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MATHEMATICS

TARGET : JEE- Advanced 2021

CAPS-10

Limit

ANSWER KEY OF CAPS-10

1.	(C)	**2.**	(C)	**3.**	(D)	**4.**	(C)	**5.**	(D)
6.	(C)	**7.**	(A)	**8.**	(D)	**9.**	(A)	**10.**	(B)
11.	(BC)	**12.**	(BC)	**13.**	(AB)	**14.**	(BC)	**15.**	(A)
16.	(B)	**17.**	(B)	**18.**	(B)	**19.**	(C)	**20.**	(D)
21.	(C)	**22.**	(A)	**23.**	(B)	**24.**	(D)	**25.**	(B)
26.	(0)	**27.**	(2)	**28.**	(**0**)	**29.**	(5)	**30.**	(16)
31.	(29)	**32.**	(13)	**33.**	$A \to Q$, $B \to Q$, $C \to Q$, $D \to Q$				
34.	**(A→S; B→R; C→P; D→Q)**			**35.**	(A→Q; B→P; C→R,S; D→S;				

SCQ (Single Correct Type) :

1. If $[x]$ is the greatest integer less than or equal to x, then $\lim\limits_{x\to 0}\left[\dfrac{x^2}{\sin x \tan x}\right] =$

(A) –1 (B) 2 (C) 0 (D) 1

Ans. (C)

Sol. $\dfrac{x^2}{\sin x \tan x} = \dfrac{x^2}{\left(x - \dfrac{x^3}{6}\right)\left(x + \dfrac{x^3}{3}\right)}$ (using expansion)

$= \dfrac{1}{1 + \dfrac{1}{6}x^2 + ...}$

$\Rightarrow \lim\limits_{x\to 0}\left[\dfrac{x^2}{\sin x \tan x}\right] = 0$

2. If $\lim\limits_{t\to x}\dfrac{e^t f(x) - e^x f(t)}{(t-x)(f(x))^2} = 2$ and $f(0) = \dfrac{1}{2}$, then find the value of f '(0).

(A) 4 (B) 2 (C) 0 (D) 1

Ans. (C)

Sol. Using L'Hospital's rule, we have

$$\lim_{t\to x}\frac{e^t f(x)-e^x f(t)}{(t-x)(f(x))^2}=2$$

$$\Rightarrow \frac{e^x f(x)-e^x f'(x)}{(f(x))^2}=2$$

$$\Rightarrow \frac{d}{dx}\left(\frac{e^x}{f(x)}\right)=2$$

$$\Rightarrow \frac{e^x}{f(x)}=2x+c$$

$$\Rightarrow f(x)=\frac{e^x}{2x+c}$$

$$\Rightarrow f(0)=\frac{1}{c}=\frac{1}{2} \quad \Rightarrow c=2$$

$$\therefore \ \Rightarrow f(x)=\frac{e^x}{2(1+x)}$$

3. The value of $\lim\limits_{x\to 0}\dfrac{\left(1+3x+2x^2\right)^{\frac{1}{x}}-\left(1+3x-2x^2\right)^{\frac{1}{x}}}{x}$ is ______.

(A) $-e^3$ (B) $3e^2$ (C) $\frac{1}{3}e^{-2}$ (D) $4e^3$

Ans. **(D)**

Sol. L = $\lim\limits_{x\to}\dfrac{\left(1+3x+2x^2\right)^{\frac{1}{x}}-\left(1+3x-2x^2\right)^{\frac{1}{x}}}{x}\left(\dfrac{0}{0}\right)$

$$\lim_{x\to 0}\frac{\left(1+3x+2x^2\right)^{\frac{1}{x}}-e^3}{x}-\lim_{x\to 0}\frac{\left(1+3x-2x^2\right)^{\frac{1}{x}}-e^3}{x}$$

$$=L_1-L_2$$

$$L_1=\lim_{x\to 0}\frac{\left(1+3x+2x^2\right)^{\frac{1}{x}}-e^3}{x}-\lim_{x\to 0}\frac{e^{\frac{\ln\left(1+3x+2x^2\right)}{x}}-e^3}{x}=-\frac{5}{2}e^3$$

similarly, $L_2=\dfrac{13}{2}e^3$

$\therefore$ L = $4e^3$

4. If $f(x)=\dfrac{\tan x}{x}$ then $\lim\limits_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}=$ (where [.] denotes the greatest integer function and {.} denotes fractional part)

(A) 3 (B) log3 (C) e^3 (D) Does not exist

Ans. **(C)**

Sol. As $x \to 0$, $[f(x)] = 1$ and $\{f(x)\} = 0$

so the given limit is in the form 1^∞

5. Let $f(x) = \left|\cos^{-1}(\sin x)\right| - \left|\sin^{-1}(\cos x)\right|$ and $\frac{3\pi}{2} \le x \le 2\pi$ $\lim\limits_{x\to 2\pi} \frac{\sin x - \tan x}{[f(x)]^3} =$ ______.

(A) $\frac{-1}{16}$ (B) $\frac{1}{32}$ (C) $\frac{-1}{8}$ (D) $\frac{1}{16}$

Ans. **(D)**

Sol. $\lim\limits_{x\to 2\pi} \frac{\sin x - \tan x}{(4\pi - 2x)^3} = \lim\limits_{x\to 0} \frac{\tan x - \sin x}{(2x)^3} = \frac{1}{16}$

6. The value of $\ell im_{x\to 1} \left(\frac{p}{1-x^p} - \frac{q}{1-x^q}\right)$; p, q $\in$ N equals

(A) $\frac{p+q}{2}$ (B) $\frac{pq}{2}$ (C) $\frac{p-q}{2}$ (D) $\sqrt{\frac{p}{q}}$

Ans. **(C)**

Sol. $\ell im_{x\to 1} \frac{p - q + qx^p - px^q}{1 - x^p - x^q + x^{p+q}} \left(\frac{0}{0} \text{ form}\right) = \ell im_{x\to 1} \frac{pqx^{p-1} - pqx^{q-1}}{-px^{p-1} - qx^{q-1} + (p+q)x^{p+q-1}} \left(\frac{0}{0} \text{ form}\right)$

$= \ell im_{x\to 1} \frac{pq(p-1)x^{p-2} - pq(q-1)x^{q-2}}{-p(p-1)x^{p-2} - q(q-1)x^{q-2} + (p+q)(p+q-1)x^{p+q-2}} = \frac{p-q}{2}$

7. $\ell im_{n\to\infty} \frac{1^2\, n + 2^2\,(n-1) + 3^2\,(n-2) + \ldots.. + n^2 . 1}{1^3 + 2^3 + 3^3 + \ldots\ldots + n^3}$ is equal to:

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{6}$

Ans. **(A)**

Sol. $\ell im_{n\to\infty} \frac{\sum_{r=1}^{n} r^2(n-r+1)}{\sum_{r=1}^{n} r^3}$

$= \ell im_{n\to\infty} \frac{\sum_{r=1}^{n}(n+1)r^2 - \sum_{r=1}^{n} r^3}{\sum_{r=1}^{n} r^3} = \ell im_{n\to\infty} \left(\frac{(n+1)\frac{(n)(n+1)(2n+1)}{6}}{\frac{n^2(n+1)^2}{4}} - 1\right)$

$= \frac{1/3}{1/4} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

8. The value of $\ell im_{x\to\frac{1}{\sqrt{2}}} \frac{\cos^{-1}(2x\sqrt{1-x^2})}{x - \frac{1}{\sqrt{2}}}$ equals

(A) $2\sqrt{2}$ (B) $-2\sqrt{2}$ (C) 0 (D) does not exist

Ans. **(D)**

Sol. Let q = $\sin^{-1} x$ $\therefore$ as $x \to \frac{1}{\sqrt{2}}$ $\therefore$ $\theta \to \frac{\pi}{4}$

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}(2\sin\theta\cos\theta)}{\sin\theta - \frac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}(\sin 2\theta)}{\sin\theta - \frac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}} \frac{\cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)}{\sin\theta - \frac{1}{\sqrt{2}}}$$

Left hand limit = $\lim_{\theta\to\frac{\pi}{4}^-} \frac{\cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)}{\sin\theta - \frac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}^-} \frac{\frac{\pi}{2} - 2\theta}{\sin\theta - \frac{1}{\sqrt{2}}} \left(\frac{0}{0}\text{ form}\right) = \lim_{\theta\to\frac{\pi}{4}^-} \frac{-2}{\cos\theta} = -2\sqrt{2}$

Right hand limit = $\lim_{\theta\to\frac{\pi}{4}^+} \frac{\cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)}{\sin\theta - \frac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}^+} \frac{\cos\left(\cos\left(2\theta - \frac{\pi}{2}\right)\right)}{\sin\theta - \frac{1}{\sqrt{2}}} = \lim_{\theta\to\frac{\pi}{4}^+} \frac{2\theta - \frac{\pi}{2}}{\sin\theta - \frac{1}{\sqrt{2}}}$

$= \lim_{\theta\to\frac{\pi}{4}^+} \frac{2}{\cos\theta} = 2\sqrt{2} \Rightarrow$ LHL $\neq$ RHL $\therefore$ Limit does not exist

9. The value of $\lim_{x\to\pi} \frac{1}{(x-\pi)}\left(\sqrt{\frac{4\cos^2 x}{2+\cos x}} - 2\right)$ is

(A) 0 (B) 2 (C) – 2 (D) does not exist

Ans. **(A)**

Sol. Let f(x) = $\sqrt{\frac{4\cos^2 x}{2+\cos x}}$ $\Rightarrow$ $f(\pi) = 2$

$\therefore \lim_{x\to\pi} \frac{1}{x-\pi}\left(\sqrt{\frac{4\cos^2 x}{2+\cos x}} - 2\right) = \lim_{x\to\pi} \frac{f(x) - f(\pi)}{x - \pi} = f'(\pi)$

Now $f'(x) = \frac{4}{2\sqrt{\frac{4\cos^2 x}{2+\cos x}}} \frac{(2+\cos x)(-2\cos x\sin x) - \cos^2 x(-\sin x)}{(2+\cos x)^2}$

$f'(p) = 0$

10. For $n \in N$, let x_n be defined as $\left(1+\frac{1}{n}\right)^{n+x_n}$ = e, then $\lim_{n\to\infty} x_n$ equals

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{e}$ (D) 0

Ans. **(B)**

Sol. We have $\left(1+\frac{1}{n}\right)^{n+x_n} = e$ ....(1)

Taking log on both sides of equation (1), we get

$\Rightarrow$ $(n + x_n)\, ln\left(1+\frac{1}{n}\right) = 1 \Rightarrow$ $n + x_n = \frac{1}{\ln\left(1+\frac{1}{n}\right)}$ $\Rightarrow$ $x_n = \frac{1}{\ln\left(1+\frac{1}{n}\right)} - n$ ....(1)

Let $\dfrac{n+1}{n} = u \quad \Rightarrow \quad nu = n + 1 \quad \Rightarrow \quad n = \dfrac{1}{u-1}$

$$\therefore \quad \underset{n\to\infty}{\text{Lim}}\ x_n = \underset{u\to1}{\text{Lim}}\left(\frac{1}{\ln u} - \frac{1}{u-1}\right) = \underset{u\to1}{\text{Lim}}\ \frac{(u-1)-\ln u}{(u-1)\ln u}\left(\frac{0}{0}\right) \text{ form}$$

$$= \underset{u\to1}{\text{Lim}}\ \frac{1-\frac{1}{u}}{\frac{u-1}{u}+\ln u} = \underset{u\to1}{\text{Lim}}\ \frac{\frac{1}{u^2}}{\frac{1}{u^2}+\frac{1}{u}} = \frac{1}{2} \textbf{ Ans. }]$$

MCQ (One or more than one correct) :

11. If $\lim\limits_{x\to1}\left(2-x+a[x-1]+b[1+x]\right)$ exists, then a and b can take the values _____, (where [ ] denotes the greatest integer function).

(A) $a=\frac{1}{3},\ b=1$ (B) $a=1,\ b=-1$ (C) $a=9,\ b=-9$ (D) $a=9,\ b=\frac{2}{3}$

Ans. **(B,C)**

Sol. Since the greatest integer function [x] is discontinuous (sensitive) at integral values of x, then for a given limit to exist both left and right hand limits must be equal.

$$\text{LHL} = \lim_{x\to1^-}\left(2-x+a[x-1]+b[1+x]\right) = 2-1+a(-1)+b(1) = 1-a+b$$

$$\text{RHL} = \lim_{x\to1^-}\left(2-x+a[x-1]+b[1+x]\right) = 2-1+a(0)+b(2) = 1+2b$$

On comparing we have –a = b

12. If $A = \lim\limits_{x\to0}\dfrac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$ and $B = \lim\limits_{x\to0}\dfrac{[|x|]}{x}$, where [ ] represents GIF, then _____.

(A) A = 1 (B) A does not exist (C) B = 0 (D) B = 1

Ans. **(B,C)**

Sol. $A = \lim\limits_{x\to0}\dfrac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)}$

$$\lim_{x\to0^-}\frac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)} = \lim_{x\to0^-}\frac{x}{-x} = -1$$

$$\lim_{x\to0^+}\frac{\sin^{-1}(\sin x)}{\cos^{-1}(\cos x)} = \lim_{x\to0^+}\frac{x}{x} = 1 \qquad \Rightarrow \text{A does not exist}$$

$$B = \lim_{x\to0}\frac{[|x|]}{x} = 0 \text{ as } [|x|] = 0 \text{ when } x\to0$$

13. $\lim\limits_{x\to\infty}\left[mx\sin\dfrac{1}{x}\right]$ is equal to (where [.] denotes the greatest integer function and $m\in I$

(A) m if m < 0 (B) m – 1 if m > 0 (C) m – 1 if m < 0 (D) m if m > 0

Ans. **(AB)**

Sol. $\lim\limits_{x\to\infty}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}} = 1^-$

14. If f : R $\rightarrow$ R be a differentiable function and f(0) = 0 and f'(0) = 1, then

$$\lim_{x\to 0} \frac{\left[f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\ldots+f\left(\frac{x}{n}\right)\right]}{x}$$, where n $\in$ N, equals

(A) 0

(B) $1 + \frac{1}{2} + \frac{1}{3} + \ldots. + \frac{1}{n}$

(C) ${}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - \ldots\ldots\ldots + (-1)^{n-1}\frac{{}^nC_n}{n}$

(D) does not exist

Ans. **(BC)**

Sol. $$\lim_{x\to 0} \frac{f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\ldots+f\left(\frac{x}{k}\right)}{x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\left[f'(x)+\frac{1}{2}f'\left(\frac{x}{2}\right)+\frac{1}{3}f'\left(\frac{x}{3}\right)+\ldots+\frac{1}{k}f'\left(\frac{x}{k}\right)\right] = 1+\frac{1}{2}+\frac{1}{3}+\ldots.+\frac{1}{n}=\int_0^1 (1+x+x^2+\ldots+x^{n-1})\,dx$$

$$=\int_0^1 \frac{1-x^n}{1-x}dx = \int_0^1 \frac{1-(1-x)^n}{x}dx = {}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - \ldots\ldots + (-1)n-1 \,.\, \frac{{}^nC_n}{n}$$

Comprehension Type Question:

Comprehension # 1

Let $f(x) = \dfrac{\sin^{-1}(1-\{x\})\times\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\times(1-\{x\})}$ where $\{x\}$ denotes the fractional part of x.

15. L = $\lim\limits_{x\to 0^+} f(x)$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{2\sqrt{2}}$ (C) $\frac{\pi}{\sqrt{2}}$ (D) $\sqrt{2}\pi$

Ans. **(A)**

16. R = $\lim\limits_{x\to 0^-} f(x)$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{2\sqrt{2}}$ (C) $\frac{\pi}{\sqrt{2}}$ (D) $\sqrt{2}\pi$

Ans. **(B)**

17. Which of the following is not true ?

(A) cos L < cos R (B) tan (2L) > tan R (C) sin L > sin R (D) None of these

Ans. **(B)**

Sol. **(15 to 17)**

We have $f(x) = \dfrac{\sin^{-1}(1-\{x\})\times\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\times(1-\{x\})}$

$$\therefore \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sin^{-1}(1-\{0+h\})\cos^{-1}(1-\{0+h\})}{\sqrt{2\{0+h\}}(1-\{0+h\})}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-h)\cos^{-1}(1-h)}{\sqrt{2h}(1-h)} \quad = \lim_{h\to 0}\frac{\sin^{-1}(1-h)}{(1-h)}\lim_{h\to 0}\frac{\cos^{-1}(1-h)}{\sqrt{2h}}$$

In second limit put $1 - h = \cos\theta$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-h)}{(1-h)}\lim_{\theta\to 0}\frac{\cos^{-1}(\cos\theta)}{\sqrt{2(1-\cos\theta)}} \quad = \lim_{h\to 0}\frac{\sin^{-1}(1-h)}{(1-h)}\lim_{\theta\to 0}\frac{\theta}{2\sin(\theta/2)} \qquad (\because \theta > 0)$$

$= \sin^{-1} 1 \times 1 = \pi/2$ and $\lim_{x\to 0^-} f(x) \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-\{0-h\})\cos^{-1}(1-\{0-h\})}{\sqrt{2\{0-h\}}\,(1-\{0-h\})} \quad = \lim_{h\to 0}\frac{\sin^{-1}(1-h-1)\cos^{-1}(1+h-1)}{\sqrt{2(-h+1)}\,(1+h-1)}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}h}{h}\lim_{h\to 0}\frac{\cos^{-1}h}{\sqrt{2(1-h)}} = 1\frac{\pi/2}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

Comprehension # 2

Let $f(x) = \lim\limits_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n$, $g(x) = \lim\limits_{n\to\infty}\left(1-x+x\sqrt[n]{e}\right)^n$. Now, consider the function y = h(x), where

$h(x) = \tan^{-1}\left(g^{-1}f^{-1}(x)\right)$

18. $\lim\limits_{n\to 0^+}\frac{\ln(f(x))}{\ln(g(x))}$ is equal to ______.

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 0 (D) 1

Ans. **(B)**

Sol. $\lim\limits_{x\to 0^+}\frac{\ln f(x)}{\ln g(x)} = \lim\limits_{x\to 0}\frac{\frac{-x}{2}}{x} = -\frac{1}{2}$

19. Domain of the function y = h(x) is ______.

(A) $(0,\infty)$ (B) R (C) (0, 1) (D) [0, 1]

Ans. **(C)**

Sol. Domain h(x) is (0,1)

20. Range of the function y = h(x) is ______.

(A) $\left(0,\frac{\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2},0\right)$ (C) R (D) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Ans. **(D)**

Sol. $h(x) = \tan^{-1}\left(\ln\left(\frac{\ln\ell}{x^2}\right)\right) x < x < 1$

$1 < \frac{1}{x^2} < \infty \Rightarrow 0 < \ln\frac{1}{x^2} < \infty \qquad \therefore -\infty < \ln\left(\ln\left(\frac{1}{x^2}\right)\right) < \infty$

$\therefore$ Range of f (x) is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Comprehension # 3

Column-I	Column-II
(I) If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$, then	(i) $a=2$ (P) $b\in R$
(II) $\lim\limits_{x\to 0}\left(1+ax+bx^2\right)^{\frac{2}{x}}=e^3$, then	(ii) $a=\dfrac{3}{2}$ (Q) $b=2$
(III) $\lim\limits_{x\to 0}\left(\dfrac{ae^x-b}{x}\right)=2$	(iii) $a=1$ (R) $b=-1$
(IV) $\lim\limits_{x\to-\infty}\left\{\sqrt{\left(x^2-x+1\right)}-ax-b\right\}=0$	(iv) $a=-1$ (S) $b=\dfrac{1}{2}$

21. Which of the following is correct?

(A) (III) – (ii) – (R) (B) (III) – (iv) – (R) (C) (III) – (i) – (Q) (D) (III) – (ii) – (S)

Ans. **(C)**

Sol. $\lim\limits_{z\to 0}\left(\dfrac{ae^x-b}{x}\right)=2$

$\Rightarrow \lim\limits_{x\to 0}\left(\dfrac{ae^x-a+a-b}{x}\right)=2 \quad \Rightarrow a=b \text{ and } \lim\limits_{x\to 0}\left(\dfrac{ae^x-a}{x}\right)=2 \quad \Rightarrow a=2=b$

22. Which of the following is correct?

(A) (II) – (ii) – (P) (B) (II) – (ii) – (R) (C) (II) – (iii) – (Q) (D) (II) – (iv) – (S)

Ans. **(A)**

Sol. Limit $L=\lim\limits_{x\to 0}\left(1+ax+bx^2\right)^{\frac{2}{x}}=e^3$

$\Rightarrow e^{\lim\limits_{x\to 0}\left(1+ax+bx^2-1\right)^{\frac{2}{x}}}=e^3 \Rightarrow \lim\limits_{x\to 0}\left(2a+2bx\right)=3 \quad \Rightarrow \quad a=\dfrac{3}{2}$ and b can be anything

23. Which of the following is correct?

(A) (I) – (ii) – (R) (B) (I) – (iii) – (R) (C) (I) – (iii) – (S) (D) (I) – (iv) – (P)

Ans. **(B)**

Sol. Limit $L=\lim\limits_{x\to 0}\dfrac{x^2+1-(x+1)(ax+b)}{x+1}=0$

Means power of x in denominator is greater than that in the numerator Or coefficient of x^2 and x in the numerator will be zero

$\Rightarrow 1-a=0 \quad \Rightarrow a=1$ and $a+b=0 \quad \Rightarrow b=-1=-1$