 + \{3g'(1) - g''(1) - 1\}x + 3g'(1)$

Let $g'(1) = a$ and $g''(1) = b$ then $g(x) = x^3 + bx^2 + (3a - b - 1)x + 3a$

Differentiating both sides w.r.t. x

$\therefore\ g'(x) = 3x^2 + 2bx + (3a - b - 1)$

Put x = 1 $\Rightarrow$ $g'(1) = 3 + 2b + 3a - b - 1 \Rightarrow a = b + 3a + 2 \Rightarrow 2a + b = -2$ .... (1)

$g''(x) = 6x + 2b$

Put x = 1

$g''(1) = 6 + 2b \Rightarrow b = 6 + 2b \Rightarrow b = -6$ .... (2)

$\therefore$ From equation (1), we get a = 2

$\therefore\ g(x) = x^3 - 6x^2 + 11x + 6$

Given $f(x) = xg(x) - 12x + 1 = x^4 - 6x^3 + 11x^2 - 6x + 1 = (x^2 + 1)^2 - 2x^2 + 11x^2 - 6x^3 - 6x$

$= (x^2 + 1)^2 - 6x(x^2 + 1) + (3x)^2$

$f(x) = (x^2 - 3x + 1)^2 = \{h(x)\}^2$ (Given)

$\because\ h(0) = 1$

$\therefore\ h(x) = x^2 - 3x + 1$