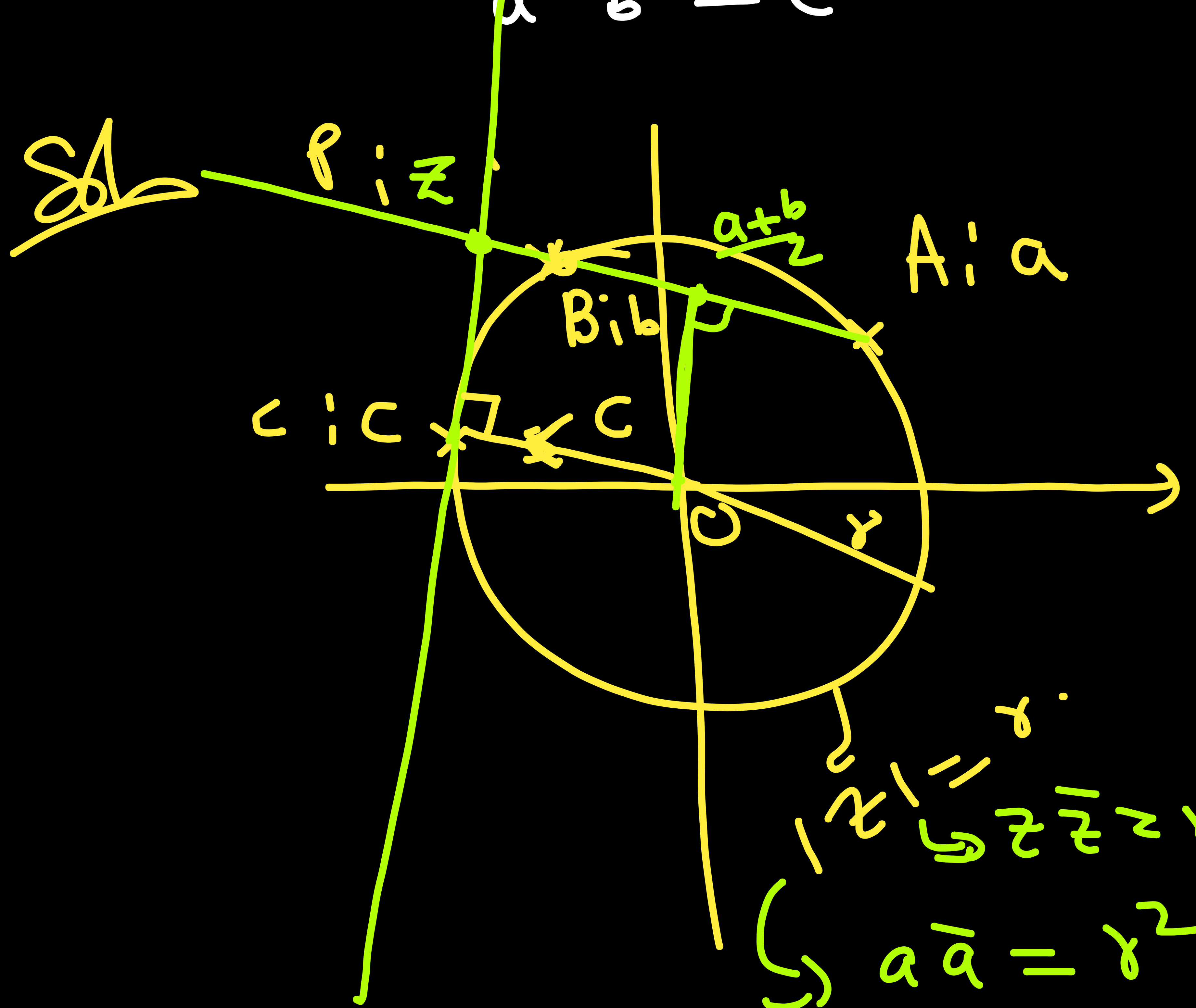
Que) Let 3-points denoted by a, b, c on complex plane lie on a circle with centre origin and radius r, If the tangent at 'c' cuts the chord joining a, b at z then find 'λ' if

$$z = \frac{a^{-1}+b^{-1}-\lambda c^{-1}}{a^{-1}b^{-1}-c^{-2}}$$

Sol



$|z| = r$

$\Rightarrow z\bar{z} = r^2$

$\Rightarrow a\bar{a} = r^2$

$\bar{a} = \frac{r^2}{a}$

$|||^{ly}$ $\bar{b} = \frac{r^2}{b}$ & $\bar{c} = \frac{r^2}{c}$.

$PC \perp OC \Rightarrow \frac{z-c}{c} + \frac{\bar{z}-\bar{c}}{\bar{c}} = 0.$

$AP \parallel AB \Rightarrow \frac{z-a}{b-a} - \frac{\bar{z}-\bar{a}}{\bar{b}-\bar{a}} = 0.$

$$\left(\frac{\bar{b}-\bar{a}}{b-a}\right)(z-a) - \bar{z} + \bar{a} = 0.$$

$$+\quad \left(\frac{\bar{c}}{c}\right)(z-c) + \bar{z} - \bar{c} = 0.$$

$$z\left(\frac{\bar{b}-\bar{a}}{b-a} + \frac{\bar{c}}{c}\right) - a\left(\frac{\bar{b}-\bar{a}}{b-a}\right) - \bar{c} + \bar{a} - \bar{c} = 0.$$

$$z\left(\frac{\bar b-\bar a}{b-a}+\frac{\bar c}{c}\right)-a\left(\frac{\bar b-\bar a}{b-a}\right)+\bar a-2\bar c=0.$$

$$\Rightarrow z\left(\frac{\frac{\cancel{r^2}}{b}-\frac{\cancel{r^2}}{a}}{b-a}+\frac{\cancel{r^2}}{c^2}\right)-a\left(\frac{\frac{\cancel{r^2}}{b}-\frac{\cancel{r^2}}{a}}{b-a}\right)+\frac{\cancel{r^2}}{a}-2\frac{\cancel{r^2}}{c}=0$$

$$\Rightarrow z\left(\frac{\cancel{a-b}^{\,-1}}{\cancel{(b-a)}ab}+\frac{1}{c^2}\right)-a\left(\frac{\cancel{a-b}}{\underset{-1}{\cancel{b-a}}}\cdot\frac{1}{ab}\right)+\frac{1}{a}-\frac{2}{c}=0$$

$$z\left(\frac{1}{c^2}-\frac{1}{ab}\right)+\frac{1}{b}+\frac{1}{a}-\frac{2}{c}=0$$

$$z=\frac{2c^{-1}-a^{-1}-b^{-1}}{c^{-2}-a^{-1}b^{-1}}=\frac{a^{-1}+b^{-1}-2c^{-1}}{a^{-1}b^{-1}-c^{-2}}\Rightarrow$$

Comparing

$\lambda=2$

$\boxed{\text{Que-2}}$ Let $b_1+b_2+b_3+b_4=0$ where $b_i \in R$, such that sum of no two being zero and $\sum_{i=1}^{4} b_i z_i = 0$, where z_i are arbitrary complex numbers such that no three of them are collinear, If these four points are concyclic then find $\left|\frac{b_1 b_2}{b_3 b_4}\right| \cdot \left|\frac{z_1-z_2}{z_3-z_4}\right|^2$

Sol

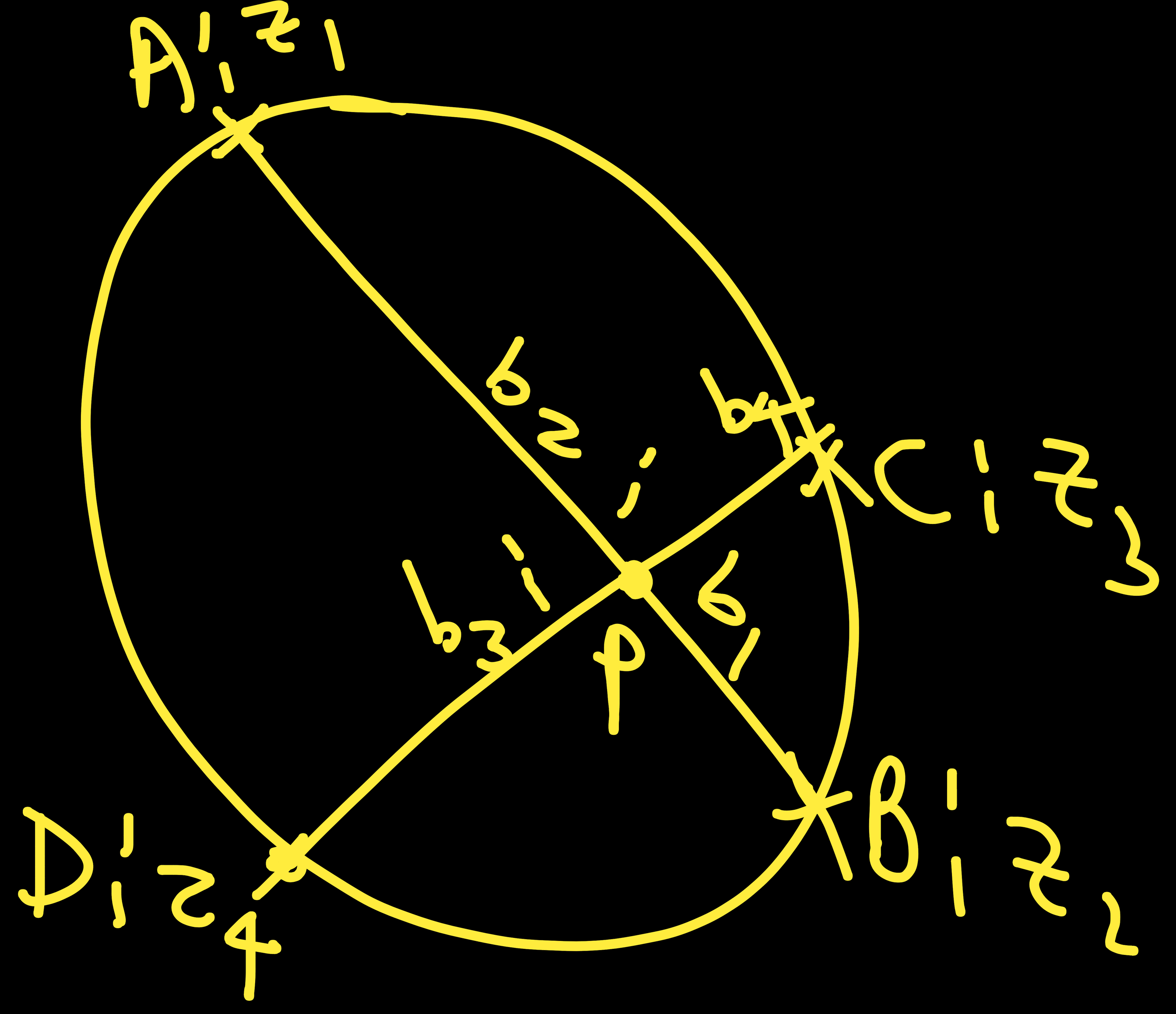
$b_1+b_2+b_3+b_4=0. \Rightarrow b_1+b_2 = -(b_3+b_4)$ —— ①

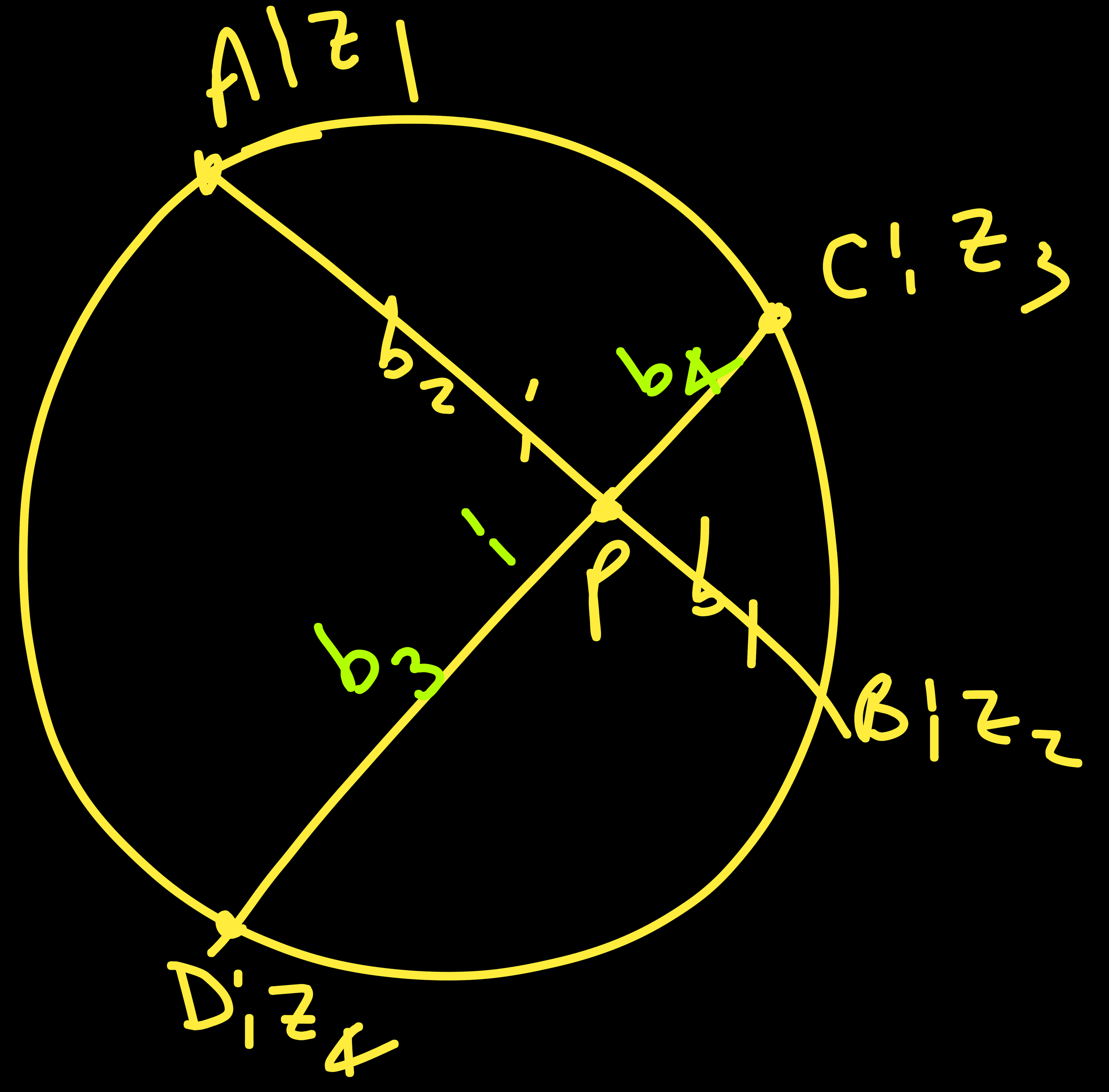
& $b_1z_1+b_2z_2+b_3z_3+b_4z_4=0. \Rightarrow b_1z_1+b_2z_2 = -(b_3z_3+b_4z_4)$. —— ②

$\frac{②}{①} \Rightarrow \frac{b_1z_1+b_2z_2}{b_1+b_2} = \frac{b_3z_3+b_4z_4}{b_3+b_4} = z_p.$

$\Rightarrow$ P is point of intersection of lines AB & CD. also $\frac{PA}{PB} = \frac{b_2}{b_1}$

& $\frac{PC}{PD} = \frac{b_4}{b_3}$





as A, B, C, D are concyclic

$$\Rightarrow PA \cdot PB = PC \cdot PD.$$

$$\Rightarrow \left(\left|\frac{b_2}{b_1+b_2}\right| \cdot AB\right) \cdot \left(\left|\frac{b_1}{b_1+b_2}\right| \cdot AB\right) = \left(\left|\frac{b_4}{b_3+b_4}\right| \cdot CD\right) \cdot \left(\left|\frac{b_3}{b_3+b_4}\right| \cdot CD\right).$$

$$\Rightarrow \frac{|b_1 b_2|\, AB^2}{\cancel{|b_1+b_2|^2}} = \frac{|b_3 b_4|\, CD^2}{\cancel{|b_3+b_4|^2}}.$$

$$|b_1 b_2|\, |z_1-z_2|^2 = |b_3 b_4| \cdot |z_3-z_4|^2$$

$$\Rightarrow \left|\frac{b_1 b_2}{b_3 b_4}\right| \cdot \left|\frac{z_1-z_2}{z_3-z_4}\right|^2 = \boxed{1}$$

$\downarrow$

R. Ans

Que If $\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=2\omega^2$ and $\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=2\omega$ then find the value of $\sum\frac{1}{a+1}$, given $a,b,c,d\in R$ and ω is non-real cube root of unity.

Soln.

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}+\frac{1}{d+\omega}=2\omega^2=\frac{2}{\omega}$$

&

$$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}+\frac{1}{d+\omega^2}=2\omega=\frac{2}{\omega^2}$$

Then find $\sum\frac{1}{a+1}$

Consider

$$\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}+\frac{1}{d+x}=\frac{2}{x}.$$

$\hookrightarrow$ roots: $\underline{\omega,\omega^2}$, $-,-$.

$$\frac{\sum(x+a)(x+b)(x+c)}{\prod(x+a)}=\frac{2}{x}$$

$$\Rightarrow x\cdot\left(\sum\left(x^3+(\sum a)x^2+(\sum ab)x+abc\right)\right)=2\cdot\prod(x+a)=2\cdot\left(x^4+(\sum a)x^3+(\sum ab)x^2+(\sum abc)x+\prod a\right)$$

$$\Rightarrow x\left(4x^3+3(\sum a)x^2+2(\sum ab)x+\sum abc\right)=2\left(x^4+(\sum a)x^3+(\sum ab)x^2+(\sum abc)x+\prod a\right)$$

$$2x^4+(\sum a)x^3-(\sum abc)x-2\prod a=0\longrightarrow$$

$$2x^4 + \left(\sum a\right)x^3 - \left(\sum abc\right)x - 2\prod a = 0 \longrightarrow \omega,\ \omega^2,\ \alpha,\ \beta.$$

$$S_1 = \omega+\omega^2+\alpha+\beta = -\left(\frac{\sum a}{2}\right) \Rightarrow \alpha+\beta = 1 - \frac{\sum a}{2}.$$

$$S_2 = (\omega+\omega^2)(\alpha+\beta) + \omega^3 + \alpha\beta = 0 \Rightarrow -(\alpha+\beta) + 1 + \alpha\beta = 0$$

$$S_3 = (\omega+\omega^2)\alpha\beta + (\alpha+\beta)\omega\cdot\omega^2 = \frac{\sum abc}{2} \Rightarrow -\alpha\beta + \alpha + \beta =$$

$$S_4 = \omega\cdot\omega^2\cdot\alpha\cdot\beta = -\frac{2\prod a}{2} \Rightarrow \alpha\beta = -\prod a.$$

$$1 + \alpha\beta - \alpha - \beta = 0$$

$$(1-\alpha)\cdot(1-\beta) = 0$$

i.e. either $\alpha = 1$ or $\beta = 1$

$\Rightarrow$ Equation will have '1' as a root

1 will satisfy

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} = \frac{2}{x}$$

$$\Rightarrow \sum \frac{1}{a+1} = \frac{2}{1} = \boxed{2}$$

Que Form an equation with roots as $\cot^2\left(\frac{r\pi}{11}\right)$, for $r = 1, 2, 3, 4, 5$.

Soln Roots $\longrightarrow \cot^2\left(\frac{r\pi}{11}\right)$, $r = 1, 2, 3, 4, 5$.

$\hookrightarrow \cot^2\left(\frac{\pi}{11}\right), \cot^2\left(\frac{2\pi}{11}\right), \cot^2\left(\frac{3\pi}{11}\right), \cot^2\left(\frac{4\pi}{11}\right), \cot^2\left(\frac{5\pi}{11}\right).$

Concept :→ Consider $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$

$= {}^nC_0 \cdot c^n + {}^nC_1 \cdot c^{n-1} \cdot i \cdot s - {}^nC_2 \cdot c^{n-2} \cdot s^2 - {}^nC_3 \cdot c^{n-3} \cdot i\, s^3 + \cdots$

$$\cdots + \cdots$$

Equating Imaginary part

$\sin(n\theta) = {}^nC_1 \cdot \cos^{n-1}(\theta) \cdot \sin(\theta) - {}^nC_3 \cdot \cos^{n-3}(\theta) \cdot \sin^3(\theta) + \cdots$

$n = 11$

$\hookrightarrow \sin(11\theta) = {}^{11}C_0 \cdot \cos^{10}(\theta) \cdot \sin(\theta) - {}^{11}C_3 \cdot \cos^8(\theta) \cdot \sin^3(\theta) + \cdots$

$\sin(11\theta) = \sin^{11}(\theta)\left(\cot^{10}(\theta) - {}^{11}C_3 \cot^8(\theta) + {}^{11}C_5 \cot^6(\theta) - \cdots\right)$

$$\sin(11\theta) = \sin^{11}(\theta)\left(\cot^{10}(\theta) - {}^{11}C_3 \cdot \cot^8(\theta) + {}^{11}C_5 \cot^6(\theta) - {}^{11}C_7 \cot^4(\theta) + {}^{11}C_9 \cot^2(\theta) - {}^{11}C_{11}\right)$$

$\theta = \frac{\pi}{11}$

$x = \cot^2\left(\frac{\pi}{11}\right)$

$\cot^2(\theta) = x$

$$\sin(11\theta) = \sin^{11}(\theta) \cdot \left(x^5 - {}^{11}C_3 \cdot x^4 + {}^{11}C_5 \cdot x^3 - {}^{11}C_7 x^2 + {}^{11}C_9 \cdot x - 1\right)$$

$\theta = \frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \frac{4\pi}{11}, \frac{5\pi}{11}$

$$0 = \sin^{11}(\theta)\left(x^5 - {}^{11}C_3 \text{---} \text{---}\right)$$

$$\Rightarrow x^5 - {}^{11}C_3 x^4 + {}^{11}C_5 x^3 - {}^{11}C_7 x^2 + {}^{11}C_9 x - 1 = 0.$$

$\cot^2\left(\frac{\pi}{11}\right), \cot^2\left(\frac{2\pi}{11}\right), \text{---}, \text{---}, \cot^2\left(\frac{5\pi}{11}\right)$

Que Let $\prod_{r=1}^{n}(a_r^2+b_r^2) = A^2+B^2$, where $a_i, b_i, A, B \in R$, $\forall i=1,2,3,---,n$ then find A, B in terms of a_i's & b_i's.

Soln Given $(a_1^2+b_1^2)(a_2^2+b_2^2) - - - (a_n^2+b_n^2) = A^2+B^2$.

$$\prod_{r=1}^{n}(a_r^2+b_r^2) = \prod_{r=1}^{n}(a_r+ib_r)(a_r-ib_r)$$

$$= \left(\underbrace{\prod(a_r+ib_r)}_{z}\right)\left(\underbrace{\prod(a_r-ib_r)}_{\bar{z}}\right)$$

$$= \left|\prod_{r=1}^{n}(a_r+ib_r)\right|^2 = \left|(a_1+ib_1)(a_2+ib_2)------(a_n+ib_n)\right|^2.$$

$$= \left|\left(\prod b_1\right)\left(\prod\left(\frac{a_1}{b_1}+i\right)\right)\right|^2 = \left(\prod b_1^2\right)\left|\prod(i-c_1)\right|^2$$

where $\frac{a_1}{b_1} = c_1$, $\frac{a_2}{b_2} = c_2$, etc.

$$= \left(b_1^2 b_2^2 \cdots b_n^2\right) \left|(i-c_1)\cdot(i-c_2) \cdots\cdots (i-c_n)\right|^2$$

$$= \left(\prod b_1^2\right) \left| i^n - i^{n-1}\left(\Sigma c_1\right) + i^{n-2}\left(\Sigma c_1 c_2\right) - i^{n-3}\left(\Sigma c_1 c_2 c_3\right) + \cdots \right|^2 .$$

$$= \left(\prod b_1^2\right) \left| \left(1 - \Sigma c_1 c_2 + \Sigma c_1 c_2 c_3 c_4 - \cdots\right) + i\left(\Sigma c_1 - \Sigma c_1 c_2 c_3 + \cdots\right) \right|^2$$

$$= \prod b_1^2 \left(1 - \Sigma c_1 c_2 + \Sigma c_1 c_2 c_3 c_4 - \cdots\right)^2 + \left(\prod b_1^2\right)\left(\Sigma c_1 - \Sigma c_1 c_2 c_3 + \Sigma c_1 c_2 c_3 c_4 c_5 - \cdots\right)^2$$

Que) Let n is odd integer > 3, and not a multiple of 3, T.P.T. x^3+x^2+x divides $(1+x)^n - x^n - 1$

Sol

$$x^3+x^2+x \mid (1+x)^n - x^n - 1$$

$$x(x^2+x+1)$$

root $0, \omega, \omega^2$

let $f(x) = (1+x)^n - x^n - 1$

$x=0 \rightarrow f(0) = 1^n - 0 - 1 = 0$

$x=\omega \rightarrow (1+\omega)^n - \omega^n - 1$

$= (-\omega^2)^n - \omega^n - 1 = -\omega^{2n} - \omega^n - 1$

$= -(\omega^{2n} + \omega^n + 1) = 0.$

$x=\omega^2 \rightarrow (1+\omega^2)^n - \omega^{2n} - 1$

$= (-\omega)^n - \omega^{2n} - 1$

$= -(\omega^n + \omega^{2n} + 1) = 0$

$\rightarrow x^3+x^2+x$ will divide $(1+x)^n - x^n - 1.$

H.P.

Que Let z_1 and z_2 be two complex numbers with $|z_1| = |z_2|$, Then find least value of $\frac{(z_1+z_2)^2}{z_1 z_2} + (\arg(z_1) - \arg(z_2))^2$.

Soln

$|z_1| = |z_2|$, Let $z_1 = r \cdot e^{i\theta_1}$, $z_2 = r \cdot e^{i\theta_2}$

$$\frac{(z_1+z_2)^2}{z_1 z_2} + (\arg(z_1) - \arg(z_2))^2$$

$$= \frac{z_1^2 + z_2^2 + 2z_1 z_2}{z_1 z_2} + (\theta_1 - \theta_2)^2.$$

$$= \frac{z_1}{z_2} + \frac{z_2}{z_1} + 2 + (\theta_1 - \theta_2)^2$$

$$= \frac{\cancel{r} \cdot e^{i\theta_1}}{\cancel{r} \cdot e^{i\theta_2}} + \frac{\cancel{r} \cdot e^{i\theta_2}}{\cancel{r} e^{i\theta_1}} + 2 + (\theta_1 - \theta_2)^2$$

$\theta_1 - \theta_2 = \alpha$

$$= e^{i\alpha} + e^{-i\alpha} + 2 + \alpha^2.$$

$$= 2\cos(\alpha) + 2 + \alpha^2.$$

$$= 4 + 2(\cos(\alpha) - 1) + \alpha^2$$

$$= 4 - 4 \cdot \sin^2\left(\frac{\alpha}{2}\right) + \alpha^2.$$

$$= 4 + 4\left(\left(\frac{\alpha}{2}\right)^2 - \sin^2\left(\frac{\alpha}{2}\right)\right).$$

$$\geq 4$$

R- min. value $= \boxed{4}$

$x \geq \sin(x)$ for all $x \geq 0$

$x^2 \geq \sin^2(x)$

Que Find the greatest and the least value of $|z|$ if it satisfy

$$\left|z-\frac{4}{z}\right|=2.$$

Sol let $z = r \cdot e^{i\theta}$

$$\left|z-\frac{4}{z}\right|=2$$

$$\left|r\cdot e^{i\theta}-\frac{4}{r}\cdot e^{-i\theta}\right|=2$$

$$\Rightarrow \left|\left(r-\frac{4}{r}\right)\cos(\theta)+i\left(r+\frac{4}{r}\right)\sin(\theta)\right|=2$$

$( )^2$

$$\left(r-\frac{4}{r}\right)^2\cdot\cos^2(\theta)+\left(r+\frac{4}{r}\right)^2\sin^2(\theta)=4$$

$$\Rightarrow r^2+\frac{16}{r^2}-8\left(\cos^2(\theta)-\sin^2(\theta)\right)=4$$

$$r^2+\frac{16}{r^2}=4+8\underline{\cos(2\theta)}.$$

$$-4\le \underbrace{r^2+\frac{16}{r^2}\le 12}$$

$$r^2+\frac{16}{r^2}\le 12.$$

$$(r^2)^2-12r^2+16\le 0$$

$$\frac{12-\sqrt{12^2-4.16}}{2}\le r^2\le\frac{12+\sqrt{12^2-4.16}}{2}$$

$5+1$

$$6-2\sqrt{5}\le r^2\le 6+2\sqrt{5}$$

$$(\sqrt{5}-1)^2\le r^2\le(\sqrt{5}+1)^2.$$

$$\sqrt{5}-1\le r\le\sqrt{5}+1.$$

$r_{max.}=\sqrt{5}+1$ & $r_{min}=\sqrt{5}-1.$ Ans

Que If z is a complex number and $z^2+az+b=0$ $(a\neq 0)$ has both roots of unit modulus T.P.T. (i) $|a|\le 2$ (ii) $|b|=1$ (iii) $\arg(b)=2\arg(a)$.

Sol

$z^2+az+b=0 \longrightarrow z_1+z_2=-a$, $z_1z_2=b$.

$\hookrightarrow z_1, z_2$

$|z_1|=|z_2|=1$ (given)

$\hookrightarrow z_1\bar{z}_1=1 \Rightarrow \bar{z}_1=\frac{1}{z_1}$

(i) $|a|=|z_1+z_2|\le |z_1|+|z_2|=1+1$

$|a|\le 2$. H.P.

(ii) $|b|=|z_1z_2|=|z_1|\cdot|z_2|=1$. H.P.

(iii)

$\Rightarrow -\bar{a}=\bar{z}_1+\bar{z}_2=\frac{1}{z_1}+\frac{1}{z_2}=\frac{z_1+z_2}{z_1z_2}$

$\Rightarrow -\bar{a}=\frac{-a}{b}$

$\Rightarrow b=\frac{a}{\bar{a}}$

$\arg(b)=\arg\left(\frac{a}{\bar{a}}\right)=\arg\left(\frac{a^2}{|a|^2}\right)$

$=\arg(a^2)=2\arg(a)$.

H.P.

$\boxed{\text{Que}}$ Two lines $z i - \bar{z} i + 2 = 0$ and $z(1+i) + \bar{z}(1-i) + 2 = 0$ intersect at P then find complex number corresponding to a point on 2^{nd} line which is at a distance 2-units from P.

Soln

$$P \leftarrow \begin{cases} L_1 \equiv z i - \bar{z} i + 2 = 0 \xrightarrow{z = x+iy} 2\,\text{Re}(iz) + 2 = 0 \\ L_2 \equiv z(1+i) + \bar{z}(1-i) + 2 = 0 \end{cases}$$

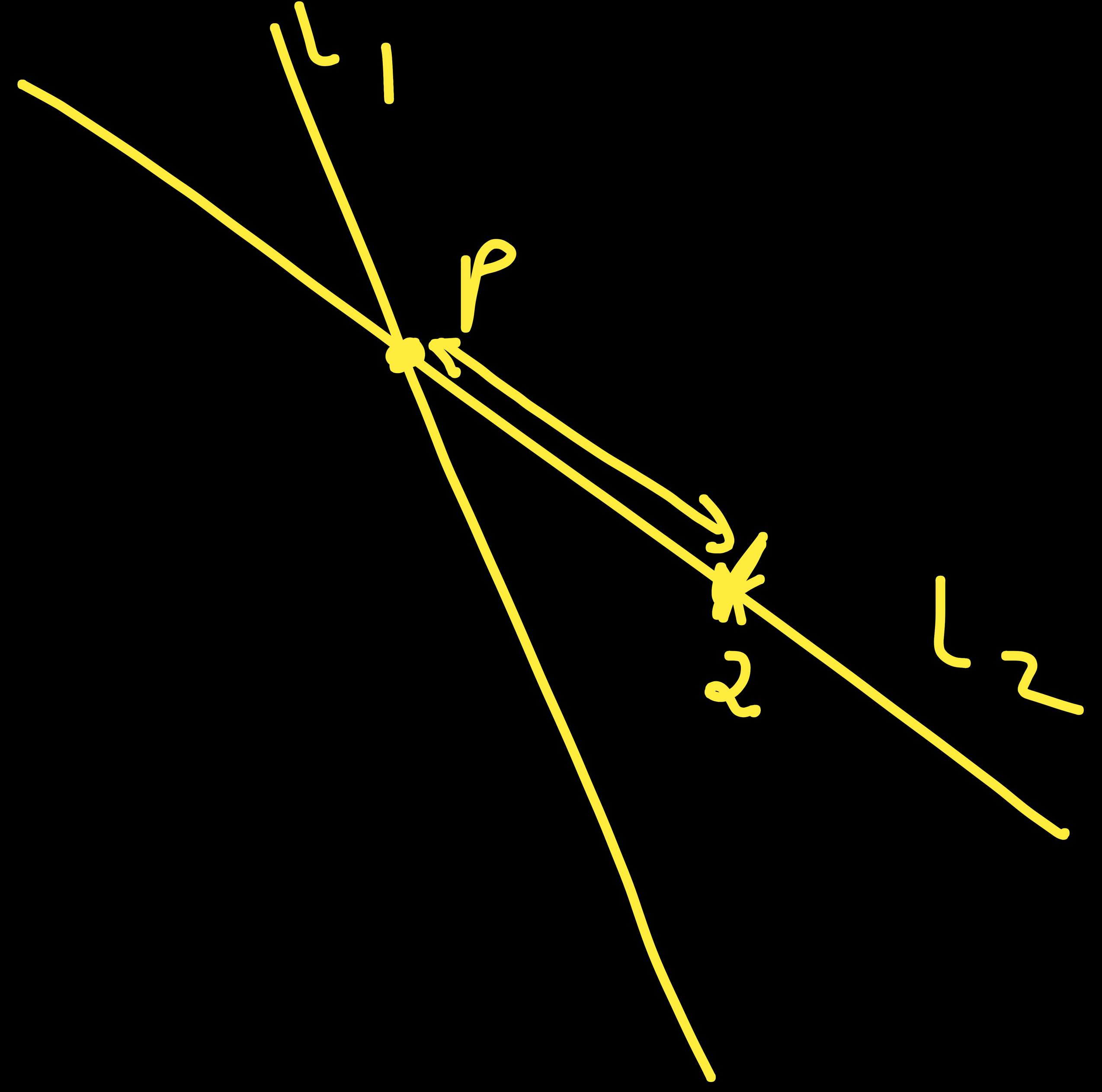
$2\,\text{Re}(ix - y) + 2 = 0 \Rightarrow -2y + 2 = 0$

$y = 1$

$z = x + iy$: $\cancel{2}\,\text{Re}\left((x+iy)(1+i)\right) + \cancel{2} = 0$

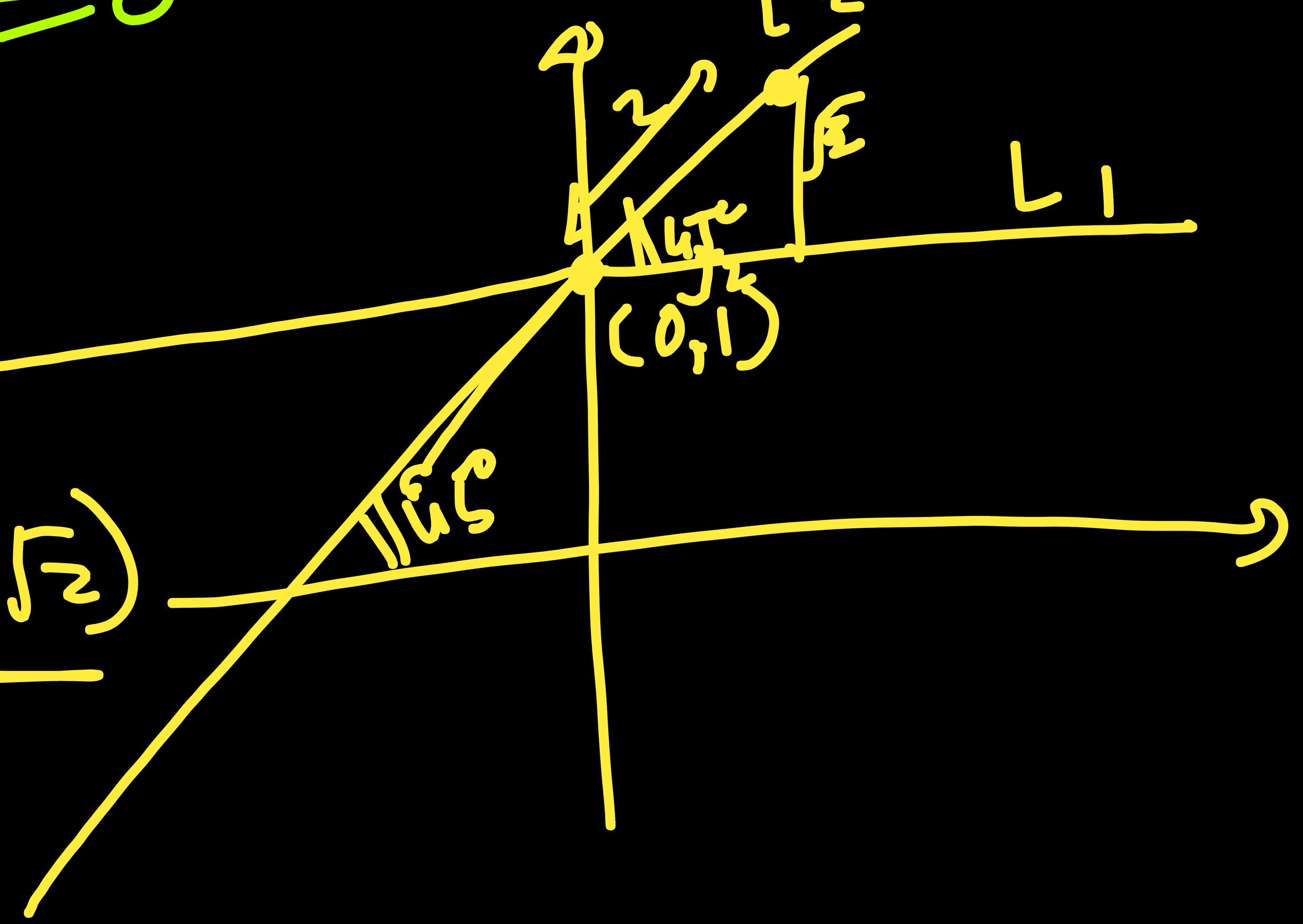
$\hookrightarrow x - y + 1 = 0 \Rightarrow x = 0$

$P : (0, 1)$



$RP_1 \equiv (\sqrt{2}, 1+\sqrt{2}) \equiv \sqrt{2} + i(1+\sqrt{2})$

$RP_2 \equiv (-\sqrt{2}, 1-\sqrt{2}) \equiv \underline{-\sqrt{2} + i(1-\sqrt{2})}$



$$L_1 \equiv zi - \bar{z}i + 2 = 0$$

$$L_2 \equiv z(1+i) + \bar{z}(1-i) + 2 = 0$$

$$-$$

$$z + \bar{z} = 0.$$

$$\bar{z} = -z$$

$$zi + zi + 2 = 0$$

$$\cancel{2}iz = -2 = \cancel{2}i^2$$

$$z = \underline{\underline{i}}$$

Let R.P. is z'

$$z'(1+i) + \bar{z}'(1-i) + 2 = 0$$

$$|z' - i| = 2.$$

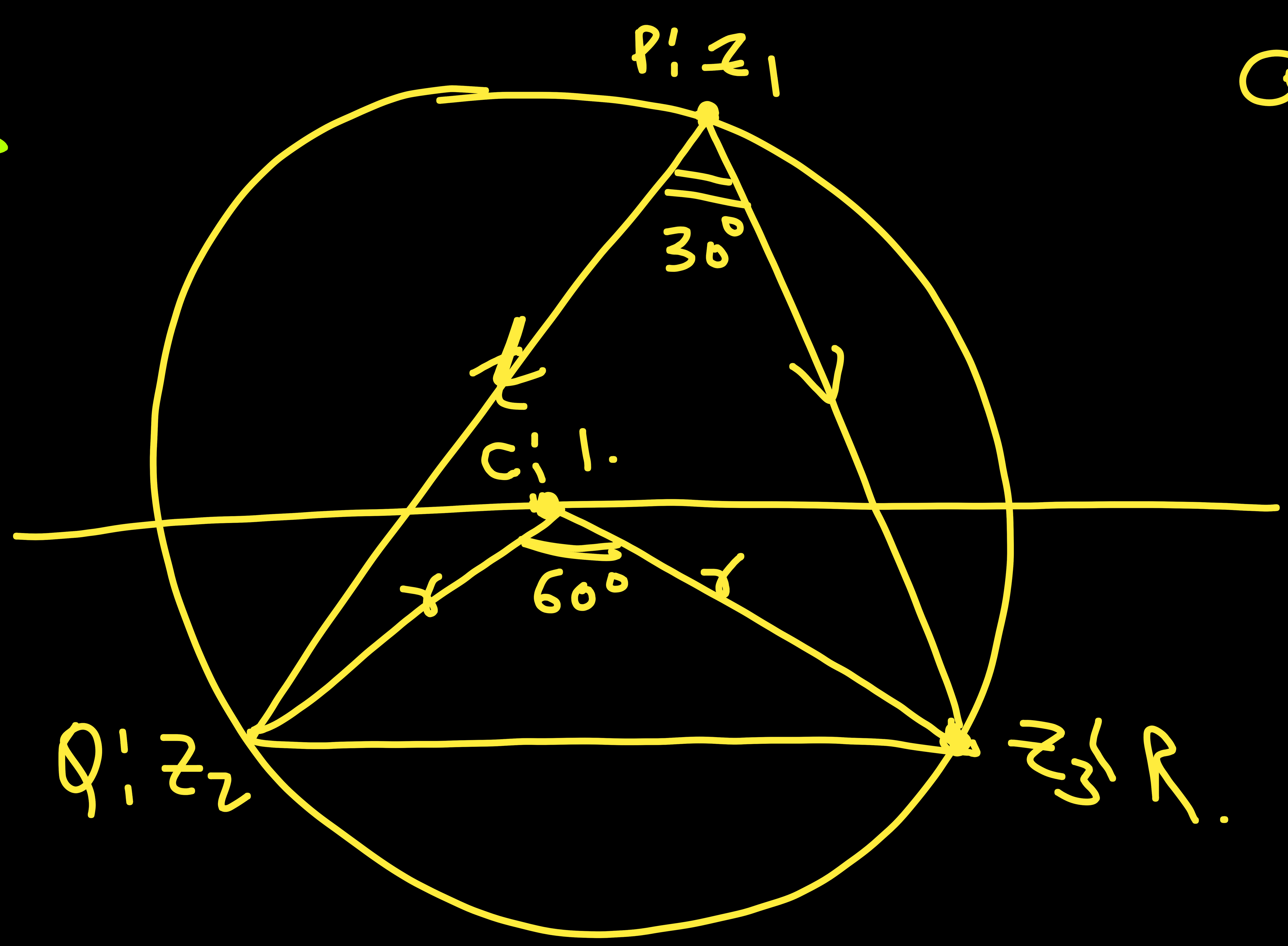
$$z\bar{z}' + 1 - i\bar{z}' + iz' = 4.$$

D.I.Y. . .

Que Let z_1, z_2, z_3 are complex numbers such that $|z_1-1| = |z_2-1| = |z_3-1|$ and $\arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = \frac{\pi}{6}$ then find the value of

$$z_2(z_2-1) - z_3(z_2+1) + (z_3+1)(z_3-1)$$

Sol



$$\arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = \frac{\pi}{6}$$

Clearly ΔCQR is an equilm Δ.

(Using condition $\Sigma z_1^2 = \Sigma z_1 z_2$)

$$1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_2 \cdot 1 + z_3 \cdot 1.$$

$$z_2(z_2-1) + (z_3^2-1) - z_3(z_2+1) = 0.$$

$$2 + \underbrace{z_2(z_2-1) + (z_3^2-1) - z_3(z_2+1)}$$

R. value $= \boxed{-2}$ Ans