Comprehension # 4

Given $g(x) = \sin\frac{1}{x}, h(x) = x^2 + 2x + (\lambda + 1)$ and $u(x) = \frac{1}{x} + \cos\frac{1}{x^2}$ Let

$$f(x) = \lim_{n\to\infty} \frac{x^{2n+1} g(x) + h(x)}{x^{2n} + 3x.u(x)}, n \in I$$

On the basis of above information, answer the following questions:

24. $\lim_{x\to\infty} f(x)$ is equal to :

(A) 0 (B) 2 (C) 5 (D) 1

Ans. (D)

Sol. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\left(\frac{1}{x}\right)} = 1$

25. If $\lim_{x\to 0} f(x) = 2$, then the value of λ is ;

(A) 10 (B) 5 (C) 2 (D) does not exist

Ans. (B)

Sol. $$f(x) = \lim_{x\to\infty} \frac{x^{2n+1}\sin\frac{1}{x} + (x^2 + 2x + (\lambda + 1)}{x^{2n} + 3.x\left(\frac{1}{x} + \cos\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{x.(x^2)^n \sin\frac{1}{x} + (x^2 + 2x + (\lambda + 1))}{(x^2)^n + 3.x\left(\frac{1}{x} + \cos\frac{1}{x^2}\right)}$$

$$f(x) = \begin{cases} \dfrac{x^2 + 2x + (\lambda + 1)}{3\left(1 + x\cdot\cos\frac{1}{x^2}\right)} \\[2ex] \dfrac{x.(x^2)^n \sin\frac{1}{x} + (x^2 + 2x + (\lambda + 1))}{1 + 3(1 + x\cos 1)} \\[2ex] I\cdot\sin\frac{1}{x} \end{cases}$$

$\because \lim_{x\to 0} f(x) = 2 \Rightarrow \frac{\lambda + 1}{3} = 2 \Rightarrow \lambda = 5$

Numerical based Questions :

26. If $\lim\limits_{x\to 0}\dfrac{\sin x + ae^{x} + be^{-x} + c\ln(1+x)}{x^3}$ exists, then find the value of a + b + c

Ans. **(0)**

Sol. $L = \lim\limits_{x\to\infty}\dfrac{\sin x + ae^{x} + be^{-x} + c\ln(1+x)}{x^3}$

$$= \lim_{x\to\infty}\frac{\left(x-\frac{x^3}{3!}\right)+a\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)+b\left(1-\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)+c\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)}{x^3}$$

$$= \lim_{x\to\infty}\frac{(a+b)+(1+a-b+c)x+\left(\frac{a}{2}+\frac{b}{2}-\frac{c}{2}\right)x^2+\left(-\frac{1}{3!}+\frac{a}{3!}-\frac{b}{3!}+\frac{c}{3!}\right)x^3}{x^3}$$

$\therefore a+b=0, 1+a-b+c=0, \dfrac{a}{2}+\dfrac{b}{2}-\dfrac{c}{2}=0$ and $L=-\dfrac{1}{3!}+\dfrac{a}{3!}-\dfrac{b}{3!}+\dfrac{c}{3}$

Solving first three equations, we get c= 0, $a=-\dfrac{1}{2}, b=\dfrac{1}{2}$

$\therefore\ a+b+c=0$

27. Let $K>0$ and $\lambda=\lim\limits_{n\to 0}\dfrac{K\left(1-4\sqrt{K^2-x^2}\right)}{x^2\sqrt{K^2-x^2}}$ is finite then the value of λK is

Ans. **(2)**

Sol. Let x = K sinθ

$\lambda=\lim\limits_{\theta\to 0}\dfrac{1}{K^2}\left(\dfrac{1-4K\cos\theta}{\sin^2\theta\cos\theta}\right)$

λ is finite $K=\dfrac{1}{4}\Rightarrow\lambda=\dfrac{2}{k}$

28. If f(x) = $\begin{cases} x-1, & x\ge 1 \\ 2x^2-2, & x<1 \end{cases}$, g(x) = $\begin{cases} x+1, & x>0 \\ -x^2+1, & x\le 0 \end{cases}$ and h(x) = |x|,

then $\lim\limits_{x\to 0}$ f(g(h(x))) is equal to

Ans. **0**

Sol. $\lim\limits_{x\to 0}$ f(g (h (x)))

L.H.L. $x\to 0^-$

$\lim\limits_{x\to 0^-} h(x) = 0^+$

$\lim\limits_{x\to 0^+} f(g(x))$

then $\lim\limits_{x\to 0^+} g(x) = 1^+$

$\lim\limits_{x\to 1^+} f(x) = 1 - 1 = 0$

R.H.L. $x\to 0^+$

$\lim\limits_{x\to 0^+} h(x) = 0^+$

so $\lim\limits_{x\to 0^+} f(g(x)) = 0$

L.H.L. = R.H.L. = 0

29. Let $\lim\limits_{x\to\infty} x\, ln \left(e\left(1+\frac{1}{x}\right)^{1-x}\right)$ equal $\frac{m}{n}$ where m and n are relatively prime positive integer.

Find (m + n).

Ans. 5

Sol. $l = \lim\limits_{x\to\infty} x\, \ell n\left(\frac{e(1+(1/x))}{(1+(1/x))^x}\right)$ (∞ x 0 form)

$= \lim\limits_{x\to\infty} x\left(1+\ell n\left(1+\frac{1}{x}\right) - x\ \ell n\left(1+\frac{1}{x}\right)\right)$

put $x = \frac{1}{t}$; as $x \to \infty$, $t \to 0$

Hence $l = \lim\limits_{t\to 0} \frac{1}{t}\left(1+\ell n(1+t) - \frac{\ell n(1+t)}{t}\right)$

$= \lim\limits_{t\to 0}\left[\ell n(1+t)^{1/t} + \frac{t-\ell n(1+t)}{t^2}\right]$

$= 1 + \lim\limits_{y\to 0}\left(\frac{e^y-1-y}{y^2}\right)$ where $ln(1 + t) = y$; $1 + t = e^y$, hence $t = e^4 - 1$

$= 1 + \frac{1}{2} = \frac{3}{2} = \frac{m}{n}$ $\Rightarrow$ (m + n) = 5 Ans.

30. Let $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = x^2 + x - 2$.

If $\lim\limits_{x\to 1} \frac{f(x)}{g(x)} = 1$ and $\lim\limits_{x\to -2} \frac{f(x)}{g(x)} = 4$, then find the value of $\frac{c^2+d^2}{a^2+b^2}$

Ans. 16

Sol. $g(x) = x^2 + x - 2 = (x + 2)(x - 1)$

Now $\lim\limits_{x\to 1} \frac{f(x)}{(x+2)(x-1)}$ and $\lim\limits_{x\to -2} \frac{f(x)}{(x+2)(x-1)}$ both exist

hence $f(x) = (x - 1)(x + 2)(k_1x + k_2)$

$\therefore$ $\lim\limits_{x\to 1}(k_1x+k_2) = 1$

$\Rightarrow$ $k_1 + k_2 = 1$ ....(1)

and $\lim\limits_{x\to -2}(k_1x+k_2) = 4$

$\Rightarrow$ $k_2 - 2k_1 = 4$ ....(2)

On solving (1) and (2), we get

$k_1 = -1$ and $k_2 = 2$

Hence cubic $f(x) = (x - 1)(x + 2)(2 - x)$

$= -x^3 + x^2 + 4x - 4$

$\therefore$ $a = -1$, $b = 1$, $c = 4$, $d = -4$

Hence $\frac{c^2+d^2}{a^2+b^2} = \frac{(4)^2+(-4)^2}{(-1)+(1)^2} = 16$ **Ans.**

31. If $\lim_{x\to3}\left(\frac{\sqrt{2x+3}-x}{\sqrt{x+1}-x+1}\right)^{\frac{x-1-\sqrt{x^2-5}}{x^2-5x+6}}$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c $\in$ N, then find the least value of $(a^2+b^2+c^2)$.

Ans. **29**

Sol. Rationalising the N^r and D^r of the base, we get

$$\text{Base} = \lim_{x\to3}\left(\frac{\sqrt{2x+3}-x}{\sqrt{x+1}-x+1}\right) = \lim_{x\to3}\left(\frac{\left(\sqrt{2x+3}-x\right)\left(\sqrt{2x+3}+x\right)\left(\sqrt{x+1}+(x-1)\right)}{\left(\sqrt{x+1}-(x-1)\right)\left(\sqrt{2x+3}+x\right)\left(\sqrt{x+1}+(x-1)\right)}\right)$$

$$=\lim_{x\to3}\left(\frac{\left(2x+3-x^2\right)\left(\sqrt{x+1}+(x-1)\right)}{\left(x+1-(x-1)^2\right)\left(\sqrt{2x+3}+x\right)}\right) = \lim_{x\to3}\frac{2}{3}\left(\frac{2x+3-x^2}{3x-x^2}\right)$$

$$=\frac{2}{3}\lim_{x\to3}\left(\frac{x^2-2x-3}{x(x-3)}\right) = \frac{2}{3}\lim_{x\to3}\left(\frac{(x-3)(x+1)}{x(x-3)}\right) = \frac{2}{3}\left(\frac{4}{3}\right) = \frac{8}{9}$$

Again rationalising the exponent, we get

$$\lim_{x\to3}\frac{x-1-\sqrt{x^2-5}}{x^2-5x+6} = \lim_{x\to3}\frac{\left(x-1-\sqrt{x^2-5}\right)\left(x-1+\sqrt{x^2-5}\right)}{\left(x^2-5x+6\right)\left(x-1+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to3}\frac{(x^2-2x+1-x^2+5)}{(x-2)(x-3)(x-1+\sqrt{x^2-5})} = \lim_{x\to3}\frac{-2(x-3)}{(x-2)(x-3)(x-1+\sqrt{x^2-5})}$$

$$=\lim_{x\to3}\frac{-2}{(x-2)(x-1+\sqrt{x^2-5})} = \frac{-2}{(1)(4)} = \frac{-1}{2}$$

$\therefore \quad L = \left(\frac{8}{9}\right)^{\frac{-1}{2}} = \frac{9^{\frac{1}{2}}}{8^{\frac{1}{2}}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4} \Rightarrow a = 3,\ b = 2, c = 4$

Hence $(a^2+b^2+c^2) = 9 + 4 + 16 = 29$ **Ans.**

32. Let $\lim_{x\to1}\frac{x^6+1+\frac{5}{3}\left(x^4+x^2\right)-\frac{8}{3}(x^5+x)}{x^5+x+6x^3-4x^4-4x^2} = \frac{p}{q}$ where p, q $\in$ N.
Find the smallest value of (p + q).

Ans. **13**

Sol. $L = \frac{1}{3}\lim_{x\to1}\frac{3x^6-8x^5+5x^4-8x+3}{x(x-1)^4}\left(\frac{0}{0}\right) = \frac{1}{3}\lim_{x\to1}\frac{(x-1)^4(3x^2+4x+3)}{x(x-1)^4}\left(\frac{0}{0}\right)$

$= \frac{1}{3}\lim_{x\to1}\frac{3x^2+4x+3}{x} = \frac{10}{3}$ **Ans.**

Note that if f (x) is a polynomial of degree 6 such that f (1) = f ' (1) = f '' (1) = f '''(1) = 0
then $f(x) = (x-1)^4(ax^2+bx+c)$

Alternatively : Using L'Hospital's rule 4 times, we get

$$L = \lim_{x\to 1} \frac{6x^5 + \frac{5}{3}\left(4x^3 + 2x\right) - \frac{8}{3}(5x^4 + 1)}{5x^4 + 1 + 18x^2 - 16x^3 - 8x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{30x^4 + \frac{5}{3}\left(12x^2 + 2\right) - \frac{8}{3}(20x^3 + 0)}{20x^3 + 36x - 48x^2 - 8} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{120x^3 + 40x - 160x^2}{60x^2 + 36 - 96x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{360x^2 + 40 - 320x}{120x - 96} = \frac{80}{24} = \frac{10}{3} = \frac{p}{q} \qquad \Rightarrow \boxed{p + q = 13} \qquad \textbf{Ans.}$$

Alternatively : $\displaystyle \lim_{x\to 1} \frac{x^6 + 1 + \frac{5}{3}\left(x^4 + x^2\right) - \frac{8}{3}(x^5 + x)}{x^5 + x + 6x^3 - 4x^4 - 4x^2}$

$$\frac{1}{3}\lim_{x\to 1} \frac{3\left(x^3 + \frac{1}{x^3}\right) + 5\left(x + \frac{1}{x}\right) - 8\left(x^2 + \frac{1}{x^2}\right)}{\left(\left(x^2 + \frac{1}{x^2}\right) + 6 - 4\left(x + \frac{1}{x}\right)\right)}$$

$$= \frac{1}{3}\lim_{t\to 2} \frac{3\left(t^3 - 3t\right) + 5t - 8\left(t^2 - 2\right)}{t^2 - 2 + 6 - 4t} \quad \left(\text{Put } x + \frac{1}{x} = t\right)$$

$$= \frac{1}{3}\lim_{t\to 2} \frac{3t^3 - 8t^2 - 4t + 16}{t^2 - 4t + 4} \left(\frac{0}{0}\right)$$

$$= \frac{1}{3}\lim_{t\to 2} \frac{(9t^2 - 16t - 4)}{2t - 4} \left(\frac{0}{0}\right)$$

$$= \frac{1}{3}\lim_{t\to 2} \frac{(18t - 16)}{2} = \frac{10}{3} = \frac{p}{q} \text{ (Given)}$$

$\Rightarrow \quad (p + q) = 10 + 3 = 13$]

Matrix Match Type :

33. Match the following

Column – I	**Column - II**
(A) $\displaystyle \lim_{x\to 0} \frac{\cos^2 x - 1}{x + \sin x \cos x}$	(P) 8
(B) $\displaystyle \lim_{x\to 0} \frac{\tan[-\pi^2]x^2 - [-\pi^2]x^2}{\sin^2 x}$ [.] denotes greatest integer function	(Q) 0
(C) $\displaystyle \lim_{x\to 0} \left[\frac{x^2}{\sin x \tan x}\right]$ where [] denotes G.I. function	(R) 4
(D) $\displaystyle \lim_{x\to 0} \frac{2\sin^2 x}{2 - \sqrt{2(1 + \cos x)}}$	(S) –1

Ans. A $\to$ Q, B $\to$ Q, C $\to$ Q, D $\to$ P

Sol. (A) $\lim_{x\to 0} -\frac{\sin^2 x}{x+\sin x\cos x}$

$\lim_{x\to 0} - \frac{\sin x}{\frac{x}{\sin x}+\cos x} = 0$

(B) $\lim_{x\to 0} \frac{\tan(-10)x^2 + 10x^2}{\sin^2 x}$

$\lim_{x\to 0} \frac{\tan(-10)x^2}{\sin^2 x} + \lim_{x\to 0} \frac{10x^2}{\sin^2 x}$

$\lim_{x\to 0} \frac{\tan(-10)x^2}{(-10x^2)} \times \frac{(-10)x^2}{(\sin^2 x)} + \lim_{x\to 0} \frac{10x^2}{\sin^2 x}$

$= -10 + 10 = 0$

(C) Let f(x) = x2 – sin x tan x

f '(x) = 2x – sin x sec2x – cosx tan x

f '(x) = 2x – sin x (sec2x + 1)

f ''(x) = 2 – cos x (sec2 x + 1) – sin x. 2 sec2x tan x

f ''(x) = 2 – (secx + tan x) – 2 secx tan2x

So in neighbourhood of 0, sec x + cos x > 2

f ''(x) < 0, f '(x) is decreasing f '(x) < f(0), x > 0

f '(x) < 0, f(x) is decreasing when x > 0

f(x) < f(0), x2 < sin x + tanx, x > 0

$0 < \frac{x^2}{\sin x \tan x} < 1 \quad = \lim_{x\to 0}\left[\frac{x^2}{\sin x \tan x}\right] = 0$

(D) $\lim_{x\to 0} \frac{(2\sin^2 x)\,(2+\sqrt{2(1+\cos)}x}{(4-2-2\cos x)}$

34. **Column-I** **Column-II**

	Column-I		Column-II
(A)	If $L = \lim_{x\to -1} \frac{\sqrt[3]{(7-x)}-2}{(x+1)}$, then 12 L =	(P)	– 2
(B)	If $L = \lim_{x\to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)}$, then $\frac{-L}{4}$ =	(Q)	2
(C)	If $L = \lim_{x\to 1} \frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$, then 20L =	(R)	1
(D)	If $L = \lim_{x\to \infty} \frac{\log x^n - [x]}{[x]}$, where $n \in N$, then – 2L =	(S)	–1

(Note : [x] denotes greatest integer less than or equal to x)

Ans. **(A→S; B→R; C→P; D→Q)**

Sol. (A) Let x + 1 = h

Then, $\lim_{x\to -1}\frac{\sqrt[3]{(7-x)}-2}{(x+1)} = \lim_{h\to 0}\frac{(8-h)^{1/3}-2}{h} = \lim_{h\to 0}\frac{2\left(1-\frac{h}{8}\right)^{1/3}-2}{h}$

$= 2\lim_{h\to 0}\frac{\left(1-\frac{1}{3}\frac{h}{8}\right)-1}{h} = -\frac{1}{12}$

(B) We have $\lim_{x\to \pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

$= \lim_{x\to\pi/4}\frac{\tan x(\tan x-1)(\tan x+1)}{\cos(x+\pi/4)} = \lim_{x\to\pi/4}\frac{\tan x(\sin x-\cos x)(\tan x+1)}{\cos x\cos(x+\pi/4)}$

$= \lim_{x\to\pi/4}\frac{\tan x(\cos x-\sin x)(\tan x+1)}{\cos x\cos(x+\pi/4)}$

$= -\sqrt{2}\lim_{x\to\pi/4}\frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)(\tan x+1)}{\cos x\cos(x+\pi/4)}$

$= -\sqrt{2}\lim_{x\to\pi/4}\frac{\tan x(\tan x+1)}{\cos x}$

(C) $\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3} = \lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$= \lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$

$= \lim_{x\to 1}\frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}$

$= \lim_{x\to 1}\frac{2-3}{(2+3)(\sqrt{1}+1)} = -\frac{1}{10}$

(D) $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]} = \lim_{x\to\infty}\frac{\log x^n}{[x]}-\lim_{x\to\infty}\frac{[x]}{[x]} = 0$

$= \lim_{x\to\infty}\frac{\log x^n}{[x]}-\lim_{x\to\infty}\frac{[x]}{[x]}$

$= 0 - 1 = -1$

35. **Column-I** **Column-II**

(A) If $\lim_{x \to \infty} \left(\sqrt{x^2 - x - 1} - ax - b\right) = 0$, where a > 0, then there exists at least one a and b for which point (a, 2b) lies on the line. — (P) $y = -3$

(B) If $\lim_{x \to 0} \frac{(1+a^3) + 8e^{1/x}}{1 + (1-b^3)e^{1/x}} = 2$, then there exists at least one a and b for which point (a, b^3) lies on the line. — (Q) $3x - 2y - 5 = 0$

(C) If $\lim_{x \to \infty} \left(\sqrt{(x^4 - x^2 + 1)} - ax^2 - b\right) = 0$, then there exists at least one a and b for which point (a, –4b) lies on the line. — (R) $15x - 2y - 11 = 0$

(D) If $\lim_{x \to -a} \frac{x^7 + a^7}{x + a} = 7$, where a < 0, ten there exists at least one a for which point (a, 2) lies on the line. — (S) $y = 2$

Ans. **(A→Q; B→P; C→R,S; D→S;)**

Sol. Here, a > 0, if a ≤ 0, if a ≤ 0, then limit = ∞

$$\therefore \quad \lim_{x \to \infty} \frac{\left(\sqrt{(x^2 - x + 1)} - ax - b\right)\left(\sqrt{(x^2 - x + 1)} + ax + b\right)}{\left(\sqrt{(x^2 - x + 1)} - ax - b\right)}$$

$$\Rightarrow \quad \lim_{x \to \infty} \frac{(x^2 - x + 1) - (ax + b)^2}{\sqrt{(x^2 - x + 1)} + ax + b} \quad \Rightarrow \lim_{x \to \infty} \frac{(1 - a^2)x^2 - (1 + 2ab)x + (1 - b^2)}{\sqrt{(x^2 - x + 1)} + ax + b}$$

This is possible only when $1 - a^2 = 0$ and $1 + 2ab = 0$

$\therefore \quad a = \pm 1$

$\Rightarrow \quad a = 1 \quad (\because a > 0)$ ....(1)

$\Rightarrow \quad b = -1/2$ $\Rightarrow (a, 2b) = (1, -1)$.

(B) Divide numerator and denominator by $e^{1/x}$, then

$$\lim_{x \to \infty} \frac{(1 + a^3)e^{1/x}}{e^{1/x} + (1 - b^3)} = 2 \quad \Rightarrow \quad \frac{0 + 8}{0 + 1 - b^3} = 2 \quad \Rightarrow \quad 1 - b^3 - 4$$

$\therefore \quad b^3 = -3 \Rightarrow b = -3^{1/3}$

Then, $a \in R$ $\Rightarrow$ $(a, b^3) = (a, -3)$

(C) $\lim_{x \to \infty} \left(\sqrt{(x^4 - x^2 + 1)} - ax^2 - b\right) = 0$

Put $x = \frac{1}{t}$

$$\therefore \quad \lim_{t \to 0} \left(\sqrt{\left(\frac{1}{t^4} - \frac{1}{t^2} + 1\right)} - \frac{a}{t^2} - b\right) = 0$$

$$\lim_{t \to 0} \frac{\sqrt{(1 - t^2 + t^4)} - a - bt^2}{t^2} = 0 \qquad ......(1)$$

Since R.H.S. is finite, numerator must be equal to 0 at t → 0.

$\therefore \quad 1 - a = 0 \qquad\qquad \therefore \quad a = 1$

From equation (1), $\lim\limits_{t \to 0} \dfrac{\sqrt{(1 - t^2 + t^4)} - 1 - bt^2}{t^2} = 0$

$$\lim_{t \to 0} \left(-1 + t^2\right)\left(\frac{1 - t^2 + t^4)^{1/2} - (1)^{1/2}}{(1 - t^2 + t^4) - 1}\right) = b$$

$\Rightarrow \quad (-1)\left(\dfrac{1}{2}\right) = b \Rightarrow b = -\dfrac{1}{2} \Rightarrow (a,\ 2b) = (1, 2)$

(D) $\quad \lim\limits_{x \to -a} \dfrac{a^7 - (-x)^7}{a - (-x)} = 7 \Rightarrow 7a^6 = 7 \Rightarrow a^6 = 1 \Rightarrow a = -1.$