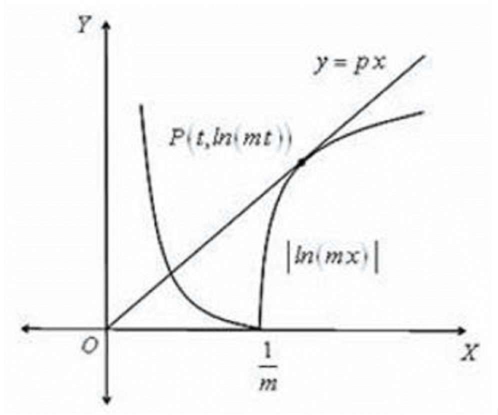
Sol.



$\frac{1}{t} = \frac{\ln mt}{t} \to t = \frac{e}{m}$

$\therefore p = \frac{m}{e}$ for two roots

For a single root, $p > \frac{m}{e}$

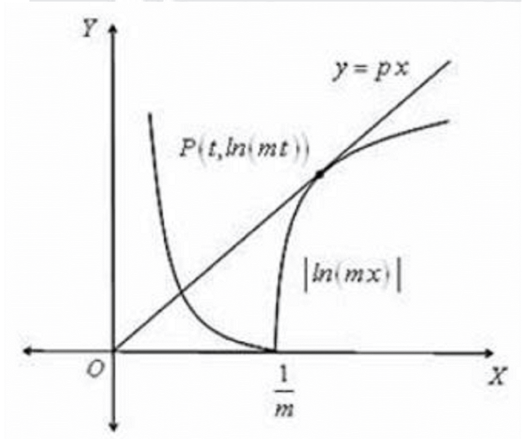
for 3 roots, $\therefore 0 < p < \frac{m}{e}$

18. The equation $|\ln(mx)|$ = px, where 'm' is a positive constant has exactly two root for

(A) $p = \frac{m}{e}$ (B) $p = \frac{e}{m}$ (C) $0 < p \le \frac{e}{m}$ (D) $0 < p \le \frac{m}{e}$

Ans. **(A)**

Sol. Slope of tangent at P = slope of OP



$\frac{1}{t} = \frac{\ln mt}{t} \to t = \frac{e}{m}$

$\therefore p = \frac{m}{e}$ for two roots

For a single root, $p > \frac{m}{e}$

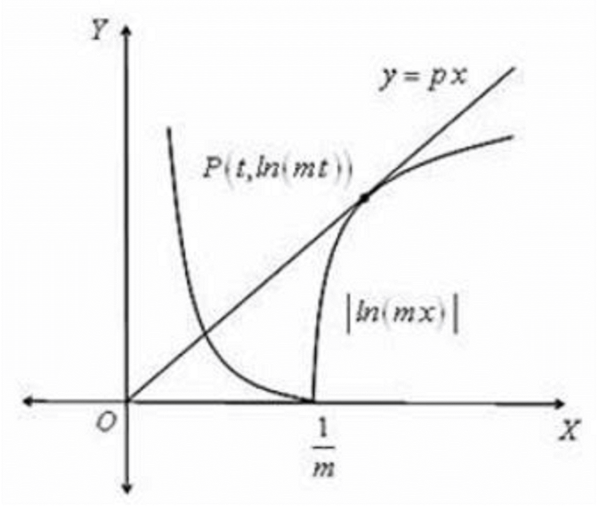
for 3 roots, $\therefore 0 < p < \frac{m}{e}$

19. The equation $|\ln(mx)|$ = px, where 'm' is a positive constant has exactly three root for

(A) $p < \frac{m}{e}$ (B) $0 < p < \frac{m}{e}$ (C) $0 < p < \frac{e}{m}$ (D) $p < \frac{e}{m}$

Ans. **(B)**

Sol. Slope of tangent at P = slope of OP



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For a single root, $p > \frac{m}{e}$

for 3 roots, $\therefore 0 < p < \frac{m}{e}$

Comprehension # 2

f(x) is a polynomial function f : → → such that f(2x) = f′(x) f″(x) .

20. The value of f(3) is___ .

(A) 4 (B) 12 (C) 15 (D) 18

Ans. **(B)**

Sol. Suppose degree of f(x) is n, then the degree of

f' is n–1 and degree of f" is n–2

So, n = (n–1)+(n–2) $\Rightarrow$ n = 3

Hence, f(x) = $ax^3 + bx^2 + cx + d$

From, f(2x) = f'(x) f"(x), we have

$$8ax^3 + 4bx^2 + 2cx + d = \left(3ax^2 + 2bx + c\right)\left(6ax + 2b\right)$$

Comparing coefficient of terms, we have

$a = \frac{4}{6}, b = 0, c = 0$ and d = 0

$\Rightarrow f(x)=\frac{4x^3}{9}$

(1) f(3) = 12

(2) one-one and onto

(3) $\frac{4x^3}{9}=x\Rightarrow x=0,\pm\frac{3}{2}$

21. f(x) is ______.

(A) one-one and onto (B) one-one and into

(C) many-one and onto (D) many-one and into

Ans. **(A)**

Sol. Suppose degree of f(x) is n, then the degree of

f' is n–1 and degree of f" is n–2

So, n = (n–1) + (n–2) $\Rightarrow$ n = 3

Hence, f(x) = $ax^3 + bx^2 + cx + d$

From, f(2x) = f'(x) f"(x), we have

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(1) f(3) = 12

(2) one-one and onto

(3) $\frac{4x^3}{9}=x\Rightarrow x=0,\pm\frac{3}{2}$

22. Equation f(x) = x has ______.

(A) Three real and distinct roots (B) one real root

(C) Four real and distinct roots (D) Two real and distinct roots

Ans. **(A)**

Sol. Suppose degree of f(x) is n, then the degree of

f' is n–1 and degree of f" is n–2

So, n = (n–1)+(n–2) $\Rightarrow$ n = 3

Hence, f(x) = $ax^3 + bx^2 + cx + d$

From, f(2x) = f'(x) f"(x), we have

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(1) f(3) = 12

(2) one-one and onto

(3) $\frac{4x^3}{9}=x \Rightarrow x=0, \pm\frac{3}{2}$

Numerical based Questions :

23. Suppose a room, containing $12000 m^3$ of air is originally free of carbon monoxide. Beginning at t = 0, smoke containing 4% carbon monoxide is introduced into the room at a rate of $1.0 m^3$ / min and the well circulated mixture is allowed to leave the room at the same rate. Extended exposure to carbon monoxide concentration as much as 0.00012 is harmful to the human body. If T (in min) is the approximate time at which this concentration is reached, then $\left[\frac{T}{4}\right]$ (in minutes) (where [.] denotes the greatest integer function) is _______.

$(\ln(0.997))=-3.005\times10^{-3}$

Ans. **(9)**

Sol. Let Q(t) = quantity of carbon monoxide in the room at any time t, then the concentration,

$x(t)=\frac{Q(t)}{1200}$

$\Rightarrow x=0.04\left[1-e^{\frac{-T}{12000}}\right]$

$\Rightarrow 0.00012=0.04\left[1-e^{\frac{-T}{12000}}\right]$

$\Rightarrow T=-12000\ln(0.997)=36\,\text{min}$

24. The line y = mx +1 touches the curves $y=-x^4+2x^2+x$ at two distinct points P (x_1, y_1) and Q (x_1, y_1) . The value of $x_1^2+x_2^2+y_1^2+y_2^2$ is ____

Ans. **(6)**

Sol. The solution of $-x^4+2x^2+x=mx+1$ is x_1 , x_1 and x_2 and x_2

$\Rightarrow x^4-2x^2+(m-1)x+1=(x-x_1)^2(x-x_2)^2$

By comparing, we get $x_1=1$, $x_2=-1$, m =1

$\Rightarrow$ P(1,2) and Q(−1,0)

25. If a curve is represented parametrically by the equations

$$x = \sin\left(t+\frac{7\pi}{12}\right) + \sin\left(t-\frac{\pi}{12}\right) + \sin\left(t+\frac{3\pi}{12}\right), \qquad y = \cos\left(t+\frac{7\pi}{12}\right) + \cos\left(t-\frac{\pi}{12}\right) + \cos\left(t+\frac{3\pi}{12}\right)$$

then find the value of $\frac{d}{dt}\left(\frac{x}{y}-\frac{y}{x}\right)$ at $t=\frac{\pi}{8}$.

Ans. **8**

Sol. We have

$$x = \sin\left(t+\frac{7\pi}{12}\right) + \sin\left(t-\frac{\pi}{12}\right)+\sin\left(t+\frac{3\pi}{12}\right) = 2\sin\left(t+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\sin\left(t+\frac{\pi}{4}\right) = 2\sin\left(t+\frac{\pi}{4}\right)$$

|||ly

$$y = \cos\left(t+\frac{7\pi}{12}\right)+\cos\left(t-\frac{\pi}{12}\right)+\cos\left(t+\frac{3\pi}{12}\right) = 2\cos\left(t+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\cos\left(t+\frac{\pi}{4}\right) = 2\cos\left(t+\frac{\pi}{4}\right)$$

Now, $\frac{x}{y} = \tan\left(t+\frac{\pi}{4}\right) = \frac{1+\tan t}{1-\tan t}$ and $\frac{y}{x} = \frac{1}{\tan\left(t+\frac{\pi}{4}\right)} = \frac{1-\tan t}{1+\tan t}$

$$\therefore \left(\frac{x}{y}-\frac{y}{x}\right) = \left(\frac{1+\tan t}{1-\tan t}\right) - \left(\frac{1-\tan t}{1+\tan t}\right) = \frac{(1+\tan t)^2-(1-\tan t)^2}{1-\tan^2 t} = \frac{4\tan t}{1-\tan^2 t} = 2\tan 2t$$

Hence $\frac{d}{dt}\left(\frac{x}{y}-\frac{y}{x}\right) = \frac{d}{dt}(2\tan 2t) = 4\sec^2 2t\Big]_{t=\frac{\pi}{8}} = 4\sec^2\frac{\pi}{4} = 8$ Ans.]

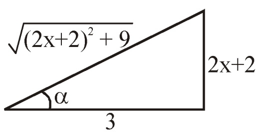
26. Find the derivative with respect to x of the function:

Let $f(x) = \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)$, find the value of $\frac{d(\tan f(x))}{d(\tan^{-1} x)}$ when $x = \frac{1}{\sqrt{2}}$.

Ans. **1**

Sol. Let $y = \tan\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)$ and $z = \tan^{-1} x$

$$y = \tan\sin^{-1}\left(\frac{2x+2}{\sqrt{(2x^2+2)^2+9}}\right)$$



$$= \tan\left(\tan^{-1}\frac{2x+2}{3}\right) = \frac{2x+2}{3}$$

$$\frac{dy}{dx}=\frac{2}{3};\ \frac{dz}{dx}=\frac{1}{1+x^2}$$

$$\therefore\ \frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{dx}{dz}=\frac{2}{3}\cdot\frac{1+x^2}{1}$$

$$\left.\frac{dy}{dz}\right|_{x=1}=1\quad]$$

27. If y = y(x) and it follows the relation $e^{xy} + y\cos x = 2$, then find y '' (0).

Ans. **2**

Sol. We have

$$e^{xy} + y\cos x = 2 \qquad ....(1)$$

Put x = 0, we get

$1 + y = 2$

$\therefore$ $y = 1$

$\Rightarrow$ (0, 1) lies on the given curve.

On differentiating (1) w.r.t. x, we get

$$\Rightarrow\quad e^{xy}\left[x\frac{dy}{dx}+y\right] + y(-\sin x) + \cos x\,\frac{dy}{dx} = 0 \qquad ....(2)$$

As (0, 1) satisfying it, we get

$$\Rightarrow\quad 0+1+\frac{dy}{dx}=0 \qquad \Rightarrow \qquad y'(0) = -1 \text{ Ans.}$$

Again differentiating (2), we get

$$\Rightarrow\quad e^{xy}\left[x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}\right]+e^{xy}\left[x\frac{dy}{dx}+y\right]^2-\left[y\cos x+\sin x\frac{dy}{dx}\right]+\cos x\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}=0$$

As (0, 1) satisfy it, we get

$\Rightarrow$ $-2 + 1 - 1 + y''(0) = 0$

$\therefore$ $y''(0) = 2$ Ans.

28. If $\lim\limits_{x\to 0}\dfrac{1-\cos 3x\cdot\cos 9x\cdot\cos 27x.........\cos 3^n x}{1-\cos\frac{1}{3}x\cdot\cos\frac{1}{9}x\cdot\cos\frac{1}{27}x.........\cos\frac{1}{3^n}x} = 3^{10}$, find the value of n.

Ans. **(4)**

Sol. Let $U_n = \lim\limits_{x\to 0}\dfrac{1-\cos 3x\cdot\cos 3^2x\cdot\cos 3^3x.........\cos 3^n x}{x^2}$

and $V_n = \lim\limits_{x\to 0}\dfrac{1-\cos\frac{x}{3}\cdot\cos\frac{x}{3^2}\cdot\cos\frac{x}{3^3}.........\cos\frac{x}{3^n}}{x^2}$

$U_n = \lim\limits_{x\to 0}\dfrac{-D\left(\cos(3x)\cdot\cos(3^2x)\cdot\cos(3^3x).........\cos(3^nx)\right)}{2x}$

now let $y = \cos 3x \cdot \cos 3^2x \cdot \cos 3^3x$ .... $\cos 3^nx$

$\ln y = \ln \cos 3x + \ln \cos 3^2x +$ .... $\ln \cos 3^3x + \ln \cos 3^nx$

$$\frac{1}{y}\frac{dy}{dx} = -[3 \tan 3x + 3^2 \tan 3^2x + ... + 3^n \tan 3^nx]$$

$$\frac{dy}{dx} = -\prod_{r=1}^{n} \cos 3^r x[3\tan 3x + 3^2 \tan 3^2 x + ... + 3^n \tan 3^n x]$$

$$\therefore \quad U_n = \lim_{x\to 0} \frac{3\tan 3x + 3^2 \tan 3^2 x + ... + 3^n \tan 3^n x}{2x} = \frac{3^2 + (3^2)^2 + (3^3)^2 + ... + (3^n)^2}{2}$$

$$U_n = \frac{3^2[3^{2n} - 1]}{(3^2 - 1)\cdot 2} \quad ....(1)$$

|||ly replacing 3^r by $\frac{1}{3^r}$ we get

$$V_n = \frac{\frac{1}{3^2}\left[1 - \frac{1}{3^{2n}}\right]}{\left(1 - \frac{1}{3^2}\right)\cdot 2} = \frac{(3^{2n} - 1)}{3^{2n}(3^2 - 1)\cdot 2} \quad ....(2)$$

$$\therefore \quad \frac{U_n}{V_n} = 3^{2n+2} = 3^{10} \text{ (given)}$$

$\therefore \quad 2n + 2 = 10 \quad \Rightarrow \quad \boxed{n = 4}$ Ans.]

Matrix Match Type :

29. Match the follwoing

Column-I	**Column-II**
(A) The number of non-differentiable points on the curve $y = \left\|e^{\|x\|} - 4\right\|$ is	(p) 1
(B) Length of the latus rectum of the parabola defined by x = cos t –sin t and y = sin 2t is	(q) 0
(C) The number of real solution of the equation $x^{2\log_3(x+3)} = 16$ is	(r) 3
(D) If in a ΔABC, $2R = r_1 - r$, then 2 cos A is equal to	(s) 2

(A) A → (r); B → (p); C → (p); D →(q) (B) A → (q); B → (p); C → (q); D →(q)

(C) A → (r); B → (r); C → (q); D →(q) (D) A → (p); B → (p); C → (q); D →(q)

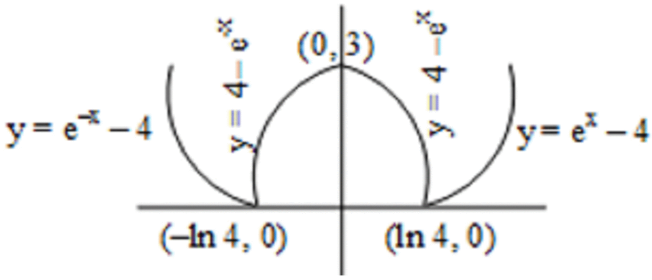
Ans. **(A)**

Sol. (A) From graph it is clear that number of non differentiable point = 3

(B) x = cos t – sin t, y = sin 2t

$\Rightarrow x^2 = -(y - 1)$

Length of latus rectum = 1



(C) $x^{2\log_x (x+3)} = 16 \quad \Rightarrow x > 0, x \neq 1$ and $(x+3)^2 = 16$

$\Rightarrow x = \pm 4 - 3 = 1, -7$

$\Rightarrow$ No solution $\Rightarrow$ number of solution = 0

(D) $2R = r_1 - r = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow 1 = 2\sin\frac{A}{2}\cos\frac{B+C}{2} = 2\sin^2\frac{A}{2}$

$\Rightarrow 1 = 2\sin^2\frac{A}{2} = 0$

$\Rightarrow \cos A = 0$

$\Rightarrow 2\cos A = 0$

30. Column-I contains function defined on R and Column-II contains their properties. Match them.

Column-I	**Column-II**
(A) $\lim\limits_{n\to\infty}\left(\dfrac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}\right)^n$ equal	(P) e
(B) $\lim\limits_{x\to 0^+}\dfrac{1}{(1+\operatorname{cosec} x)^{\frac{1}{\ln(\sin x)}}}$ equals	(Q) e^2
(C) $\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\cos^{-1}x\right)^{1/x}$ equals	(R) $e^{-2/\pi}$
	(S) $e^{\pi/6}$

(A) A → (S); B → (Q); C → (R) (B) A → (S); B → (P); C → (R)

(C) A → (Q); B → (P); C → (R) (D) A → (P); B → (Q); C → (S)

Ans. **(B)**

Sol. (A) $l = \lim\limits_{n\to\infty}\dfrac{e^{n\left(1+\tan\frac{\pi}{2n}-1\right)}}{e^{n\left(1+\sin\frac{\pi}{3n}-1\right)}} = \dfrac{e^{\lim\limits_{n\to\infty} n\tan\frac{\pi}{2n}}}{e^{\lim\limits_{n\to\infty} n\sin\frac{\pi}{3n}}} = \dfrac{e^{\pi/2}}{e^{\pi/3}} = e^{\pi/6}$ Ans. $\Rightarrow$ (S)

(B) Consider, $l = \lim\limits_{x\to 0}\left(1+\dfrac{1}{\sin x}\right)^{\frac{1}{\ln(\sin x)}}$ $\Rightarrow L = \dfrac{1}{l}$

but $\sin x = t$ as $x \to 0, t \to 0$

$\therefore \quad l = \left(1+\frac{1}{t}\right)^{\frac{1}{\ln t}}$ of the form (∞^0)

$\ln l = \frac{1}{\ln t}\ln\left(1+\frac{1}{t}\right)$; $\ln l = \underset{t\to 0}{\text{Lim}}\ \frac{\ln(t+1)-\ln(t)}{\ln t}$

$\therefore \quad l = e^{\underset{t\to 0}{\text{Lim}}\frac{\ln(t+1)-\ln t}{\ln t}} = e^{0-1} = \frac{1}{e}$ (Using L'Hospital's Rule)

$\therefore \quad L = \frac{1}{l} = e$ Ans. $\Rightarrow$ (P)

(C) $L = e^{\underset{x\to 0}{\text{Lim}}\frac{\frac{2}{\pi}\cos^{-1}x-1}{x}} = e^{l}$ where $l = \underset{x\to 0}{\text{Lim}}\ \frac{\frac{-2}{\pi\sqrt{1-x^2}}}{1} = -\frac{2}{\pi}$

$\Rightarrow L = e^{-\frac{2}{\pi}}$ Ans. $\Rightarrow$ (R).]