competishun — Power of real gurus **MATHEMATICS** TARGET : JEE- Advanced 2021	**CAPS-12** **MOD & AOD-1**

ANSWER KEY OF CAPS-12

1.	(B)	**2.**	(B)	**3.**	(C)	**4.**	(B)	**5.**	(D)
6.	(A)	**7.**	(A)	**8.**	(C)	**9.**	(D)	**10.**	(D)
11.	(AC)	**12.**	(BD)	**13.**	(AC)	**14.**	(ABC)	**15.**	(ABC)
16.	(AC)	**17.**	(D)	**18.**	(A)	**19.**	(B)	**20.**	(B)
21.	(A)	**22.**	(A)	**23.**	(9)	**24.**	(6)	**25.**	(8)
26.	(1)	**27.**	(2)	**28.**	(4)	**29.**	(A)	**30.**	(B)

SCQ (Single Correct Type) :

1. The curves $C_1 : y = x^2 - 3$; $C_2 : y = kx^2$, $k < 1$ intersect each other at two different points. The tangent drawn to C_2, at one of the points of intersection A= (a, y_1) (a > 0) meets C_1 again at $B(1, y_2)(y_1 \neq y_2)$. Then the value of a = ?

(A) 4 (B) 3 (C) 2 (D) 1

Ans. **(B)**

Sol. $C_1 \,\&\, C_2 \Rightarrow A\left(\sqrt{\frac{3}{1-k}}, \frac{3k}{1-k}\right) = \left(a, ka^2\right) \equiv \left(a, a^2 - 3\right)$

Tangent to C_2 at A is $y - (a^2 - 3) = 2\,ka(x-a)$ ....(1)

$\Rightarrow B = (1, -2)$

From expression (1)

$$-2 - a^2 + 3 = 2a\left(1 - \frac{3}{a^2}\right)(1-a)$$

$\Rightarrow a = 3, a = -2, a = 1$

but $a \neq 1$ and a > 0

$\therefore$ a = 3

2. Tangent at P_1 other than origin on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on, then $\frac{\text{area of } \Delta P_1P_2P_3}{\text{area of } \Delta P_2P_2P_4}$ equals

(A) 1 : 20 (B) 1 : 16 (C) 1 : 18 (D) 1 : 2

Ans. **(B)**

Sol. Let $P_1 = \left(t_1, t_1^3\right), P_2 = \left(t_2, t_2^3\right), P_3 = \left(t_3, t_3^3\right)$.......

Solving tangent equation at P_1 with the curve again we get $t_2 = -2t_1$. Repeating the process we have $t_3 = 4t_1, t_4 = -8t_1$.......

$$\therefore \frac{\Delta P_1P_2P_3}{\Delta P_2P_3P_4} = \begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix} \div \begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \frac{1}{16}$$

3. The portion of the tangent to the curve $x = \sqrt{a^2 - y^2} + \frac{a}{2}\log\frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$ between the point of contact and the x-axis is of length

(A) a^2 (B) 2a (C) a (D) 3a

Ans. **(C)**

Sol. Given, $x = \sqrt{a^2 - y^2} + \frac{a}{2}\log\frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$

Thus, y = a sin 2θ

$$x = a\cos 2\theta + \frac{a}{2}\log\left(\tan^2\theta\right)$$

$$\frac{dx}{d\theta} = -a\sin 2\theta(2) + \frac{a}{2}\frac{1}{(\tan^2\theta)}\left(2\tan\theta\sec^2\theta\right)$$

$$= -2a\sin 2\theta + \frac{2a}{\sin 2\theta}$$

$$= \frac{2a}{\sin 2\theta}\left(1 - \sin^2 2\theta\right)$$

$$= \frac{2a\cdot\cos^2 2\theta}{\sin 2\theta}$$

$$\therefore \frac{dy}{dx} = \frac{2a\cos 2\theta}{2a.\frac{\cos^2 2\theta}{\sin 2\theta}} = \tan 2\theta = m$$

$\Rightarrow$ Length of tangent $= |y_1|\sqrt{1 + \frac{1}{m^2}}$

$= \left|a\sin 2\theta\right|\sqrt{1 + \frac{1}{\tan^2 2\theta}}$ $= a\sin 2\theta\cdot\operatorname{cosec}2\theta$ $= a$

4. The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Then 'a' can be.

(A) 0 (B) 1 (C) 2 (D) 3

Ans. **(B)**

Sol. Given parabola, $y = -a^2x^2 + 5ax - 4$ ...(1)

Given curve $y = \frac{1}{1-x}$

At $x = 2, y = \frac{1}{1-2} = -1$

Point of contact of chord is (2, −1)

Also equation of tangent at (2, −1) to $y = \frac{1}{1-x}$

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}$$

$$\frac{dy}{dx_{|x=2|}} = 1$$

Equation of tangent

(y+1) = (1) (x – 2)

Now Equation (1) and (2) meet at a point

$\therefore\ x - 3 = -a^2x^2 + 5ax - 4$

$\Rightarrow -a^2x^2 + (5a-1)x - 1 = 0$

Let x_1, x_2 be the two values of x

$\therefore$ Sum of roots $(x_1 + x_2) = \frac{-1+5a}{a^2}$ ...(3)

Also given, the chord is bisected at x = 2

$\Rightarrow \frac{x_1 + x_2}{2} = 2$ $\qquad \therefore \frac{5a-1}{2a^2} = 2$

$\Rightarrow 5a - 1 = 4a^2$

$\Rightarrow 4a^2 - 5a + 1 = 0$

$\Rightarrow a = 1, \frac{1}{4}$

5. Angle between the curves $y = [|\sin x| + |\cos x|]$ where [.] denotes the greatest integer function and $x^2 + y^2 = 5$ is $\tan^{-1}(k)$ then k can be -

(A) -2 (B) 3 (C) $\frac{1}{3}$ (D) 2

Ans. **(D)**

Sol. $1 < |\sin x| + |\cos x| < \sqrt{2}$ $\quad [|\sin x| + |\cos x|] = 1$

P(2, 1) and Q (–2, 1) are the point of intersections, angle at p is $\tan^{-1}2$

6. Let f (x) = $\displaystyle\lim_{h\to 0}\frac{(\sin(x+h))^{\ln(x+h)}-(\sin x)^{\ln x}}{h}$ then $f\left(\frac{\pi}{2}\right)$ is

(A) equal to 0 (B) equal to 1 (C) $\ln\frac{\pi}{2}$ (D) non existent

Ans. **(A)**

Sol. Let g (x) = $(\sin x)^{\ln x}$ = $e^{\ln x \cdot \ln(\sin x)}$

$$f(x) = g'(x) = (\sin x)^{\ln x}\left[\cot x(\ln x)+\frac{\ln(\sin x)}{x}\right]$$

Hence $f\left(\frac{\pi}{2}\right)=g'\left(\frac{\pi}{2}\right)$ = 1(0 + 0) = 0 Ans.]

7. If $y=\frac{x}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)$ where $a\in(-\infty,-1)\cup(1,\infty)$ then $y'\left(\frac{\pi}{2}\right)$ equals

(A) $\frac{1}{a}$ (B) $\frac{2}{a}$ (C) $\frac{1}{2a}$ (D) a

Ans. **(A)**

Sol. Let $\sqrt{a^2-1}=C_1$ and $a+\sqrt{a^2-1}=C_2 \Rightarrow C_2-C_1=a$

$$\therefore \quad y=\frac{x}{C_1}-\frac{2}{C_1}\tan^{-1}\left(\frac{\sin x}{C_2+\cos x}\right)$$

$$\frac{dy}{dx}=\frac{1}{C_1}-\frac{2}{C_1}\frac{\left[(C_2+\cos x)\cos x+\sin^2 x\right]}{\left[1+\left(\frac{\sin x}{C_2+\cos x}\right)^2\right]\cdot(C_2+\cos x)^2}=\frac{1}{C_1}-\frac{2}{C_1}\cdot\frac{C_2\cos x+1}{(C_2+\cos x)^2+\sin^2 x}$$

$$\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{2}}=\frac{1}{C_1}-\frac{2}{C_1}\cdot\frac{1}{C_2^2+1}=\frac{1}{C_1}\left[\frac{C_2^2+1-2}{C_2^2+1}\right]=\frac{1}{C_1}\cdot\frac{C_2^2-1}{C_2^2+1}$$

$$\therefore \left.\frac{dy}{dx}\right]_{x=\frac{\pi}{2}}=\frac{2a^2+2a\sqrt{a^2-1}-2}{\sqrt{a^2-1}\left(2a^2-\cancel{1}+2a\sqrt{a^2-1}+\cancel{1}\right)}=\frac{2a^2+2a\sqrt{a^2-1}-2}{\sqrt{a^2-1}\left(2a\left(a+\sqrt{a^2-1}\right)\right)}$$

$$=\frac{2(a^2-1)+2a\sqrt{a^2-1}}{\sqrt{a^2-1}\left(2a\left(a+\sqrt{a^2-1}\right)\right)}=\frac{2\sqrt{a^2-1}\left(\sqrt{a^2-1}+a\right)}{\left(\sqrt{a^2-1}\right)(2a)\left(a+\sqrt{a^2-1}\right)}=\frac{1}{a}\text{ Ans.]}$$

8. Let $f(x)=\begin{vmatrix}\cos x & \sin x & \cos x\\ \cos 2x & \sin 2x & 2\cos 2x\\ \cos 3x & \sin 3x & 3\cos 3x\end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right)=$

(A) 0 (B) – 12 (C) 6 (D) 12

Ans. **(C)**

Sol. Differentiate column wise, where $\Delta_1 = -2$; $\Delta_2 = 0$ and $\Delta_3 = 8$.

9. If a curve is represented parametrically by the equations $x = f(t)$ and $y = g(t)$ then $\left(\frac{d^2y}{dx^2}\right)\Big/\left(\frac{d^2x}{dy^2}\right)$ is equal to (where $f'(t) \neq 0$, $g'(t) \neq 0$)

(A) 1 (B) $\frac{g'(t)}{f'(t)}$ (C) $-\left(\frac{g'(t)}{f'(t)}\right)^2$ (D) $-\left(\frac{g'(t)}{f'(t)}\right)^3$

Ans. **(D)**

Sol. We know that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

$$\therefore \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right) = \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy} \quad \text{(chain rule)} = \frac{-1}{\left(\frac{dy}{dx}\right)^2}\cdot\left(\frac{d^2y}{dx^2}\right)\left(\frac{1}{\frac{dy}{dx}}\right)$$

$$\therefore \frac{d^2x}{dy^2} = \frac{\frac{-d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}; \quad \text{Hence } \frac{\frac{d^2y}{dx^2}}{\frac{d^2x}{dy^2}} = -\left(\frac{dy}{dx}\right)^3 = -\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^3 = -\left(\frac{g'(t)}{f'(t)}\right)^3.$$

Alternatively : $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$; $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)\cdot\frac{dt}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$

And $\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{f'(t)}{g'(t)} \Rightarrow \frac{d^2x}{dy^2} = \frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)\cdot\frac{dt}{dy} = \frac{g'(t)f''(t) - f'(t)g''(t)}{(g'(t))^3}$

$$\therefore \left(\frac{d^2y}{dx^2}\right)\Big/\left(\frac{d^2x}{dy^2}\right) = -\left(\frac{g'(t)}{f'(t)}\right)^3$$

10. If $f(x) = \sin^{-1}\{[3x+2] - \{3x + (x - \{2x\})\}\}$, $x \in \left(0, \frac{\pi}{12}\right)$ and $gof(x) = x$, $x \in \left(0, \frac{\pi}{12}\right)$

then $g'\left(\frac{\pi}{6}\right)$ is equal to :

Note : {y} and [y] denote fractional part function and greatest integer function respectively.

(A) $\frac{\sqrt{3}}{8}$ (B) $\frac{-1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{-\sqrt{3}}{4}$

Ans. **(D)**

Sol. $f(x) = \sin^{-1}\{[3x+2] - \{3x + (x - \{2x\})\}\}$

$x \in \left(0, \frac{\pi}{12}\right) \Rightarrow \quad 2x \in \left(0, \frac{\pi}{6}\right) \quad \therefore \{2x\} = 2x$

and $3x \in \left(0, \frac{\pi}{4}\right)$

$f(x) = \sin^{-1}\{2 - \{3x - x\}\} = \sin^{-1}\{2 - 2x\}$ $(\text{As } \{2-2x\} = 1-2x)$

$f(x) = \sin^{-1}(1 - 2x) = y \Rightarrow 1 - 2x = \sin y \Rightarrow x = \frac{1-\sin y}{2} = f^{-1}(y)$

$\therefore f^{-1}(x) = \frac{1-\sin x}{2} = g(x)$

$g'(x) = \frac{-1}{2}\cos x$

$g'\left(\frac{\pi}{6}\right) = \frac{-1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{-\sqrt{3}}{4}$

MCQ (One or more than one correct) :

11. The equation of a straight line which is tangent to one point and normal to the other point on the curve $x = 2t^2 + 1$ & $y = 4t^3$ is/are

(A) $\sqrt{2}x - y = \frac{31\sqrt{2}}{27}$ (B) $\sqrt{2}x - y = \frac{39\sqrt{2}}{27}$

(C) $\sqrt{2}x + y = \frac{31\sqrt{2}}{27}$ (D) $\sqrt{2}x + y = \frac{39\sqrt{2}}{27}$

Ans. **(AC)**

Sol. $\frac{dy}{dx} = 3t$

Equation of tangent at 't' is $(y - 4t^3) = 3t(x - 2t^2 - 1)$ ... 1

This is a normal at another point 't_1'

$\therefore$ Slope of normal at $\left(2t_1^2 + 1, 4t_1^3\right) = 3t$

i.e., $\frac{-1}{3t_1} = 3t$ ....(2)

& $\left(2t_1^2 + 1, 4t_1^3\right)$ lies on (1)

$\Rightarrow t_1 = \frac{-t}{2}$ .....(3)

From (2) & (3), $t = \pm\frac{\sqrt{2}}{3}$

$\therefore$ Required straight lines are $\sqrt{2}x - y - \frac{31\sqrt{2}}{27}$ & $\sqrt{2}x + y = \frac{31\sqrt{2}}{27}$

12. Suppose ' f ' and ' g ' are functions having second derivatives f " and g" everywhere, if f (x).g(x) =1 for all ' x ' and f ' and g' are never zero, then $\frac{f''(x)}{f'(x)} - \frac{g''(x)}{g'(x)}$ equals

(A) $\frac{-2f'(x)}{f(x)}$ (B) $\frac{-2g'(x)}{g(x)}$ (C) $\frac{-f'(x)}{f(x)}$ (D) $\frac{2f'(x)}{f(x)}$

Ans. **(BD)**

Sol. $g = \frac{1}{f} \quad \Rightarrow g' = \frac{-1}{f^2} f'$

$$g'' = -\left[\frac{-2}{f^3} f'^2 + \frac{1}{f^2} f''\right] = \frac{2}{f^3} f'^2 - \frac{f''}{f^2}$$

$$\Rightarrow \frac{f''}{f'} - \frac{g''}{g'} = \frac{f''}{f'} - \frac{\frac{2}{f^3} f'^2 - \frac{f''}{f^2}}{\frac{-1}{f^2} f'} = \frac{f''}{f'} - \left(\frac{-2f'}{f} + \frac{f''}{f}\right) = \frac{2f'}{f}$$

In a similar manner, we can show that the same is equal to $\frac{-2g'}{g}$

13. Let ' f ' be a real valued function defined on the interval (0, ∞) by $f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\, dt$.

Then which of the following statement(s) is/are true?

(A) f '(x) exists for all $x \in (0,\infty)$ and f ' is continuous on $(0,\infty)$, but not differentiable on $(0, \infty)$

(B) f "(x) exists for all $x \in (0,\infty)$

(C) There exists $\alpha > 1$ such that |f '(x)| < | f (x)| for all $x \in (\alpha,\infty)$

(D) There exists $\beta > 0$ such that |f(x)| + |f '(x)| < β for all $x \in (0,\infty)$

Ans. **(AC)**

Sol. $f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$ for x > 0

$\therefore$ f' is continuous on (0, ∞) but f' is not differentiable at $x = 2n\pi + \frac{3\pi}{2}$ ('n' is a non-negative integer) so f' is not differentiable on (0, ∞)

Both f (x) and f'(x) are positive for all x > 1 and f'(x) < $\sqrt{2}+1$ for x > 1

Since, $0 \le \int_0^x \sqrt{1+\sin t}\, dt < \left(\sqrt{2}+1\right)x$ for x > θ

f(x) $\to +\infty$ as x $\to +\infty$

So, there exists $\alpha > 1$ such that x >$\alpha \Rightarrow$ f (x) > $\sqrt{2}+1$ and this implies that

$f'(x) < \sqrt{2}+1 < f(x)$ or $|f'(x)| < |f(x)|$ for x > α

f (x) $\to -\infty$ as x $\to 0^+$ and f'(x) $\to +\infty$ as x $\to 0^+$

$\therefore$ |f (x)| + f'(x) $\to +\infty$ as x $\to 0^+$

Also, f (x) $\to \infty$ as $x \to \infty$ and f'(x) is bounded on $(1, \infty)$

Hence, $|f(x)| + f'(x) \to +\infty$ as $x \to +\infty \Rightarrow$ f(x) is unbounded.

14. Let a curve be given parametrically $x(t) = 3t^2,\ y(t) = 2t^3,\ t \in R$

Suppose a line L is tangent at one point and normal at another point of the curve. Then which of the following statements is/are correct?

(A) There are two possible situation for the line L

(B) A possible equation of line L is $\sqrt{2}x - y - 2\sqrt{2} = 0$

(C) A possible equation of line L is $\sqrt{2}x + y - 2\sqrt{2} = 0$

(D) A possible equation of line L is $x + \sqrt{2}y - 2\sqrt{2} = 0$

Ans. **(ABC)**

Sol. Suppose L is tangent at t_1, normal of t_2

$$\therefore \left(\frac{dy}{dx}\right)_{t=t_1} = t_1, \left(\frac{dy}{dx}\right)_{t=t_2} = \frac{-1}{t_2}, t_1t_2 = -1$$

$$t_1 = \frac{2t_1^3 - 2t_2^3}{3t_1^2 - 3t_2^2} = \frac{2\left(t_1^2 + t_2^2 + t_1t_2\right)}{3\left(t_1 + t_2\right)}$$

$$= \frac{2\left(t_1^2 + \frac{1}{t_1^2} - 1\right)}{3\left(t_1 - \frac{1}{t_1}\right)} = \frac{2\left(t_1^4 - t_1^2 + 1\right)}{3t_1\left(t_1^2 - 1\right)} \Rightarrow t_1^2 = 2 \Rightarrow t_1 = \pm\sqrt{2}$$

$$t_1 = \sqrt{2} \Rightarrow \frac{y - 4\sqrt{2}}{x - 6} = \sqrt{2} \Rightarrow y = \sqrt{2}x - 2\sqrt{2}$$

$$t_2 = -\sqrt{2} \Rightarrow \frac{y + 4\sqrt{2}}{x - 6} = -\sqrt{2} \Rightarrow y + \sqrt{2}x = -2\sqrt{2}$$

15. Two functions f & g have first & second derivatives at x = 0 & satisfy the relations,

$f(0) = \frac{2}{g(0)}$, $f'(0) = 2g'(0) = 4g(0)$, $g''(0) = 5f''(0) = 6f(0) = 3$ then

(A) if $h(x) = \frac{f(x)}{g(x)}$ then $h'(0) = \frac{15}{4}$

(B) if $k(x) = f(x) \cdot g(x) \sin x$ then $k'(0) = 2$

(C) $\lim_{x \to 0} \frac{g'(x)}{f'(x)} = \frac{1}{2}$

(D) none

Ans. **(ABC)**

Sol. $$h'(0) = \frac{g(0)\,f'(0) - f(0)\,g'(0)}{\left(g(0)\right)^2} = \frac{g(0)\,4g(0) - \frac{2}{g(0)} \cdot 2g(0)}{\left(g(0)\right)^2}$$

$$= 4 - \frac{4}{(g(0))^2} \quad = 4 - (f(0))^2 \quad = 4 - \left(\frac{1}{2}\right)^2 \quad = \frac{15}{4}$$

$k'(0) = f(0)\ \ g(0)\ \cos 0 + f'(0)\ \ g(0)\ \ \sin 0 \times f(0)\ \ g'(0)\ \ \sin 0$

$k'(0) = f(0)\ \ g(0) = 2$

$\lim\limits_{x \to 0} \frac{g'(x)}{f'(x)} = \frac{1}{2}$.

16. Which of the following statements are true?

(A) If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $(dy/dx) = 1$.

(B) If $f(x) \equiv a_0 x^{2m+1} + a_1 x^{2m} + a_2 x^{2m-1} + \ldots\ldots + a_{2m+1} = 0$ $(a_0 \neq 0)$ is a polynomial equation with rational co-efficients then the equation $f'(x) = 0$ must have a real root.$(m \in N)$.

(C) If $(x - r)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots + a_0$ repeated m times where $1 \leq m \leq n$ then r is a root of the equation $f'(x) = 0$ repeated $(m - 1)$ times.

(D) If $y = \sin^{-1}(\cos \sin^{-1} x) + \cos^{-1}(\sin \cos^{-1} x)$ then $\frac{dy}{dx}$ is dependent on x.

Ans. **(AC)**

Sol. (A) $dy/dx = 1$

(D) y reduces to $\frac{\pi}{2}$ $\Rightarrow$ False.]

Comprehension Type Question:

Comprehension # 1

To find the point of contact $P = (x_1, y_1)$ of a tangent to the graph of $y = f(x)$ passing through origin O, we equate the slope of tangent to $y = f(x)$ at 'P' to the slope of OP. Hence, we solve the equation $f'(x_1) = \frac{f(x_1)}{x_1}$ to get x_1 and y_1 .

17. The equation $|\ln(mx)| = p\,x$, where 'm' is a positive constant has a single root for

(A) $0 < p < \frac{m}{e}$ (B) $p < \frac{e}{m}$ (C) $0 < p < \frac{e}{m}$ (D) $p > \frac{m}{e}$

Ans. **(D)**