Que Find the remainder when $(3053)^{956} - (2417)^{333}$ is divided by 9.

Soln

$$(3053)^{956} - (2417)^{333}$$

$$= (9K+2)^{956} - (9K_1 - 4)^{333}$$

$$= 9 \cdot I_1 + 2^{956} - (9I_1 - 4^{333})$$

$$= 9I_3 + 2^{956} + 2^{666}$$

$$= 9I_3 + (2^3)^{318} \cdot 2^2 + (2^3)^{222}$$

$$= 9I_3 + 4(9-1)^{318} + (9-1)^{222}$$

$$= 9I_1 + 4(9K_1 + 1) + (9K+1) = 9I' + 5$$ Ans

$8 = 9 - 1$

$= 2^3$

$\boxed{\text{Que}}$ P.T. $\dfrac{1}{{}^{2n-1}C_r} = \dfrac{2n}{2n+1}\left(\dfrac{1}{{}^{2n}C_r} + \dfrac{1}{{}^{2n}C_{r+1}}\right)$ and hence or otherwise

find $\displaystyle\sum_{r=1}^{19} (-1)^{r-1} \cdot \frac{r^2}{{}^{19}C_{r-1}}$.

Sol T.P. $\dfrac{1}{{}^{2n-1}C_r} = \dfrac{2n}{(2n+1)}\left(\dfrac{1}{{}^{2n}C_r} + \dfrac{1}{{}^{2n}C_{r+1}}\right)$

$\Rightarrow$ R.H.S. $= \dfrac{2n}{2n+1}\left(\dfrac{r!\,2n-r!}{2n!} + \dfrac{(r+1)!\,(2n-r-1)!}{2n!}\right)$

$= \dfrac{(2n)\,r!\,(2n-r-1)!}{\cancel{(2n+1)}\,2n!}\left(\cancel{\dfrac{(2n-r)+(r+1)}{}}\right)$

$= \dfrac{r!\,(2n-r-1)!}{(2n-1)!} = \dfrac{1}{{}^{2n-1}C_r}$

$=$ L.H.S.

H.P.

$\displaystyle\sum_{r=1}^{19} (-1)^{r-1} \cdot \frac{r^2}{{}^{19}C_{r-1}} = \sum_{r=1}^{19} (-1)^{r-1} \cdot \frac{20 \cdot r}{{}^{20}C_r}$

$\displaystyle = 20 \sum_{r=1}^{19} (-1)^{r-1} \frac{r+1-1}{{}^{20}C_r}$

$\dfrac{{}^{n}C_r}{r+1} = \dfrac{{}^{n+1}C_{r+1}}{n+1}$

$\displaystyle = 20 \cdot 21 \sum_{r=1}^{19} (-1)^{r-1} \cdot \frac{1}{{}^{21}C_{r+1}} - 20 \sum_{r=1}^{19} (-1)^{r-1} \cdot \frac{1}{{}^{20}C_r}$

$$R\cdot Sum = 20\cdot 21 \sum_{r=1}^{19} \frac{(-1)^{r-1}}{^{21}C_{r+1}} - 20\cdot \sum_{r=1}^{19} (-1)^{r-1} \cdot \frac{1}{^{20}C_r}$$

$$= 20\cdot 21 \sum_{r=1}^{19} (-1)^{r-1} \left(\frac{22}{23}\left(\frac{1}{^{22}C_{r+1}} + \frac{1}{^{22}C_{r+2}}\right)\right) - 20 \sum_{r=1}^{19} (-1)^{r-1} \left(\frac{21}{22}\left(\frac{1}{^{21}C_r} + \frac{1}{^{21}C_{r+1}}\right)\right)$$

$$= \frac{20\cdot 21\cdot 22}{23}\left(\left(\underline{\frac{1}{^{22}C_2}} + \cancel{\frac{1}{^{22}C_3}}\right) - \left(\cancel{\frac{1}{^{22}C_3}} + \cancel{\frac{1}{^{22}C_4}}\right) + \left(\cancel{\frac{1}{^{22}C_4}} + \cancel{\frac{1}{^{22}C_5}}\right) \cancel{+} - \cdots \cancel{+} \left(\cancel{\frac{1}{^{22}C_{20}}} + \underline{\frac{1}{^{22}C_{21}}}\right)\right)$$

$$- \frac{20\cdot 21}{22}\left(\frac{1}{^{21}C_1} + \frac{1}{^{21}C_2} - \frac{1}{^{21}C_2} - \frac{1}{^{21}C_3} + \frac{1}{^{21}C_3} + \frac{1}{^{21}C_4} - \cdots + \frac{1}{^{21}C_{19}} + \frac{1}{^{21}C_{20}}\right)$$

Formula

$$\frac{1}{^{2n-1}C_r} = \frac{2n}{2n+1}\left(\frac{1}{^{2n}C_r} + \frac{1}{^{2n}C_{r+1}}\right)$$

$$\frac{1}{^{n}C_r} + \frac{1}{^{n}C_{r+1}} = \frac{r!\,(n-r)!}{n!} + \frac{(r+1)!\,(n-r-1)!}{n!}$$

$$= \frac{r!\,(n-r-1)!}{n!}\left(n - \cancel{r} + \cancel{r} + 1\right)$$

$$= \frac{(n+1)}{n}\left(\frac{r!\,(n-1-r)!}{(n-1)!}\right)$$

$$\frac{1}{^{n}C_r} + \frac{1}{^{n}C_{r+1}} = \frac{n+1}{n}\cdot\left(\frac{1}{^{n-1}C_r}\right)$$

$$\frac{1}{^{n-1}C_r} = \frac{n}{n+1}\left(\frac{1}{^{n}C_r} + \frac{1}{^{n}C_{r+1}}\right)$$

$$R.\ Sum = \frac{20\cdot 21\cdot 22}{23}\left(\frac{1}{^{22}C_2} + \frac{1}{^{22}C_{21}}\right) - \frac{20\cdot 21}{22}\left(\frac{1}{^{21}C_1} + \frac{1}{^{21}C_{20}}\right)$$

$$= \frac{20\cdot 21\cdot \cancel{22}}{23}\left(\frac{1}{\cancel{22}\times 21} + \frac{1}{\cancel{22}}\right) - \frac{20\cdot \cancel{21}}{\cancel{22}_{11}}\left(\frac{\cancel{2}}{\cancel{21}}\right)$$

$$= \frac{20\times \cancel{21}}{23}\left(\frac{22}{\cancel{21}}\right) - \frac{20}{11}.$$

$$= 20\left(\frac{22}{23} - \frac{1}{11}\right).$$

$\boxed{\text{Que}}$ Find $\displaystyle\sum_{r=1}^{30} \frac{2r^2-29r+1}{(31-r)\cdot {}^{31}C_r}$.

Sol

$$\sum_{r=1}^{30} \frac{2r^2-29r+1}{(31-r)\,{}^{31}C_r} = \sum_{r=1}^{30} \frac{r^2+2r+1-(31-r)\,r}{(31-r)\cdot{}^{31}C_r}$$

$$= \sum_{r=1}^{30}\left(\frac{(r+1)^2}{(31-r)\cdot{}^{31}C_r} - \frac{r}{{}^{31}C_r}\right)$$

$$= \sum_{r=1}^{30}\left(\underbrace{\frac{r+1}{{}^{31}C_{r+1}}}_{T_{r+1}} - \underbrace{\frac{r}{{}^{31}C_r}}_{T_r}\right)$$

$$= \frac{31}{{}^{31}C_{31}} - \frac{1}{{}^{31}C_1}$$

$$= \boxed{31-\frac{1}{31}}$$ Ans

$$\frac{(r+1)^2}{(31-r)\cdot{}^{31}C_r}$$

$$\frac{(r+1)^2}{(31-r)\cdot{}^{31}C_{31-r}}$$

$$= \frac{(r+1)^2}{31\cdot{}^{30}C_{30-r}} = \frac{(r+1)^2}{31\cdot{}^{30}C_r}$$

$$= \frac{(r+1)\cdot\cancel{31}}{\cancel{31}\;{}^{31}C_{r+1}}$$

$$\frac{{}^{n}C_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$$

$$r\cdot{}^{n}C_r = n\cdot{}^{n-1}C_{r-1}$$

Que Evaluate $\sum_{s=0}^{n-r} {}^{r+s}C_s \cdot {}^{n}C_{r+s}$

Sol

$$\sum_{s=0}^{n-r} {}^{r+s}C_s \cdot {}^{n}C_{r+s}$$

$$= \sum_{s=0}^{n-r} \frac{\cancel{\lfloor r+s}}{\lfloor s \, \lfloor r} \cdot \frac{\lfloor n}{\cancel{\lfloor r+s} \, \lfloor n-r-s}$$

$$= \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \sum_{s=0}^{n-r} \frac{\lfloor n-r}{\lfloor s \, \lfloor n-r-s} = {}^{n}C_r \cdot \sum_{s=0}^{n-r} {}^{n-r}C_s$$

$$= {}^{n}C_r \cdot 2^{n-r} = 2^{n-r} \cdot {}^{n}C_r.$$

A

$\boxed{\text{Que}}$ Find $\frac{{}^nC_0}{1^2} - \frac{{}^nC_1}{2^2} + \frac{{}^nC_2}{3^2} - \cdots + (-1)^n \cdot \frac{{}^nC_n}{(n+1)^2}$

Soln $R.Sum = \sum_{r=0}^{n} (-1)^r \cdot \frac{{}^nC_r}{(r+1)^2} = \frac{1}{n+1}\sum_{r=0}^{n} (-1)^r \cdot \frac{{}^{n+1}C_{r+1}}{(r+1)}$

Consider

$(1-x)^{n+1} = {}^{n+1}C_0 - \underbrace{{}^{n+1}C_1 \cdot x + {}^{n+1}C_2 \cdot x^2 - \cdots}$

${}^{n+1}C_1 \cdot x - {}^{n+1}C_2 \cdot x^2 + {}^{n+1}C_3 \cdot x^3 - \cdots = 1-(1-x)^{n+1}$

$\div x$

${}^{n+1}C_1 - {}^{n+1}C_2 \cdot x + {}^{n+1}C_3 x^2 - \cdots = \frac{1-(1-x)^{n+1}}{x}$

$\int_0^1$

${}^{n+1}C_1 - \frac{{}^{n+1}C_2}{2} + \frac{{}^{n+1}C_3}{3} - \cdots = \int_0^1 \frac{1-(1-x)^{n+1}}{x} dx = \int_0^1 \frac{1-x^{n+1}}{1-x} \cdot dx = \int_0^1 (1+x+x^2+\cdots+x^n)\, dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}$

$R.Sum = \frac{1}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)$

Que Find ${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{2n-1}C_n + {}^nC_2 \cdot {}^{2n-2}C_n - \dots + (-1)^n \cdot {}^nC_n \cdot {}^nC_n$

Sol

$${}^nC_0 \cdot {}^{2n}C_{\underline{n}} - {}^nC_1 \cdot {}^{2n-1}C_{\underline{n}} + {}^nC_2 \cdot {}^{2n-2}C_{\underline{n}} - \dots$$

$$= \text{coff of } x^n \text{ in } \left({}^nC_0(1+x)^{2n} - {}^nC_1 (1+x)^{2n-1} + {}^nC_2(1+x)^{2n-2} - \dots \pm (-1)^n \cdot {}^nC_n (1+x)^n\right)$$

$$= \text{coff of } x^n \text{ in } \left((1+x)^n \left({}^nC_0 (1+x)^n - {}^nC_1 (1+x)^{n-1} + {}^nC_2(1+x)^{n-2} - \dots\right)\right)$$

$$= \text{coff of } x^n \text{ in } \left((1+x)^n (\cancel{1}+x-\cancel{1})^n\right) = \text{coff of } x^n \text{ in } \left(x^n (1+x)^n\right) = {}^nC_0$$

$$= \boxed{1} \text{ Ans}$$

$$7 \cdot {}^nC_r = \text{coff of } x^r \text{ in } 7(1+x)^n$$

$$-5 \cdot {}^{n+1}C_r = \text{“ “ “ } -5(1+x)^{n+1}$$

$$7 \cdot {}^nC_r - 5 \cdot {}^{n+1}C_r = \text{coff of } x^r \text{ in } \left(7(1+x)^n - 5(1+x)^{n+1}\right)$$

Que In ΔABC with side length as $a=5$, $b=6$ and $c=7$ find

$$\sum_{r=0}^{n} {}^nC_r \cdot a^r \cdot b^{n-r} \cdot \cos(rB-(n-r)A), \quad \text{if } n=10.$$

$$(c(\theta)+is(\theta))^n = c(n\theta)+is(n\theta)$$

$$\left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta)+i\sin(n\theta)$$

Soln Let $S_1 = \sum_{r=0}^{n} {}^nC_r \cdot a^r \cdot b^{n-r} \cdot \cos(rB-(n-r)A)$. (say)

$$S_2 = \sum_{r=0}^{n} {}^nC_r \cdot a^r \cdot b^{n-r} \cdot \sin(rB-(n-r)A)$$

Now consider

$$S_1+iS_2 = \sum_{r=0}^{n} {}^nC_r \cdot a^r \cdot b^{n-r}\left(e^{i(rB-(n-r)A)}\right)$$

$$= \sum_{r=0}^{n} {}^nC_r \cdot \left(a \cdot e^{iB}\right)^r \cdot \left(b \cdot e^{-iA}\right)^{n-r} = \left(ae^{iB}+be^{-iA}\right)^n$$

$$S_1 + iS_2 = \left(a e^{iB} + b e^{-iA}\right)^n$$

$$= \left(a\left(\cos(B) + i\sin(B)\right) + b\left(\cos(A) - i\sin(A)\right)\right)^n$$

$$= \left(\left(\underbrace{a\cos(B) + b\cos(A)}_{= c \text{ (using Projection formula)}}\right) + i\left(\underbrace{a\sin(B) - b\sin(A)}_{= 0 \text{ (using sine-Rule)}}\right)\right)^n$$

$$= c^n \Rightarrow S_1 = c^n = \boxed{7^{10}} \quad \text{Ans}$$

$$\& \quad S_2 = 0.$$

Que Find $^{2021}C_1 - \left(1+\frac{1}{2}\right)\cdot {}^{2021}C_2 + \left(1+\frac{1}{2}+\frac{1}{3}\right){}^{2021}C_3 + \ldots + (-1)^{n-1}\left(1+\frac{1}{2}+\ldots+\frac{1}{2021}\right)\cdot {}^{2021}C_{2021}$

Sol R. Sum $= \left(\int_0^1 1\cdot dx\right){}^{2021}C_1 - \left(\int_0^1 (1+x)\,dx\right){}^{2021}C_2 + \left(\int_0^1 (1+x+x^2)\,dx\right){}^{2021}C_3 - \ldots$

$$= \int_0^1 \left(1\cdot {}^{2021}C_1 - (1+x)\cdot {}^{2021}C_2 + (1+x+x^2)\cdot {}^{2021}C_3 - \ldots\right)dx$$

$$= \int_0^1 \left(\frac{(1-x)}{1-x}\cdot {}^{2021}C_1 - \frac{1-x^2}{1-x}\cdot {}^{2021}C_2 + \frac{1-x^3}{1-x}\cdot {}^{2021}C_3 - \ldots\right)dx$$

$$= \int_0^1 \left(\frac{\left({}^{2021}C_1 - {}^{2021}C_2 + {}^{2021}C_3 - \ldots\right) - \left(x\cdot {}^{2021}C_1 - x^2\, {}^{2021}C_2 + x^3\cdot {}^{2021}C_3 - \ldots\right)}{1-x}\right)dx$$

Using $^{2021}C_0 - {}^{2021}C_1 + {}^{2021}C_2 - \dots = 0$

$1 = {}^{2021}C_1 - {}^{2021}C_2 - \dots$

$$R.S. = \int_0^1 \frac{\left(1 - {}^{2021}C_1 \cdot x + {}^{2021}C_2 \cdot x^2 - \dots\right)}{1-x}\,dx$$

$$= \int_0^1 \frac{(1-x)^{2021}}{1-x}\cdot dx = \int_0^1 (1-x)^{2020}\cdot dx = \int_0^1 x^{2020}\,dx = \left.\frac{x^{2021}}{2021}\right]_0^1 = \boxed{\frac{1}{2021}}$$

Que Find $\sum_{p=1}^{n}\left(\sum_{m=p}^{n} {}^{n}C_m \cdot {}^{m}C_p\right)$.

Sol

$$\sum_{p=1}^{n}\left(\sum_{m=p}^{n} {}^{n}C_m \cdot {}^{m}C_p\right)$$

$$\sum_{p=1}^{n}\left(\sum_{m=p}^{n} \frac{\lfloor n}{\cancel{\lfloor m}\,\lfloor n-m} \cdot \frac{\cancel{\lfloor m}}{\lfloor p\,\lfloor m-p}\right)$$

$$= \sum_{p=1}^{n} \frac{\lfloor n}{\lfloor p\,\lfloor n-p}\left(\sum_{m=p}^{n} \frac{\lfloor n-p}{\lfloor n-m\,\lfloor m-p}\right) = \sum_{p=1}^{n} {}^{n}C_p\left(\sum_{m=p}^{n} {}^{n-p}C_{m-p}\right)$$

$$\sum_{p=1}^{n} {}^{n}C_p\left(2^{n-p}\right)$$

$$(1+2)^n - {}^{n}C_0 \cdot 2^n = 3^n - 2^n$$

Ans

Que If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ then find the value of the expression

$$\sum\sum_{0\le i<j\le n} (i+j)\cdot(C_i + C_j + C_iC_j).$$

Sol

$$S_1 = \sum\sum_{0\le i<j\le n} (i+j)\left({}^nC_i + {}^nC_j + {}^nC_i\cdot{}^nC_j\right)$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(i+j)\left({}^nC_i + {}^nC_j + {}^nC_i\cdot{}^nC_j\right) = S_{i=j} + S_{i>j} + S_{i<j}$$

$S_{i<j}$ → R.Sum $(= S_1)$

$S_{i>j}$ → S_1 (as L.H.S. is symmetric w.r.t i & j).

$S_{i=j}$ → $\displaystyle\sum_{i=0}^{n} 2i\left(2\cdot{}^nC_i + {}^nC_i^2\right)$ → $4\cdot n\cdot 2^{n-1} + 2\cdot n\cdot{}^{2n-1}C_{n-1}$.

$$= \sum_{i=0}^{n} 2i\left(2\cdot{}^nC_i + {}^nC_i^2\right) + 2\cdot S_1$$

$$= \sum_{i=0}^{n}\left((n+1)\,i\cdot{}^nC_i + i\cdot 2^n + i\cdot{}^nC_i\,2^n + {}^nC_i\,\frac{n(n+1)}{2} + n\cdot 2^{n-1} + {}^nC_i\,n\cdot 2^{n-1}\right)$$

$$= (n+1)\cdot n\cdot 2^{n-1} + 2^n\cdot\frac{n(n+1)}{2} + 2^n\cdot n2^{n-1} + \frac{n(n+1)}{2}\cdot 2^n + n(n+1)\cdot 2^{n-1} + 2^n\cdot n\cdot 2^{n-1}.$$

formula

1. $$\sum_{r=0}^{n} r \cdot {}^{n}C_r = n \sum {}^{n-1}C_{r-1}$$
$$= n \cdot 2^{n-1}$$

$$\sum i \, {}^{n}C_i^2 = \sum \underline{i \, {}^{n}C_i} \cdot {}^{n}C_i$$
$$= n \sum {}^{n-1}C_{i-1} \, {}^{n}C_{n-i}$$
$$= n \cdot {}^{2n-1}C_{n-1}$$

$$\sum {}^{n}C_r^2 = {}^{2n}C_n \cdot$$

$${}^{n}C_r \cdot {}^{n}C_{n-r}$$