Que. Let $A = (a_{ij})$ be an $n \times n$ matrix of real numbers such that $\sum_{j=1}^{n} a_{ij} = 1$, for each i, Then $A - I$ is an invertible matrix. (T/F).

Soln. Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Also.
$$\left.\begin{aligned} a_{11}+a_{12}+\cdots+a_{1n} &= 1 \\ a_{21}+a_{22}+\cdots+a_{2n} &= 1 \\ &\vdots \\ a_{n1}+a_{n2}+\cdots+a_{nn} &= 1 \end{aligned}\right\} \text{(1)}$$

Consider $X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$

Now consider matrix AX.
$$AX = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}} = \begin{bmatrix} a_{11}+a_{12}+\cdots+a_{1n} \\ a_{21}+a_{22}+\cdots+a_{2n} \\ a_{31}+a_{32}+\cdots+a_{3n} \\ \vdots \\ a_{n1}+a_{n2}+\cdots+a_{nn} \end{bmatrix}_{n \times 1}$$

$$= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{(from (1))}$$

$$AX = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} = X \Rightarrow AX = X$$

$$\Rightarrow AX - X = O$$

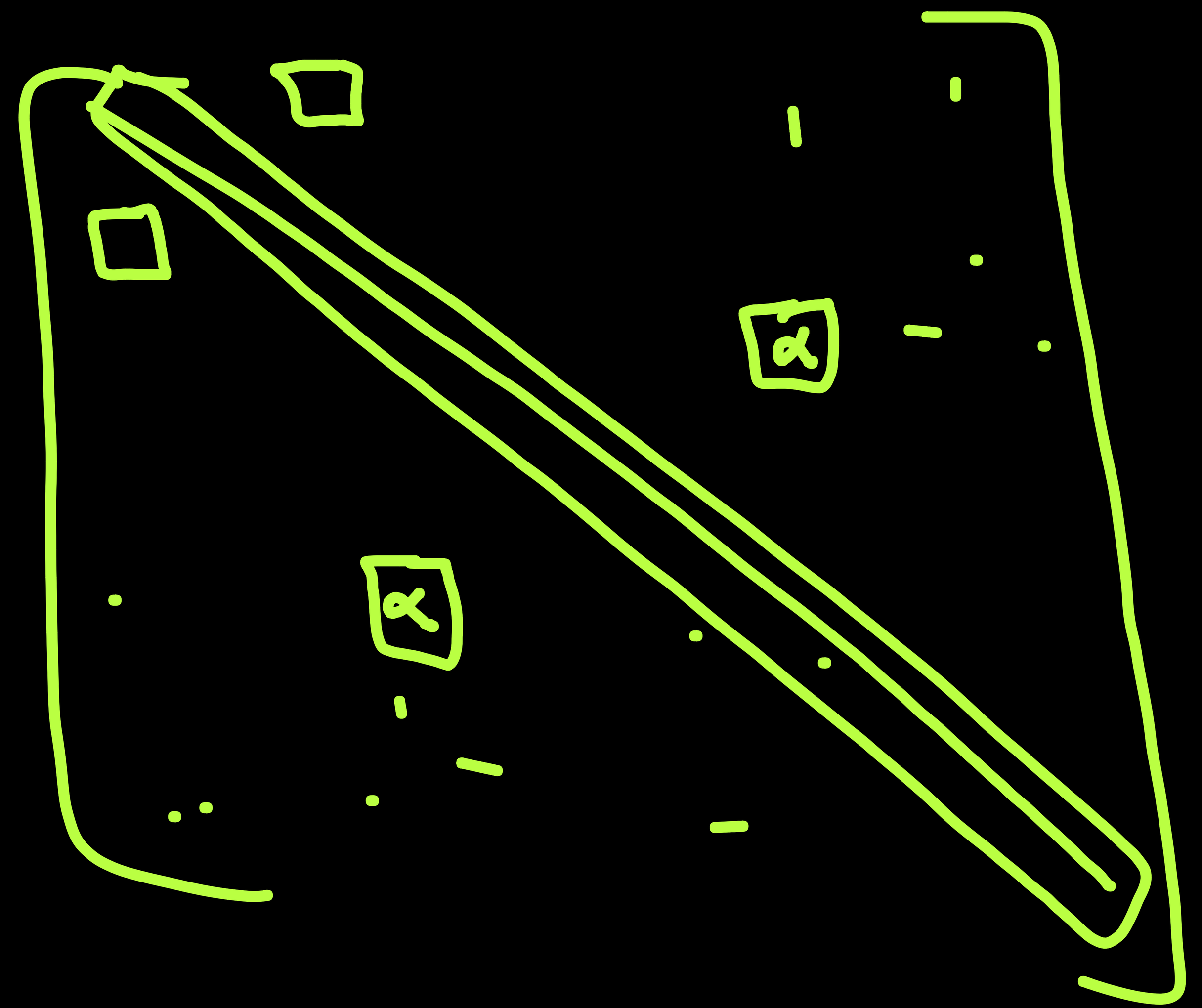
$$\Rightarrow (A - I) \cdot X = O$$

$$\Rightarrow |A - I| = 0. \Rightarrow A - I \text{ is non-invertible}$$

$\therefore$ The given statement is false.

Q. Let $A = (a_{ij})_{n\times n}$ is a symmetric matrix, where n is odd and each row of the matrix is a permutation of the integers $1, 2, 3, ----, n$, then find possible range of $tr(A)$.

Soln. Symmetric matrix means $a_{ij} = a_{ji}$
$= \alpha \left(\in \{1, 2, 3, --, n\}\right)$

$A =$ 

Note that α is present in 1^{st} row & in 2^{nd} row & in 3^{rd} row - - - -

$\Rightarrow$ In matrix 'A', element 'α' is present exactly n-times.

Now if it is present at the position i^{th} row & j^{th} column then it will also be present at the position j^{th} row and i^{th} column. (for $\forall\ i \neq j$)

So α will be present in total even times at non-diagonal positions $\Rightarrow$ It will be present odd times at diagonal position. (is 'n' is odd) $\Rightarrow$ Now as it is true for all $\alpha \in \{1, 2, 3, --, n\}$ and there are total n-diagonal entries

$\Rightarrow$ Each number out of $1, 2, 3, ---, n$ must be present exactly at one of the diagonal position

$\Rightarrow$ diagonal is also one of the permutation of $1, 2, 3, ---, n$.

$$\Rightarrow tr(A) = \sum_{r=1}^{n} r = \boxed{\frac{n(n+1)}{2}}$$

Ans

Que Let A is a square matrix and let $\exists$ two distinct natural numbers m and n such that $A^m = I$ and $A^n \neq I$ then find possible value(s) of $|I + A + A^2 + \cdots + A^{m-1}|$.

Soln Given $A^m = I$ & $A^n \neq I$

$\Rightarrow I - A^m = 0$

$\Rightarrow (I - A)(I + A + A^2 + \cdots + A^{m-1}) = 0$

$I - A^n \neq 0$

$\Rightarrow (I - A).(I + A + A^2 + \cdots + A^{n-1}) \neq 0$

$\Downarrow$

$I - A \neq 0$

$\text{Det}(I + A + A^2 + \cdots + A^{m-1}) = \boxed{0}.$

Ans

NOTE $A \cdot B = 0$

& if $A \neq 0$ then B must be singular.

Que Let A be a 2×3 matrix where B is a 3×2 matrix. If $|AB| = 4$ then find possible value(s) of $|BA|$.

Soln

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}$$

$$|AB| = 4 = \left| \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}}_{3\times 2} \right|$$

$$= \left| \begin{bmatrix} \_ & \_ \\ \_ & \_ \end{bmatrix}_{2\times 2} \right|$$

$|AB| = |A|\cdot|B|$

$$\text{Det}(BA) = \text{Det}\left(\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} \cdot \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \right)$$

$3\times 2 \quad 2\times 3$

$$= \text{Det}\left(\begin{bmatrix} a_1b_1 + b_2a_4 & b_1a_2 + b_2a_5 & - \\ a_1b_3 + b_4a_4 & - & - \\ a_1b_5 + b_6a_4 & - & - \end{bmatrix} \right)$$

$$= \text{Det}\left(\begin{bmatrix} b_1 & b_2 & 0 \\ b_3 & b_4 & 0 \\ b_5 & b_6 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= \left(\text{Det} \begin{bmatrix} b_1 & b_2 & 0 \\ b_3 & b_4 & 0 \\ b_5 & b_6 & 0 \end{bmatrix} \right) \cdot \left(\text{Det} \begin{bmatrix} a_1 & a_2 & a_3 \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$= 0 \cdot 0 = \boxed{0} \quad \text{Ans}$$

Que If A satisfy $x^2+4x-5=0$ T.P.T. (i) A^{-1} exists (ii) If $A^{-1}=\alpha A+\beta I$ then find $\alpha+\beta$ (iii) If $A^3=\alpha_1 A+\beta_1 I$ then find $\alpha_1+\beta_1$.

Sol

Given $x^2+4x-5=0$

$$A^2+4A-5I=O$$

$$A^2+4A=5I$$

det

$$|A^2+4A|=|5I|$$

$$\Rightarrow |A(A+4I)|=5^n|I|=5^n$$

$$|A|\cdot|(A+4I)|$$

$$\Rightarrow |A|\neq 0 \Rightarrow A^{-1} \text{ exists}$$

(ii)

$$A^2+4A=5I$$

$A^{-1}\cdot$

$$A+4I=5A^{-1}$$

$$A^{-1}=\frac{1}{5}A+\frac{4}{5}I$$

$$\Rightarrow \alpha=\frac{1}{5} \;\&\; \beta=\frac{4}{5}$$

$$\Downarrow$$

$$\alpha+\beta=\frac{1}{5}+\frac{4}{5}=\boxed{1}$$

Ans

(ii)

(iii) $A^2=5I-4A$

Consider

$$A^3=A\cdot(A^2)=A(5I-4A)$$

$$=5A-4A^2$$

$$=5A-4(5I-4A)$$

$$=5A-20I+16A$$

$$=21A-20I$$

$$\Downarrow$$

$$\alpha_1=21 \;\&\; \beta_1=-20.$$

$$\Rightarrow \alpha_1+\beta_1=\boxed{1}$$

Ans

Que. If A and B are two orthogonal matrices of order n such that $|A|+|B|=0$, then identify true statement.

(A) $|A+B| = |A|+|B|$ (B) $|A+B| = 0$ (C) A and B are singular matrices

(D) $A+B=O$.

Soln. Given $AA'=I$ & $BB'=I$

$$\Downarrow \qquad\qquad \Downarrow$$

$|AA'| = |I|$ $\qquad |B| = \pm 1$

$|A|\cdot|A'| = 1$

$|A|^2 = 1$

$|A| = \pm 1$

Now given

$|A|+|B|=0$.

$\Rightarrow |A|=1$ & $|B|=-1$

Consider,

$$|A+B| = |A|.|A+B| = |A'|.|A+B|$$

$$= |A'A + A'B|$$

$$= |I + A'B| = |I + B'A|$$

$$= |B'B + B'A| = |B'|.|B+A|$$

$$= -1.|A+B| = -|A+B|$$

$\Rightarrow |A+B| = 0$.

(A) ✓ (B) ✓ (C) ✗ (D) ✗

Que If $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ and $|A^n-I|=1-\lambda^n$ for $n\in N$ then find λ.

Soln

$$A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

$$A^2=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

$$=\begin{bmatrix}2&2\\2&2\end{bmatrix}=2\cdot A.$$

$$A^3=A\cdot A^2=A\cdot 2A=2A^2=2\cdot(2A)$$

$$A^3=2^2A$$

$$\vdots$$

$$A^n=2^{n-1}\cdot A=\begin{bmatrix}2^{n-1}&2^{n-1}\\2^{n-1}&2^{n-1}\end{bmatrix}$$

$$A^n-I=\begin{bmatrix}2^{n-1}&2^{n-1}\\2^{n-1}&2^{n-1}\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}2^{n-1}-1&2^{n-1}\\2^{n-1}&2^{n-1}-1\end{bmatrix}$$

$$\text{Det}(A^n-I)=(2^{n-1}-1)^2-(2^{n-1})^2$$

$$=(\cancel{2^{n-1}}-1-\cancel{2^{n-1}})(2^{n-1}-1+2^{n-1})$$

$$=-1\cdot(2^n-1)=1-2^n=1-\lambda^n$$

$$\boxed{\lambda=2}$$ Ans

Que Let A and B are square matrices of order 'n' such that $A+B = AB$, then identify correct statement(s) :→

(A) $(A+B)^3 = A^3 + B^3 + 3A^2B + 3AB^2$

(B) $A+B = B+A$

(C) $(A+B)^3 = A^3 + B^3 + ABA + A^2B + BA^2 + BAB + B^2A + AB^2$

(D) $A = B$.

Soln: $A+B = AB$.

$AB - A - B = O$

$+I$

$AB - A - B + I = I$

$\Rightarrow (A-I)(B-I) = I \Rightarrow A-I$ and $B-I$ are inverse of each other

$(B-I)(A-I) = I$

$BA - A - B + I = I \Rightarrow BA = A+B$.

$\Rightarrow AB = BA$.

(A) (B)

(C) ~~(D)~~

Concept

$PQ = \lambda . I$

$\Downarrow$

P & Q will be commutable matrices.

P^{-1}: $P^{-1}P . Q = \lambda P^{-1}$

$IQ = \lambda P^{-1} \Rightarrow Q = \lambda P^{-1}$

$\cdot P$

$QP = \lambda I$

$\boxed{\text{Que}}$ If $A \neq A^2 = I$ then $\det(I+A) =$

(A) 0 (B) 1 (C) -1 (D) 2.

Soln Given $A^2 \neq A$

$\Rightarrow A^2 - A \neq 0$

$\Rightarrow A(A-I) \neq 0$

$\Rightarrow A - I \neq 0$

Now $A^2 = I \Rightarrow A^2 - I = 0$

$(A-I)\cdot(A+I) = 0$

As $A - I \neq 0$

$\Rightarrow A+I$ will be singular matrix

$\Rightarrow |A+I| = 0.$

$\Rightarrow$ (A)

$\boxed{\text{Que}}$ Let A, $B\ (\neq I)$ are commutable square matrices such that $A^n - B^n$ is invertible for some $n\ (\in N)$, also $A^n - B^n = A^{n+1} - B^{n+1} = A^{n+2} - B^{n+2}$ then identify the correct statement (s).

$\boxed{S_1}$ $|I - A| = 0$ $\boxed{S_2}$ $|I - B| = 0$ $\boxed{S_3}$ $A + B = AB + I$ $\boxed{S_4}$ $A^2 = B^2$.

Sol^n $A, B \longrightarrow$ square matrices of same order. & $AB = BA$.

$|A^n - B^n| \neq 0$.

$A^n - B^n = A^{n+1} - B^{n+1} = A^{n+2} - B^{n+2}$.

$= C$ (say)

$|C| \neq 0 \Rightarrow C^{-1}$ exists

Consider $X^2 - (A+B)X + AB = O \rightarrow A, B$

$A^2 - (A+B)A + AB = O$

$A^n \Rightarrow A^{n+2} - (A+B)A^{n+1} + AB \cdot A^n = O$

$|||^{ly}$ $B^{n+2} - (A+B) \cdot B^{n+1} + AB \cdot B^n = O$

$-$

$$\underbrace{(A^{n+2} - B^{n+2})}_{=C} - (A+B)\underbrace{(A^{n+1} - B^{n+1})}_{=C} + AB\underbrace{(A^n - B^n)}_{=C} = O$$

$$C - (A+B)\cdot C + AB\cdot C = O$$

$$\left(I - (A+B) + AB\right)\cdot C = O$$

$\cdot C^{-1}$

$$I - (A+B) + AB = O \Rightarrow A+B = AB + I$$

$$\Rightarrow (I-A)\cdot(I-B) = O$$

as $I-A \neq O$

$I-B$ will be singular

as $I-B \neq O$

$I-A$ " " "

$$\Rightarrow |I-A| = 0 = |I-B|$$

S1 ✓ S2 ✓ S3 ✓ S4 ✗

let $U_n = \alpha^n - \beta^n$

Rough

$ax^2 + bx + c = 0$ ⟶ α, β

$a\alpha^2 + b\alpha + c = 0$

$\alpha^n \cdot$ ⟶ $a\alpha^{n+2} + b\alpha^{n+1} + c\cdot\alpha^n = 0$

IIIly $a\cdot\beta^{n+2} + b\cdot\beta^{n+1} + c\cdot\beta^n = 0$

$-$

$a(\alpha^{n+2} - \beta^{n+2}) + b(\alpha^{n+1} - \beta^{n+1}) + c(\alpha^n - \beta^n) = 0$

$\Rightarrow a\cdot U_{n+2} + bU_{n+1} + cU_n = 0$