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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-11

CONTINUITY & DIFFERENEIABILITY

ANSWER KEY OF CAPS-11

1.	(C)	**2.**	(B)	**3.**	(B)	**4.**	(A)	**5.**	(C)
6.	(C)	**7.**	(B)	**8.**	(B)	**9.**	(B)	**10.**	(B)
11.	(AD)	**12.**	(BCD)	**13.**	(ACD)	**14.**	(BC)	**15.**	(BC)
16.	(AB)	**17.**	(ABD)	**18.**	(0)	**19.**	(10)	**20.**	(1)
21.	(C)								

SCQ (Single Correct Type) :

1. The value of f(0), so that the function $f(x) = \dfrac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$ (a > 0) becomes continuous for all x, is given by -

(A) $a\sqrt{a}$ (B) $\sqrt{a}$ (C) $-\sqrt{a}$ (D) $-a\sqrt{a}$

Ans. **(C)**

Sol. $\displaystyle\lim_{x\to 0}\frac{(a^2-ax+x^2-a^2-ax-x^2)}{(a+x-a+x)}\cdot\frac{(\sqrt{a+x}+\sqrt{a-x})}{(\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2})}$

$$= \lim_{x\to 0} -\frac{2ax}{2x}\left(\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}}\right) = -\frac{a\sqrt{a}}{a} = -\sqrt{a}$$

2. Let f(x) = [cos x + sinx] , 0 < x < 2π, where [.] denotes G.I.F. The number of points of discontinuity of f(x) is-

(A) 6 (B) 5 (C) 4 (D) 3

Ans. **(B)**

Sol. $f(x) = \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]$

discontinuity may arise at the points where

$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$, $\sin\left(x+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$ and $\sin\left(x+\frac{\pi}{4}\right)=0$

$x=\frac{\pi}{2}$; $x=\frac{\pi}{2},\frac{3\pi}{2}$; $x=\frac{3\pi}{2},\frac{7\pi}{4}$ five points

$\therefore$ (B)

3. The function $f(x) = \begin{cases} x^2\left[\dfrac{1}{x^2}\right] & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$, is (where [.] denotes G.I.F.)

(A) Continuous at x = 1 (B) Continuous at x = – 1

(C) Discontinuous at x = 0 (D) Continuous at x = 2

Ans. **(B)**

Sol. $\lim\limits_{x \to 0^+} x^2\left[\dfrac{1}{x^2}\right] = \lim\limits_{x \to 0^+} x^2\left(\dfrac{1}{x^2} - \left\{\dfrac{1}{x^2}\right\}\right) \Rightarrow \lim\limits_{x \to 0^+}\left(1 - x^2\left\{\dfrac{1}{x^2}\right\}\right) = 1$

similarly $\lim\limits_{x \to 0^-} f(x) = 1$

$\therefore$ (C)

4. If $f(x) = [x^2] + \sqrt{\{x\}^2}$, where [.] and {.} denote the greatest integer and fractional part functions respectively, then-

(A) f(x) is continuous at all integral points except 0

(B) f(x) is continuous and differentiable at x = 0

(C) f(x) is discontinuous for all $x \in I - \{1\}$

(D) f(x) is not differentiable for all $x \in I$.

Ans. **(A)**

Sol. $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}\left([x^2] + \sqrt{\{x\}^2}\right) \Rightarrow \lim\limits_{x \to 1^+}(1 + 0) = 1$

$\because \lim\limits_{x \to 1^-} f(x) \Rightarrow \lim\limits_{x \to 1^-}(0 + 1) = 1$ and $f(1) = 1 \Rightarrow \lim\limits_{x \to 1} f(x) = f(1)$

$\therefore$ continuous at x = 1

similarly we check for another integers

5. If f(x) is a continuous function for all real values of x satisfying $x^2 + (f(x) - 2)x + 2\sqrt{3} - 3 - \sqrt{3}\, f(x) = 0$, then the value of $f(\sqrt{3})$ is -

(A) $\sqrt{3}$ (B) $1 - \sqrt{3}$ (C) $2(1 - \sqrt{3})$ (D) $2(\sqrt{3} - 1)$

Ans. **(C)**

Sol. As f(x) is continuous for all $x \in R$

Thus $\lim\limits_{x \to \sqrt{3}} f(x) = f\left(\sqrt{3}\right)$

since $f(x) = \dfrac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x}$, $x \neq \sqrt{3}$

$\therefore \lim\limits_{x \to \sqrt{3}} f(x) = \lim\limits_{x \to \sqrt{3}} \dfrac{x^2 - 2x + 2\sqrt{3} - 3}{\sqrt{3} - x} = \lim\limits_{x \to \sqrt{3}} \dfrac{(x - \sqrt{3})(x + \sqrt{3} - 2)}{(\sqrt{3} - x)} = 2\left(1 - \sqrt{3}\right)$

Thus $f\left(\sqrt{3}\right) = 2(1 - \sqrt{3})$

$\therefore$ (C)

6. If g(x) is a polynomial satisfying g(x)g(y) = g(x) + g(y) + g(xy) − 2 for all real x and y and g(2) = 5, then $\lim_{x\to 3} g(x)$ is _____.

(A) 9 (B) 25 (C) 10 (D) none of these

Ans. **(C)**

Sol. $g(x)g(y) = g(x) + g(y) + g(xy) - 2$

Putting x = 2, y = 1, we have

$5g(1) = 5 + g(1) + 5 - 2 \quad \Rightarrow g(1) = 2$

Putting $y = \frac{1}{x}$, we have

$g(x)g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right)$

Hence $g(x) = x^n + 1$ or $-x^n + 1$ (by comparing the coefficient of x, x^2, ...)

If $g(x) = x^n + 1$ then $g(1) = 1 + 1 = 2$ and $g(2) = 2^n + 1 = 5 \Rightarrow n = 2$

So, g(x) = x^2 +1

and if g(x) = – x^n +1, then g(1) = 0, which is not true

So g(x) = x^2 + 1

$\therefore \lim_{x\to 3} g(x) = 10$

7. A differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies f (xy) = f (x) + f(y) $\forall$ x, $y \in \mathbb{R}^+$. If f(16) = 3, then the value of f(2) is _____.

(A) $\frac{3}{8}$ (B) $\frac{3}{4}$ (C) $\sqrt{3}$ (D) $\frac{3}{2}$

Ans. **(B)**

Sol. $f(xy) = f(x) + f(y) \forall x, y \in \mathbb{R}^+$

Putting x = y = 4, we have

$f(16) = 3 = 2f(4) \Rightarrow f(4) = \frac{3}{2}$

Putting x = y = 2, we have

$f(4) = \frac{3}{2} = 2f(2) \quad \Rightarrow f(2) = \frac{3}{4}$

8. Let f(x), $f : \mathbb{R} \to \mathbb{R}$ be a non-constant continuous function such that f(2x) = (e^x+1) f(x). the value of f'(0) is ______ .

(A) $\lim_{h\to 0} \frac{f(h)}{e^h + 1}$ (B) $\lim_{h\to 0} \frac{f(h)}{e^h - 1}$ (C) $\lim_{h\to 0} \frac{f(h)}{e^h - h - 1}$ (D) $\lim_{h\to 0} \frac{f(h)}{e^{-h} - 1}$

Ans. **(B)**

Sol. Given that $f(2x) = \left(e^x + 1\right)f(x)$ and f(0) = 0

$$\Rightarrow \left(e^{x}-1\right)f(2x)=\left(e^{2x}-1\right)f(x)\Rightarrow \frac{f(2x)}{e^{2x}-1}=\frac{f(x)}{e^{x}-1}$$

$$\Rightarrow \frac{f(x)}{e^{e}-1}=\frac{f\left(\frac{x}{2}\right)}{e^{\frac{x}{2}}-1}=\frac{f\left(\frac{x}{4}\right)}{e^{\frac{x}{2^2}}-1}=.....=\frac{f\left(\frac{x}{2^n}\right)}{e^{\frac{x}{2^n}}-1}$$

$$\Rightarrow \frac{f(x)}{e^{e}-1}=\lim_{n\to\infty}\frac{f\left(\frac{x}{2^n}\right)}{e^{\frac{x}{2}}-1}=\lim_{n\to 0}\frac{f(h)}{e^{h}-1}=\lim_{n\to\infty}\frac{f(h)}{h}=\lim_{n\to\infty}\frac{f(h)-f(0)}{h}=f'(0)$$

$$\therefore f'(0)=\lim_{h\to 0}\frac{f(h)}{e^{h}-1}$$

9. If f(x) be positive, continuous and differentiable on the interval (a, b). If $\lim_{x\to a^+} f(x)=1$ and $\lim_{x\to b^-} f(x)=3^{\frac{1}{4}}$ also $f'(x)>(f(x))^3+\frac{1}{f(x)}$ then ______.

(A) $b-a>\frac{\pi}{24}$ (B) $b-a<\frac{\pi}{24}$ (C) $b-a=\frac{\pi}{12}$ (D) $b-a=\frac{\pi}{24}$

Ans. **(B)**

Sol. $\frac{f'(x)f(x)}{f(x)^4+1}>1$

Integrating both sides with respect to x from a to b

$$\Rightarrow \frac{1}{2}\left[\tan^{-1}\left(\left(f(x)\right)^2\right)\right]_a^b>(b-a)$$

$$\Rightarrow \frac{1}{2}\left\{\frac{\pi}{3}-\frac{\pi}{4}\right\}>(b-a)$$

$$\Rightarrow b-a<\frac{\pi}{24}$$

10. If f(x) = sgn ($\sin^2 x - \sin x - 1$) has exactly four points of discontinuity for $x\in(0, n\pi)$ $n\in\mathbb{N}$ then n can be

(A) only 4 (B) 4 or 5 (C) only 5 (D) 5 or 6

Ans. **(B)**

Sol. g(x) = sgn (f(x)) has discontinuty points, at the points where f(x) = 0

$$\Rightarrow \sin x=\frac{1\pm\sqrt{5}}{2}$$

$$\Rightarrow \sin x=\frac{1+\sqrt{5}}{2}\ \text{(not possible)}$$

$$\Rightarrow \sin x=\frac{1-\sqrt{5}}{2}$$

MCQ (One or more than one correct) :

11. Suppose that f(x) is a differentiable invertible function with $f'(x) \neq 0$ and $h(x) = \int_1^x f(t)\,dx$. Given that $f(1) = f'(1) = 1$ and g(x) is inverse of f(x). Let G(x) = $x^2g(x) - xh(g(x))$ $\forall x \in \mathbb{R}$. Which of the following are correct?

(A) G'(1) = 2 (B) G'(1) = 3 (C) G"(1) = 2 (D) G"(1) = 3

Ans. **(AD)**

Sol. $h'(x) = f(x) \Rightarrow h''(x) = f'(x)$

$h(1) = 0, f(1) \Rightarrow h'(1) = h''(1) = 1 = g(1)$

$f(g(x)) = x \Rightarrow f'(g(x))\,g'(x) = 1 \Rightarrow f(g(1))\,g'(1) = 1$

$\Rightarrow f'(1).g'(1) = 1 \Rightarrow g'(1) = 1$

$G(x) = x^2\,g(x) - xh\left(g(x)\right)$

$G'(x) = 2xg(x) + x^2\,g'(x) - h(g(x)) - xh'(g(x))\,g'(x)$

$= 2xg(x) + x^2\,g'(x) - h(g(x)) - x^2\,g'(x)$

$= 2xg(x) - h(g(x))$

$G''(x) = 2\,g(x) + 2\,xg'(x) - h'(g(x))\,g'(x)$

$G'(1) = 2\,g(1) - h(g(1)) = 2$

$G''(1) = 2\,g(1) + g'(1) = 3$

12. consider the function $f(x) = \begin{cases} \int_0^x \left(4 + |t-2|\right)dt, & x > 3 \\ ax^2 + bx & x \leq 3 \end{cases}$. If f(x) is differentiable at x = 3, then ____.

(A) $a + b = \frac{83}{18}$ (B) $a + b = \frac{85}{18}$ (C) $ab = \frac{7}{27}$ (D) $\frac{b}{a} = 84$

Ans. **(BCD)**

Sol. $\int_0^x \left(4 + |t-2|\right)dt = \int_0^2 (6-t)\,dt + \int_2^x (t+2)\,dt = \frac{x^2}{2} + 2x + 4$

$f(x) = \begin{cases} \frac{1}{2}x^2 + 2x + 4 & x > 3 \\ ax^2 + bx & x \leq 3 \end{cases}$

By solving, we get $a = \frac{1}{18}, b = \frac{14}{3}$

13. If $f(x) = \left|x^2 - 4|x| + 3\right|$ then _____.

(A) f(x) is non-differentiable at 5 points

(B) f(x) is non-differentiable at 4 points

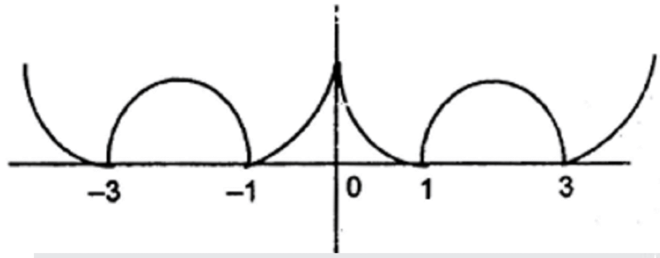
(C) f(x) has local maxima at $x = 0$

(D) f(x) has local minima at x = –1

Ans. **(ACD)**

Sol. $x^2 - 4x + 3 = 0 \quad \Rightarrow x = 1, 3$

Now graph of $\left|x^2 - 4\left|x\right| + 3\right|$ is as shown. From the graph, it is clear that f(x) is non differentiable at 5 points, local maximum at x = 0 and local minima at x = –1



14. If $f(x) = \begin{cases} a + \dfrac{\sin^3[x]}{x} & x > 0 \\ 3, & x = 0 \\ 2b + \left[\dfrac{\sin x - x}{x^3}\right] & x < 0 \end{cases}$ and f(x) is continuous at x = 0, then

(A) a = 2 (B) a = 3 (C) b = 2 (D) b = 3

Ans. **(BC)**

Sol. $\lim\limits_{x \to 0^+} f(x) = a$ as $\sin^3[x] = 0$ when $x \to 0^+$

$\lim\limits_{x \to 0^+} f(x) = 2b - 1$ and f(0) = 3

$\therefore$ a = 3 and b = 2

15. If $f(x) = -1 + |x - 2|, 0 \le x \le 4$ and $g(x) = 2 - |x|, -1 \le x \le 3$ then (fog) (x) is ______.

(A) Discontinuous at x = 0

(B) Continuous at x = 0

(C) Not differentiable at x = 0

(D) Differentiable at x = 0

Ans. **(BC)**

Sol. $f(x) = \begin{cases} 1 - x, & x \le x \le 2 \\ x - 3, & 2 \le x \le 4 \end{cases}$

$g(x) = \begin{cases} 2 + x, & -1 \le x \le 0 \\ 2 - 3, & 0 < x \le 3 \end{cases}$

$(fog)(x) = \begin{cases} 1 - g(x), & 0 \le g(x) \le 2 \\ g(x - 3), & 2 < g(x) \le 4 \end{cases}$

$= \begin{cases} -1 - x, & -1 \le x \le 2 \\ -1 + x, & 0 < x \le 2 \end{cases}$

16. Suppose f is a function that satisfies the equation f(x+y) = f(x) + f(y) + x^2y + xy^2 for all real numbers x and y. If $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$, Then _____.

(A) f(x) > 0 for x > 0 and f (x) < 0 for x < 0

(B) f’ (0) =1

(C) f”(0) =1

(D) f”’(0) = 6

Ans. **(AB)**

Sol. Observe that f(x) = 0 and $f'(x) = \lim\limits_{h \to 0} \dfrac{f(h)}{h} = 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(h) + x^2 h + xh^2}{h} = x^2 + 1$$

Hence, $f(x) = \dfrac{x^3}{3} + x$

17. Let [x] denote the greatest integer less than or equal to x. If f(x) = [x sin πx], then f(x) is :

(A) continuous at x = 0 (B) continuous in (–1, 0)

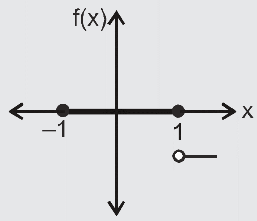
(C) differentiable at x = 1 (D) differentiable in (–1, 1)

Ans. **(ABD)**

Sol. f(x) = [x sin πx]

f(0) = 0

$f(0^-) = \lim\limits_{h \to 0^+} [-h \sin\pi(-h)] = 0$; $f(0^+) = \lim\limits_{h \to 0^+} [h \sin \pi h] = 0$



Here $f(0^-) = f(0^+) = f(0) = 0$

So continuous at x = 0

Since graph of f(x) is as shown in the figure

Now all options can be checked from graph.

Numerical based Questions :

18. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies sinx cosy (f(2x + 2y) – f(2x–2y)) = cos x sin y $(f(2x+2y) + f(2x-2y))$. If $f'(0) = \dfrac{1}{2}$, then the value of $4f''(x) + f(x)$ is ______

Ans. **(0)**

Sol. sin x cos y (f(2x+2y)–f(2x–2y)) = cosx sin y (f(2x + 2y)–f(2x–2y))

$$\Rightarrow \frac{f(2x+2y)}{f(2x-2y)} = \frac{\sin(x+y)}{\sin(x-y)} \Rightarrow \frac{f(\alpha)}{\sin\frac{\alpha}{2}} = \frac{f(\beta)}{\sin\frac{\beta}{2}} = k$$

$$\Rightarrow f(x) = k\sin\frac{x}{2} \Rightarrow 4f''(x) + f(x) = 0$$

19. Let f(x) be a real valued function not identically zero such that $f(x + y^3) = f(x) + [f(y)]^3$ $\forall$ x, y $\in$ R and $f'(0) \geq 0$, then find f(10)

Ans. f(10) = 10

Sol. Given $f(x + y^3) = f(x) + [f(y)]^3$ and $f'(0) \geq 0$

putting x = y = 0, we get

$f(0) = f(0) + (f(0))^3 \Rightarrow f(0) = 0$

also $f'(0) = \operatorname*{limit}\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \operatorname*{limit}\limits_{h \to 0} \dfrac{f(h)}{h}$

Let L = $f'(0) = \lim_{h\to 0} \frac{f(0+(h^{1/3})^3)-f(0)}{(h^{1/3})^3} = \lim_{h\to 0} \frac{\left(f(h^{1/3})\right)^3}{(h^{1/3})^3} = L^3$

or $\quad L = L^3 \quad$ or $\quad L = 0, 1, -1$ as $f'(0) \geq 0 \Rightarrow f'(0) = 0, 1$

Thus $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f\left(x+(h^{1/3})^3\right)-f(x)}{(h^{1/3})^3}$

$f'(x) = \lim_{h\to 0} \frac{f(x)+(f(h^{1/3}))^3-f(x)}{(h^{1/3})^3} \quad \Rightarrow \quad f'(x) = 0, 1$

Integrating both sides, we get

f(x) = 0 or f(x) = x + c

As f(0) = 0 , we have f(x) = 0 or f(x) = x

Now f(x) = 0 is imposible as f(x) is not identically zero

$\therefore \quad f(x) = x$ and $f(10) = 10$

20. Let f be a continuous function on R such that $f\left(\frac{1}{4x}\right) = \left(\sin e^x\right) e^{-x^2} + \frac{x^2}{x^2+1}$, then find the value of f(0).

Ans. 1

Sol. As f is continuous on R, so $f(0) = \lim_{x\to 0} f(x)$

Thus $f(0) = \lim_{n\to\infty} f\left(\frac{1}{4n}\right) = \lim_{n\to\infty}\left((\sin e^n)\, e^{-n^2} + \frac{1}{1+\frac{1}{n^2}}\right) = 0 + 1 = 1$

Matrix Match Type :

21. Match the following :

Column-1	Column-2
(A) $f(x) = (\log_e x)(x-1)^{\frac{1}{5}}$ at $x = 1$	(p) continuous
(B) $f(x) = [\cos 2\pi x]^4 \sqrt{\left\{\operatorname{som}\pi \frac{x}{2}\right\}}$, where [.] and {.} denote the greatest integer and the fractional part function, respectively, at x = 1	(q) discontinuous
(C) $f(x) = \begin{cases} \cos 2x, & x \in \mathbb{Q} \\ \sin x, & x \notin \mathbb{Q} \end{cases}$ at $x = \frac{\pi}{6}$	(r) differentiable
(D) $f(x) = \min\{1, \cos x, 1-\sin x\}$, $-\pi \le x \le \pi$ at x = 0	(s) non differentiable

(A) A–p; B–r; C–q; D–s

(B) A–r; B–p; C–s; D–r

(C) A–p, r; B–p, r; C–p, s; D–p, s

(D) A–r; B–p; C–q; D–r

Ans. **(C)**

Sol. (A) $f(x) - (\log_e x)(x-1)^{\frac{1}{5}}$

Clearly, f(x) is continuous at x = 1

$$f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\left(\frac{\log(1+h)}{h}\right)\left(h^{\frac{1}{5}}\right) = 0$$

$$f'(1^+) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{h} = \lim_{h\to 0}\left(\frac{\log(1-h)}{-h}\right)\left(-h^{\frac{1}{5}}\right) = 0$$

Hence, f(x) is differentiable at x = 0

(B) $f(x) = [\cos 2\pi x] + \sqrt{\left\{\sin\frac{\pi x}{2}\right\}}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to^{-1}}[\cos 2\pi x] + \lim_{x\to 1^-}\sqrt{\left\{\sin\frac{\pi x}{2}\right\}} = 0+1=1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to^{+1}}[\cos 2\pi x] + \lim_{x\to 1^+}\sqrt{\left\{\sin\frac{\pi x}{2}\right\}} = 0+1=1$$

Also f(1) = 1 + 0 = 1

$\therefore$ f(x) is continuous at x = 1

$$f'(1^+) = \lim_{x\to 0}\frac{[\cos 2\pi(1+h)] + \sqrt{\left\{\sin\left(\frac{\pi(1+h)}{2}\right)\right\}} - 1}{h}$$

$$= \lim_{x\to 0}\frac{[\cos 2\pi h)] + \sqrt{\left\{\cos\left(\frac{\pi h}{2}\right)\right\}} - 1}{h}$$

$$= \lim_{x\to 0}\frac{\sqrt{\left\{\cos\left(\frac{\pi h}{2}\right)\right\}} - 1}{h}$$

$$= \lim_{x\to 0}\frac{-\frac{\pi}{2}\sin\left(\frac{\pi h}{2}\right)}{2\sqrt{\cos\left(\frac{\pi h}{2}\right)}}$$

Similarly, $f'(1^-) = 0$

(C) $f(x) = \begin{cases}\cos 2x, & x\in\mathbb{Q}\\ \sin x & x\notin\mathbb{Q}\end{cases}$

f(x) is continuous when cos 2x = sin x,

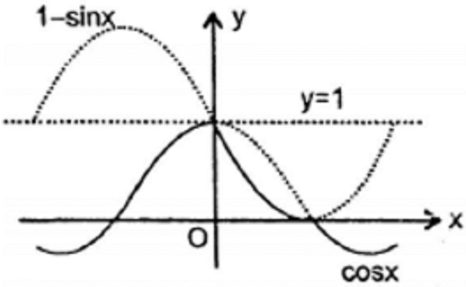
which has $x = \frac{\pi}{6}$ as one of the solutions.

Hence, it is continuous at $x = \frac{\pi}{6}$

If neighbourhood of $x = \frac{\pi}{6}$ is rational, then f'(x) = $-\sqrt{3}$

and if neighbourhood is irrational, then $f'(x) = \frac{1}{2}$

Here, $f'\left(\frac{\pi}{6}\right)^- \neq f'\left(\frac{\pi}{6}\right)^+$ $\Rightarrow$ f(x) is not differentiable at $x = \frac{\pi}{6}$

(D) 

From graph of f(x), we can say that f(x) is continuous but not differentiable at x = 0

Subjective Match Type :

22. Discuss the continuity of the function f(x) = $\lim\limits_{n\to\infty}\dfrac{(1+\sin x)^n+\log x}{2+(1+\sin x)^n}$

Ans. f(x) is discontinuous at integral multiples of π

Sol. f(x) = $\lim\limits_{n\to\infty}\dfrac{(1+\sin x)^n+\log x}{2+(1+\sin x)^n}$

(i) for $0 < \sin x \le 1$,

$$f(x) = \lim_{n\to\infty}\frac{1+\dfrac{\log x}{(1+\sin x)^n}}{\dfrac{2}{(1+\sin x)^n}+1} = 1$$

(ii) for sin x = 0,

$$f(x) = \frac{1+\log x}{3}$$

(iii) for $-1 \le \sin x < 0$,

$$f(x) = \frac{\log x}{2}$$

$$\therefore \quad f(x) = \begin{cases} \dfrac{\log x}{2} & , \quad -1\le \sin x\le 0 \\ \dfrac{1+\log x}{3} & , \quad \sin x = 0 \\ 1 & , \quad 0<\sin x\le 1 \end{cases}$$

$\therefore$ f(x) is discontinuous at integral multiples of π

23. If $g(x) = \begin{cases} \dfrac{1-a^x+xa^x.\,\ell na}{x^2a^x} & , \quad x<0 \\ \dfrac{(2a)^x - x\,\ell n2a-1}{x^2} & , \quad x>0 \end{cases}$

(where a > 0) , then find ‘a’ and g(0) so that g(x) is continuous at x = 0.

Ans. $\dfrac{1}{\sqrt{2}}$, $\dfrac{1}{8}(\ell n\, 2)^2$

Sol. $\lim\limits_{x\to 0^-}\dfrac{1-a^x+x\,.\,a^x\ell na}{x^2a^x}$

$$= \lim_{x\to 0^-} \frac{-a^x \ell na + \ell na(a^x + xa^x\,\ell na)}{x^2a^x\ell na + 2x.a^x} = \lim_{x\to 0^-} \frac{a^x(\ell na)^2}{(xa^x\ell na + 2a^x)} = \frac{(\ell na)^2}{2}$$

$$\lim_{x\to 0^+} \frac{(2a)^x - x\ell n\,2a - 1}{x^2} = \lim_{x\to 0^+} \frac{(2a)^x\ell n2a - \ell n2a}{2x}$$

$$= \lim_{x\to 0^+} \frac{(2a)^x(\ell n2a)^2}{2} = \frac{(\ell n2a)^2}{2}$$

for g(x) to be continuous $(\ell na)^2 = (\ell n2a)^2$

$\Rightarrow \quad (\ell na + \ell n2a) = 0$

$\Rightarrow \quad a = \frac{1}{\sqrt{2}}$

$\therefore \quad g(0) = \frac{1}{8}(\ell n\,2)^2$

24. If $f : R \to (-\pi, \pi)$ be a derivable function such that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$, $xy < 1$.

If $f(1) = \frac{\pi}{2}$ and $\lim_{x\to 0} \frac{f(x)}{x} = 2$, find f(x).

Ans. $2\tan^{-1}x$

Sol. $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right) \quad \Rightarrow \quad f'(x) = f'\left(\frac{x+y}{1-xy}\right)\left(\frac{(1-xy)-(-y)(x+y)}{(1-xy)^2}\right)$

$f'(x) = f'\left(\frac{x+y}{1-xy}\right)\left(\frac{1+y^2}{(1-xy)^2}\right)$

put x = 0

$f'(0) = f'(y)\,(1+y^2)$

$\frac{f'(0)}{1+y^2} = f'(y)$

Integrating both sides

$f'(0)\tan^{-1}y = f(y) + c$ ..... (1)

Now put x = 0 & y = 0 in $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$

we get $2f(0) = f(0)$

$\therefore f(0) = 0$ .... (2) and $\because \lim_{x\to 0} \frac{f(x)}{x} = 2$

$\Rightarrow \lim_{x\to 0} \frac{f(x)-f(0)}{x} = 2$

$\Rightarrow \quad f'(x) = 2 \qquad \therefore \quad f'(0) = 2$ .... (3)

from (2) & (3) & (1)

$2\tan^{-1}y = f(y) + c$

now put y = 1, we get

$2\tan^{-1}1 = f(1) + c$

$\frac{\pi}{2} = \frac{\pi}{2} + c \quad \Rightarrow \quad c = 0$

$\therefore\ f(x) = 2\tan^{-1}x$

25. Given $f(x) = \cos^{-1}\left(\text{sgn}\left(\frac{2[x]}{3x - [x]}\right)\right)$, where sgn ( ) denotes the signum function and [.] denotes the greatest integer function. Discuss the continuity and differentiability of f (x) at x = ± 1.

Ans. f is continuous & derivable at x = – 1 but f is neither continuous nor derivable at x = 1

Sol. $\because f(1) = 0$

$$\text{R.H.L.} = \lim_{h\to 0^+} f(1 + h) = \lim_{h\to 0^+} \cos^{-1}\left(\text{sgn}\left(\frac{2}{3 + 3h - 1}\right)\right) = \lim_{h\to 0^+} \cos^{-1}\left(\text{sgn}\left(\frac{2}{2 + 3h}\right)\right) = 0$$

$$\text{L.H.L.} = \lim_{h\to 0^+} f(1 - h) = \lim_{h\to 0^+} \cos^{-1}\left(\text{sgn}\left(\frac{0}{3 - 3h}\right)\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

$\because \quad f(-1) = 0$

$\therefore$ f(x) is discontinuous hence non-derivable at x = 1

$\because \quad f'(-1^+) = \quad = 0$

and $\quad f'(-1^-) = \quad = 0$

$\Rightarrow \quad f'(-1^+) = f'(-1^-) = 0$

$\therefore$ f(x) is derivable at x = – 1