Que Four ~~different integers~~ form an increasing A.P. such that one of them is sum of the squares of the remaining numbers then find the largest number.

Soln Let numbers are $a-d,\ a,\ a+d,\ a+2d$, also according to the question a, d will be integers

Given $a+2d = (a-d)^2 + a^2 + (a+d)^2$

$a+2d = 3a^2 + 2d^2$

$3a^2 - a + 2d(d-1) = 0 \longrightarrow$ for real 'a' $D \geq 0 \Rightarrow 1 - 4.3.2d(d-1) \geq 0.$

$d^2 - d - \frac{1}{24} \leq 0.$

$$\frac{1 \pm \sqrt{1+\frac{1}{6}}}{2} = \frac{1 \pm \sqrt{\frac{7}{6}}}{2}$$

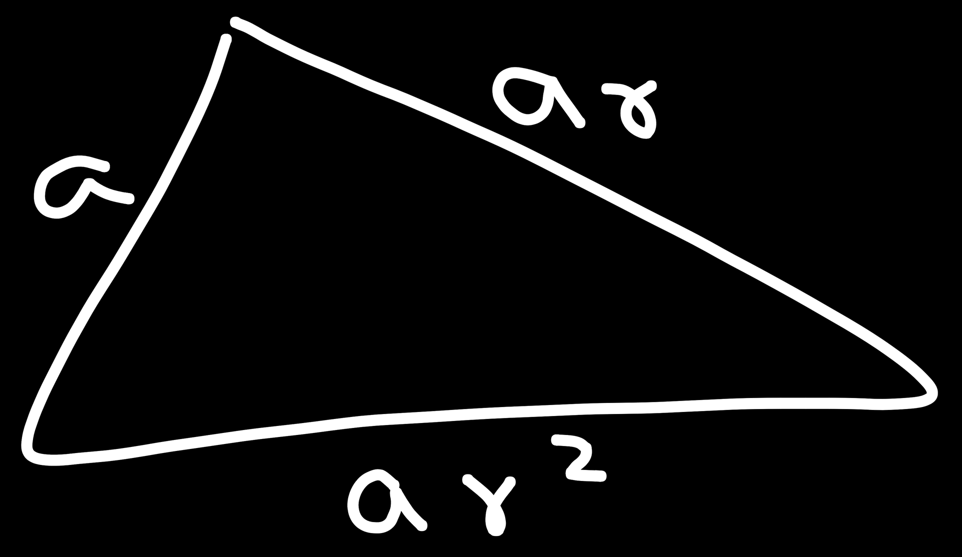
$d^2 - d - \frac{1}{24} \leq 0 \Rightarrow \frac{1-\sqrt{\frac{7}{6}}}{2} \leq d \leq \frac{1+\sqrt{\frac{7}{6}}}{2} \Rightarrow d = 0, 1$

$a + \cancel{2} = 3a^2 + \cancel{2} \Rightarrow a = 0, \frac{1}{3} \Rightarrow a = 0.$

The largest no. is $a+2d = \boxed{2}$ Ans

Que. Let length of sides of a triangle are in G.P., then find possible range of common ratio.

Soln



C-1 If $r = 1$ then sides will be a, a, a ✓.
S_1

C-2 If $r > 1$ then for Δ to be formed

$S_1 + S_2 > S_3$

$\Rightarrow a + ar > ar^2$

$\Rightarrow r^2 - r - 1 < 0 \rightarrow \frac{1 \pm \sqrt{5}}{2}$

$r \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$

$r \in \left(1, \frac{1+\sqrt{5}}{2}\right)$ $\cap$
S_2

C-3 If $r \in (0,1)$ then from C-2

$\frac{1}{r} \in \left(1, \frac{1+\sqrt{5}}{2}\right)$

$r \in \left(\frac{2}{1+\sqrt{5}}, 1\right)$

$\in \left(\frac{\sqrt{5}-1}{2}, 1\right)$
S_3

Final Soln. $\equiv r \in S_1 \cup S_2 \cup S_3$

$\boxed{r \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)}$

Que Let $(\underbrace{666\ldots6}_{2021\text{-times}})^2 + \underbrace{888\ldots8}_{2021\text{-times}} = \underbrace{aa\ldots a}_{n\text{-times}}$ then

find $a+n$, given $a, n \in \mathbb{N}$.

Soln Let $m = 2021$ then $\text{L.H.S.} = (\underbrace{666\ldots6}_{m\text{-times}})^2 + \underbrace{888\ldots8}_{m\text{-times}}$

$$\text{L.H.S.} = \left(\frac{\cancel{6}^{\,2}}{\cancel{9}_{\,3}}(\underbrace{999\ldots9}_{m\text{-times}})\right)^2 + \frac{8}{9}(\underbrace{999\ldots9}_{m\text{-times}}) = \frac{4}{9}(10^m-1)^2 + \frac{8}{9}(10^m-1)$$

$$= \frac{4}{9}(10^m-1)\left((10^m-1)+2\right)$$

$$= \frac{4}{9}(10^m-1)(10^m+1) = \frac{4}{9}(10^{2m}-1).$$

$$= \underbrace{4444\ldots4}_{2m\text{-times}}.$$

Hence $a = 4$ & $n = 2m = 2\cdot2021$
$= 4042$

hence $a+n = 4+4042 = \boxed{4046}$ Ans

Que. Find the sum of n-terms of the series

$$\frac{1}{3}+\frac{3}{3\cdot 7}+\frac{5}{3\cdot 7\cdot 11}+\frac{7}{3\cdot 7\cdot 11\cdot 15}+\text{------}$$

Soln General Term $= T_r = \dfrac{(2r-1)}{3\cdot 7\cdot 11\cdot 15\cdot\text{-----}(4r-1)}$

$$= \frac{1}{2}\left(\frac{(4r-1)-1}{3\cdot 7\cdot 11\cdot 15\text{---}(4r-5)(4r-1)}\right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{3\cdot 7\cdot 11\text{---}(4r-5)}}_{V_r}-\underbrace{\frac{1}{3\cdot 7\cdot 11\cdot\text{-----}(4r-1)}}_{V_{r+1}}\right)$$

$$S_n = T_1+\sum_{r=2}^{n} T_r$$

$$= \frac{1}{3}+\frac{1}{2}\sum_{r=2}^{n}\left(\frac{1}{3\cdot 7\text{---}(4r-5)}-\frac{1}{3\cdot 7\text{---}(4r-1)}\right)$$

$$= \frac{1}{3}+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{3\cdot 7\cdot 11\text{---}(4n-1)}\right)$$

$$= \frac{1}{2}-\frac{1}{2}\,\frac{1}{3\cdot 7\cdot 11\cdot\text{----}(4n-1)}$$ Ans

$$\boxed{S_\infty = \frac{1}{2}}$$ Ans

Que Let $a_1 = \frac{1}{2}$, $a_{k+1} = a_k^2 + a_k$, $\forall k \geq 1$ and let

$$x_n = \frac{1}{a_1+1} + \frac{1}{a_2+1} + \frac{1}{a_3+1} + \cdots + \frac{1}{a_n+1}$$ then

find the value of $[x_{100}]$, where $[\cdot]$ denotes G.I.F.

Soln $a_1 = \frac{1}{2}$, $a_{k+1} = a_k^2 + a_k \Rightarrow a_{k+1} = a_k(a_k+1)$

$$\frac{1}{a_{k+1}} = \frac{1}{a_k \cdot (a_k+1)} = \frac{1}{a_k} - \frac{1}{a_k+1}$$

$$\frac{1}{a_k+1} = \frac{1}{a_k} - \frac{1}{a_{k+1}}$$

$k=1$

$$\frac{1}{a_1+1} = \frac{1}{a_1} - \cancel{\frac{1}{a_2}}$$

$$\frac{1}{a_2+1} = \cancel{\frac{1}{a_2}} - \cancel{\frac{1}{a_3}}$$

$$\vdots$$

$$+\ \frac{1}{a_n+1} = \cancel{\frac{1}{a_n}} - \frac{1}{a_{n+1}}$$

$$x_n = \frac{1}{a_1} - \frac{1}{a_{n+1}}$$

$$x_n = 2 - \frac{1}{a_{n+1}}$$

$$x_{100} = 2 - \frac{1}{a_{101}} \Rightarrow [x_{100}] = \left[2 - \frac{1}{a_{101}}\right]$$

Now $a_1 = \frac{1}{2}$

$$a_{k+1} = a_k^2 + a_k, \quad k \geq 1$$

$\longrightarrow a_1 < a_2 < a_3 < \cdots < a_{101}$

$k=1$

$$a_2 = a_1^2 + a_1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$k=2$

$$a_3 = a_2^2 + a_2 = \frac{3}{4} + \frac{9}{16} = \frac{21}{16} > 1$$

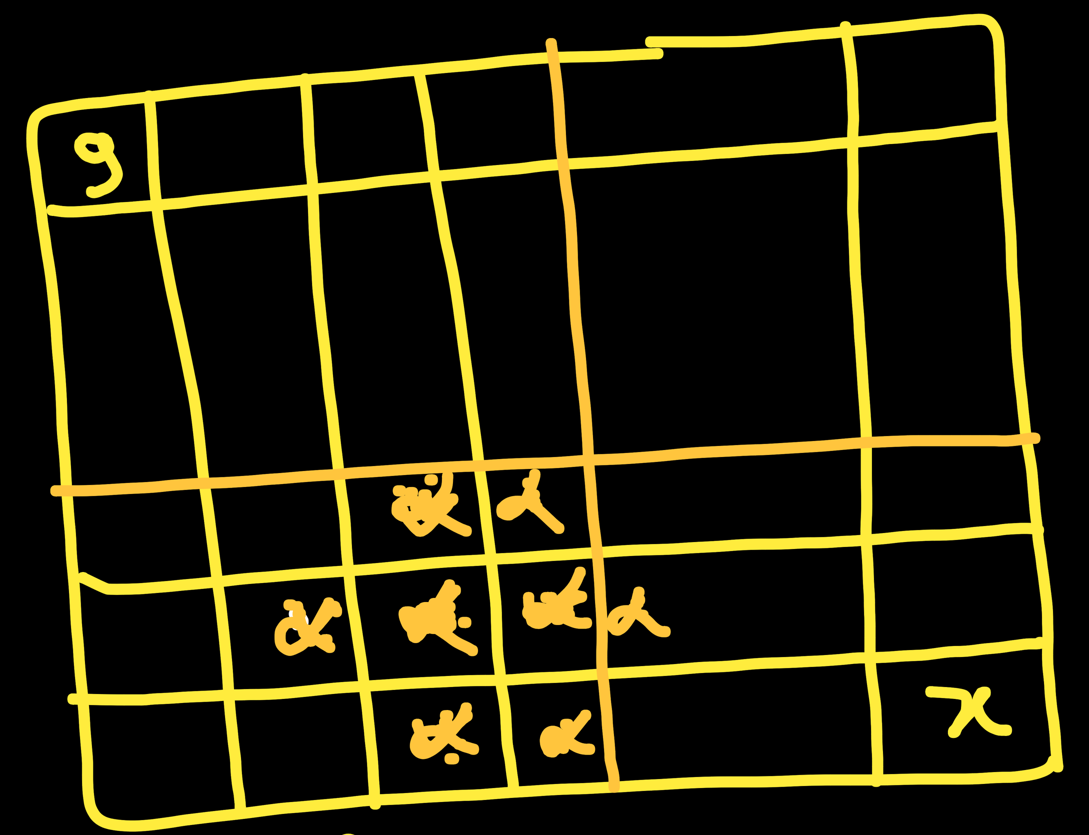
$\Rightarrow a_{101} > 1.$

Hence

$$[x_{100}] = \left[2 - \frac{1}{a_{101}}\right]$$

$$= \boxed{1} \text{ Ans}$$

Que. In the given diagram each of the box is filled by a number which is Arithmetic mean of the number in the boxes sharing a side with it. If the first square is filled with '9', then find 'x'.



grid of 2021×2021.

Soln: Concept-1

$$\min\{a_i\} \le AM\{a_i's\} \le \max.\{a_i\}$$

Concept 2 If $AM.\{a_i\} = \min.\{a_i\} = l$

then all the numbers $= l$.

Let the minimum number written in the squares is 'α' then If it is AM of a, b, c, d then all $a=b=c=d$ must be 'α' as none of these numbers can be less than 'α' so they must be $\geq \alpha$, but if any of the numbers (a, b, c, d) is $> \alpha$ then AM will also be $> \alpha \Rightarrow$ Not possible.

Now with same logic all boxes which share a side with a or b or c or d, will have same number 'α' ---- and so.

same $= \alpha$,

$\Rightarrow$ All numbers are same $= \alpha$, but 1^{st} - square contains '9'

hence $\alpha = 9$

$\Rightarrow \boxed{x = 9}$ Ans

Que Three numbers in A.P. are removed from first n-consecutive natural numbers and average of remaining numbers is found to be $\frac{43}{4}$. Find 'n' and the removed numbers if one of the removed number is perfect square.

Soln: $1, 2, 3, 4, - - - - - - -, n$

$\underbrace{3, 4, - - - - - -}$

$a, b, c \longrightarrow$ A.P.

$a-d, a, a+d$

$\hookrightarrow$ Sum $= 3a$.

minimum possible sum of removed numbers

$= 1+2+3 = 6$

max. possible sum of removed number

$= n+(n-1)+n-2 = 3n-3 = 3(n-1)$

According to question,

Sum of remaining numbers $= \frac{43}{4} \cdot (n-3)$.

$\Rightarrow$ Maximum possible sum of remaining numbers

$= \frac{n(n+1)}{2} - 6$.

$\Rightarrow$ Minimum possible sum of remaining nos.

$= \frac{n(n+1)}{2} - 3(n-1)$

Hence

$$\frac{n(n+1)}{2} - 3(n-1) \le \frac{43}{4}(n-3) \le \frac{n(n+1)}{2} - 6$$

$$2n^2 + 2n - 12n + 12 \le 43n - 129$$

$$2n^2 - 53n + 141 \le 0.$$

$$\begin{array}{r} 53 \\ 53 \\ \hline 2809 \end{array}$$

$$\frac{53 \pm \sqrt{2809 - 1128}}{4}$$

$$= \frac{53 \pm \sqrt{1681}}{4} = \frac{53 \pm 41}{4}$$

$$= \frac{94}{4}, \frac{12}{4}$$

$$= \frac{47}{2}, 3$$

$$\frac{47}{2} = 23.5$$

$$\Rightarrow n \in [3, 23.5]$$

$$43n - 129 \le 2n^2 + 2n - 24$$

$$2n^2 - 41n + 105 \ge 0.$$

$$\begin{array}{r} 41 \\ 41 \\ \hline 1681 \end{array}$$

$$\frac{41 \pm \sqrt{1681 - 840}}{4}$$

$$= \frac{41 \pm \sqrt{841}}{4} = \frac{41 \pm 29}{4}$$

$$= \frac{35}{2}, \frac{12}{4} = 3, 17.5$$

$$n \le 3 \quad n \ge 17.5$$

$$18 \le n \le 23 \Rightarrow n = 18, 19, 20, 21, 22, 23$$

Sum of remaining numbers $= \frac{43}{4}(n-3)$

$\equiv$ Integer.

$\Rightarrow$ $n-3$ must be divisible by 4.

$n = 4k+3 \Rightarrow \boxed{n = 19,\ 23}$

For $n = 19$

Sum of remaining nos.

$= \frac{43}{\cancel{4}} \times \cancel{16}\,4 = 172$

Sum of all numbers $= \frac{n(n+1)}{2} = 190$

" " re

$\Rightarrow$ $\boxed{n = 19}$

Removed Number

$a-d, \quad a, \quad a+d$

$5, 6, 7$ ✗

$4, 6, 8$ ✓

$3, 6, 9$ ✓

$2, 6, 10$ ✗

$1, 6, 11$ ✓

$(4, 6, 8)$

$(3, 6, 9)$

$(1, 6, 11)$

Que Find the sum of n-terms of the series

$$\frac{\sin(x)}{\cos(x)+\cos(2x)} + \frac{\sin(2x)}{\cos(x)+\cos(4x)} + \frac{\sin(3x)}{\cos(x)+\cos(6x)} + \text{------}$$

Soln General Term $= \dfrac{\sin(rx)}{\cos(x)+\cos(2rx)} = \dfrac{\sin(rx)}{2\cos((r+\frac{1}{2})x)\cdot\cos((r-\frac{1}{2})x)}$

$$= \frac{1}{2\sin(\frac{x}{2})}\cdot\frac{2\sin(rx)\cdot\sin(\frac{x}{2})}{2\cos((r+\frac{1}{2})x)\cdot\cos((r-\frac{1}{2})x)} = \frac{1}{4\sin(\frac{x}{2})}\left(\frac{\cos(rx-\frac{x}{2})-\cos(rx+\frac{x}{2})}{\cos((r+\frac{1}{2})x)\cdot\cos((r-\frac{1}{2})x)}\right)$$

$$= \frac{1}{4\sin(\frac{x}{2})}\left(\underbrace{\frac{1}{\cos((r+\frac{1}{2})x)}}_{T_r} - \underbrace{\frac{1}{\cos((r-\frac{1}{2})x)}}_{T_{r-1}}\right)$$

R. Sum $= S_n = \dfrac{1}{4\sin(\frac{x}{2})}\left(\dfrac{1}{\cos((n+\frac{1}{2})x)} - \dfrac{1}{\cos(\frac{x}{2})}\right)$ Ans

Que If all the terms of an A.P. are natural numbers. The sum of first 20-terms is greater than 1072 and less than 1162, find the progression if it's sixth term is 32.

Soln. $a, a+d, a+2d, \ldots \longrightarrow$ A.P. acc. to question $a, d \in I$.

$$S_{20} = \frac{20}{2}(2a+19\cdot d) = 10(2a+19d)$$

acc. to question

$$1072 \leq S_{20} \leq 1162$$

$$1072 \leq 10\cdot(2a+19d) \leq 1162.$$

$$\frac{1072}{10} \leq 64+9\cdot d \leq \frac{1162}{10}$$

$$\frac{1072-640}{10} \leq 9\cdot d \leq \frac{1162-640}{10}$$

$$\frac{432}{10} \leq 9\cdot d \leq \frac{522}{10} \Rightarrow \frac{48}{10} \leq d \leq \frac{58}{10} \Rightarrow 4.8 \leq d \leq 5.8$$

Also $T_6 = 32 = a+5\cdot d$.

$64 = 2a+10d$.

$64+9d = 2a+19d$

$\boxed{d=5} \rightarrow a = 32-25 = 7$

$\Rightarrow$ Numbers are.

$7, 12, 17, 22, \ldots$

Que Find $\displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i\cdot 3^j\cdot 3^k}$.

$(i \neq j \neq k)$

Soln Let Given Sum $= S = \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i}\cdot\frac{1}{3^j}\cdot\frac{1}{3^k}$ $(i \neq j \neq k)$

Now Consider

$$S_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i}\cdot\frac{1}{3^j}\cdot\frac{1}{3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k}\right) = \left(\frac{1}{1-\frac{1}{3}}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}.$$

$\otimes$

$$S_2 = \sum_{\substack{i=0\\(i=j=k)}}^{\infty}\frac{1}{3^i}\cdot\frac{1}{3^j}\cdot\frac{1}{3^k} = \sum_{i=0}^{\infty}\frac{1}{3^{3i}} = \frac{1}{1-\frac{1}{27}} = \frac{27}{26}.$$

$$S_3 = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i}\cdot\frac{1}{3^i}\cdot\frac{1}{3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3^k} - \sum_{k=i}\frac{1}{3^k}\right) = \frac{3}{2}\cdot\frac{9}{8} - \frac{27}{26}$$

$i = j \neq k$

$$= \frac{27}{16} - \frac{27}{26} = \frac{27\times \cancel{10}^{\,5}}{16\times\cancel{26}_{\,13}}$$

$$= \frac{27\times 5}{16\times 13}$$

$$R.\ \text{Sum} = S = S_1 - S_2 - 3 \cdot S_3$$

$$= \frac{27}{8} - \frac{27}{26} - 3 \cdot \frac{27 \cdot 5}{16 \cdot 13}$$

$$= \frac{27}{2 \times 8 \times 13}\left(26 - 8 - 15\right)$$

$$= \frac{27}{2 \times 8 \times 13} \times (3) = \boxed{\frac{81}{208}} \text{ Ans}$$