Que In a ΔABC, if $a^2 + b^2 = 101c^2$, then find the value of

$$\frac{\cot(C)}{\cot(A) + \cot(B)}$$

Formula

$$\boxed{\cot(A) = \frac{b^2 + c^2 - a^2}{4\Delta}}$$

$$= \frac{\cos(A)}{\sin(A)} = \frac{b^2 + c^2 - a^2}{2bc\sin(A)}$$

$$\Delta = \frac{1}{2} bc \sin(A)$$

Sol^n Given $a^2 + b^2 = 101 \cdot c^2$

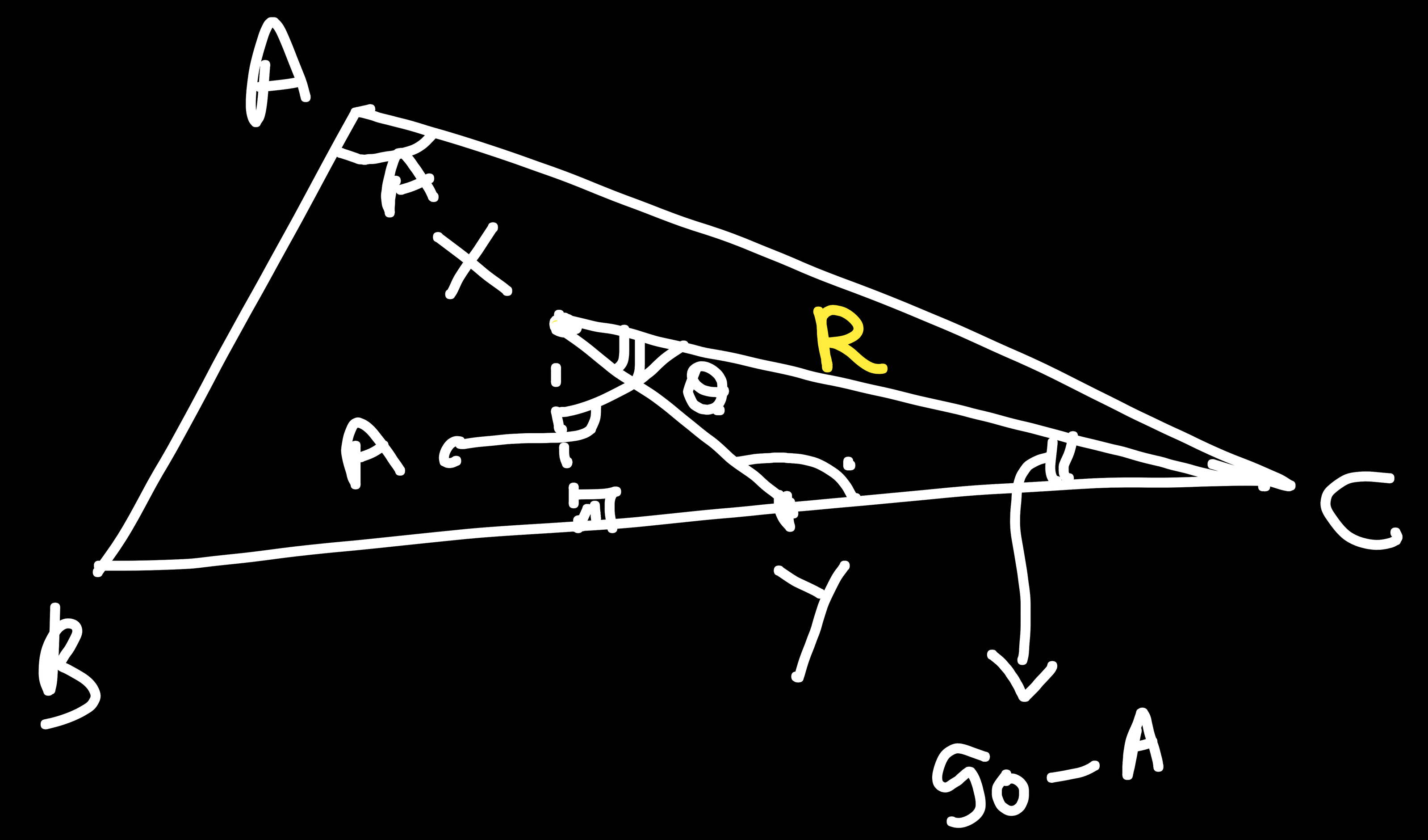
$$\frac{\cot(C)}{\cot(A) + \cot(B)} = \frac{\left(\frac{a^2 + b^2 - c^2}{\cancel{4\Delta}}\right)}{\left(\frac{\cancel{b^2} + c^2 - \cancel{a^2}}{\cancel{4\Delta}}\right) + \left(\frac{\cancel{a^2} + c^2 - \cancel{b^2}}{\cancel{4\Delta}}\right)}$$

$$= \frac{a^2 + b^2 - c^2}{2c^2} = \frac{101c^2 - c^2}{2c^2}$$

$$= \boxed{50} \text{ Ans}$$

Que Let X is the circumcentre of ΔABC and Y is a point on the side BC. Find the circumradius of ΔCXY, when Y tends to coincide with 'C'.

Soln:



Let circumradius of ΔCXY is R'

In ΔXYC

$$XC = 2R' \sin(\angle XYC)$$

$$R = 2R' \cdot \sin(\pi - (\theta + 90^\circ - A))$$

$$R = 2R' \sin(\theta + 90^\circ - A) = 2R' \cos(\theta - A)$$

$$R = 2R'$$

$$R' = \frac{R}{2\cos(A-\theta)}.$$

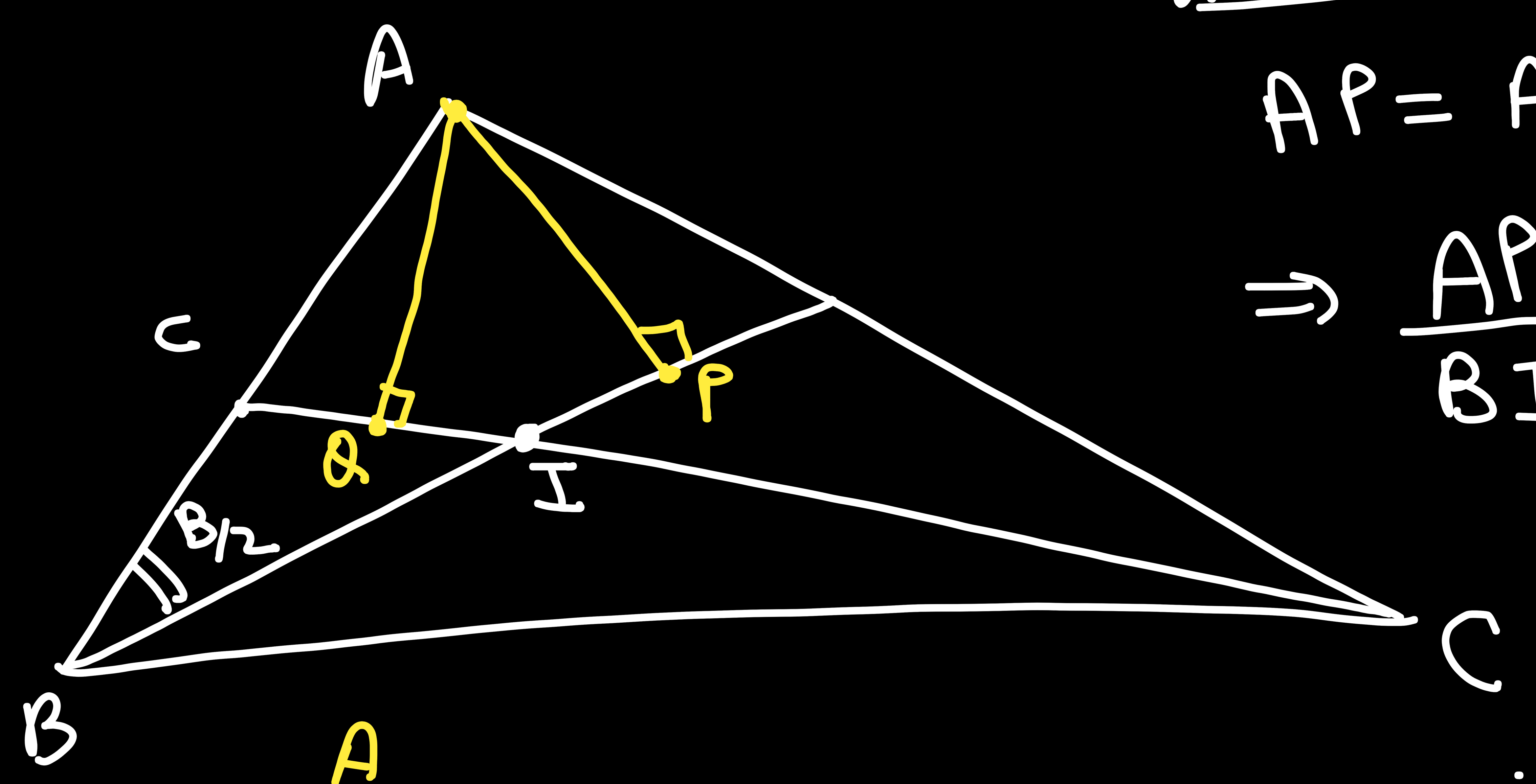
When $Y \longrightarrow C$

then $\theta \longrightarrow 0$

$$\boxed{R' = \frac{R}{2\cos(A)}}$$

Que Let ABC be a triangle with Incentre as I and P, Q be the feet of the perpendiculars from A to BI and CI respectively, If $A = 60^\circ$ then find the value of $\dfrac{AP}{BI} + \dfrac{AQ}{CI}$.

Soln:



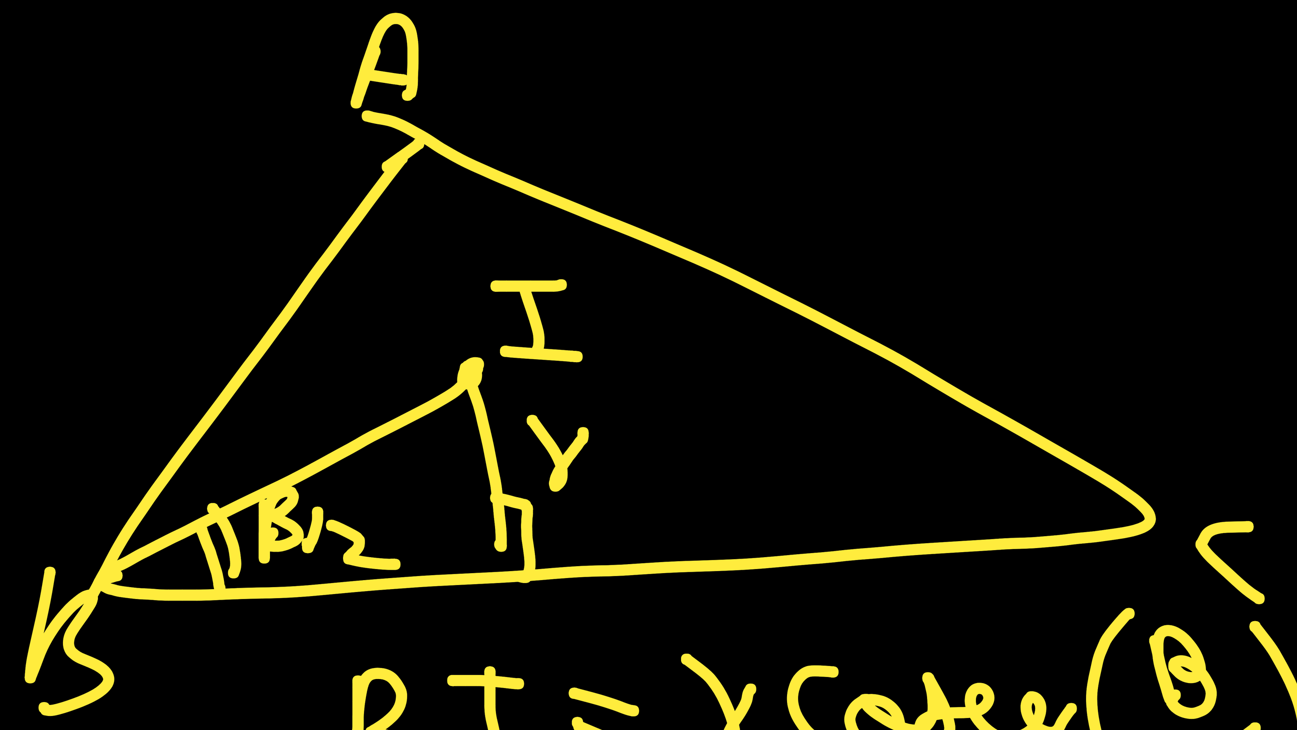
In $\triangle ABP$.

$$AP = AB \cdot \sin\left(\frac{B}{2}\right) = c \sin\left(\frac{B}{2}\right)$$

$$\Rightarrow \frac{AP}{BI} = \frac{c \cdot \sin\left(\frac{B}{2}\right)}{r \cdot \csc\left(\frac{B}{2}\right)} = \frac{c \cdot \sin^2\left(\frac{B}{2}\right)}{r} = \frac{c}{2r} \cdot (1 - \cos(B))$$

$$\frac{AP}{BI} = \frac{c - c\cos(B)}{2r}$$

similarly $\dfrac{AQ}{CI} = \dfrac{b - b\cos(C)}{2r}$



$BI = r\,\mathrm{Cosec}\left(\frac{B}{2}\right)$

$$\frac{AP}{BI} + \frac{AQ}{CI} = \frac{b + c - (c\cos(B) + b\cos(C))}{2r} = \frac{b + c - a}{2r}$$

$$= \frac{\cancel{2}(s - a)}{\cancel{2}r} = \frac{s - a}{r}$$

$$\frac{AP}{BI} + \frac{AQ}{CI} = \frac{s-a}{r}$$

$$= \cot\left(\frac{A}{2}\right)$$

$$= \cot(30^\circ)$$

$$= \boxed{\sqrt{3}}$$ Ans

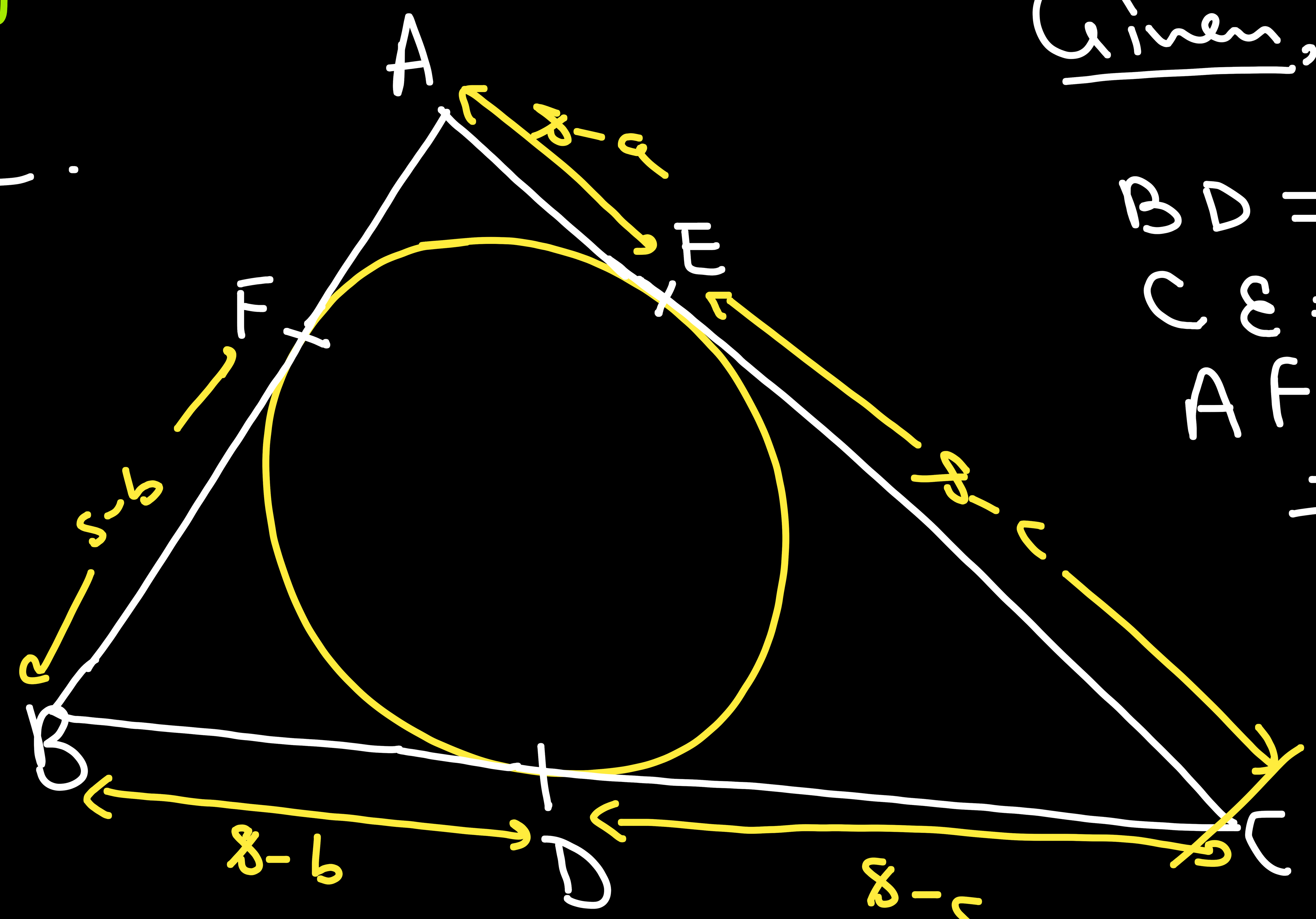
Formula

$$r = (s-a)\tan\left(\frac{A}{2}\right)$$

$$\Rightarrow \frac{s-a}{r} = \cot\left(\frac{A}{2}\right)$$

Que In ΔABC, incircle touches side BC, CA and AB at D, E & F respectively. If the radius of incircle is 4 units and if BD, CE and AF are consecutive integers, find the length of sides of the triangle ABC.

Sln.



Given, $r = 4$.

$$BD = n = s-b$$
$$CE = n+1 = s-c$$
$$AF = n+2 = s-a$$
$$+$$
$$3n+3 = 3s-2s$$

$$s = 3(n+1)$$

$$r = 4 \Rightarrow \frac{\Delta}{s} = 4 \Rightarrow \Delta^2 = 16s^2$$

$$\Rightarrow \cancel{s}\cdot(s-a)\cdot(s-b)(s-c) = 16\, s^{\cancel{2}}$$

$$(n+2)\cdot n\cdot \cancel{(n+1)} = 16\cdot 3\cdot\cancel{(n+1)}$$

$$n^2+2n = 48$$

$$\hookrightarrow n = 6, \cancel{-8}$$

$$s-a = 8$$
$$s-b = 6$$
$$s-c = 7$$

$(s-a)+(s-b) \xrightarrow{+} c = 14$

$(s-b)+(s-c) \xrightarrow{+} a = 13$

$(s-a)+(s-c) \xrightarrow{+} b = 15$

$$\boxed{a=13,\ b=15,\ c=14}$$

for equality

$$\frac{p_1}{a} = \frac{p_2}{b} = \frac{p_3}{c} = \lambda \text{ (say)}$$

$$p_1 = a\lambda, \quad p_2 = b\lambda, \quad p_3 = c\lambda.$$

We also have

$$ap_1 + bp_2 + cp_3 = 2\Delta \Rightarrow a^2\lambda + b^2\lambda + c^2\lambda = 2\Delta$$

$$\lambda = \frac{2\Delta}{\Sigma a^2}.$$

Required values of p_1, p_2, p_3 are

$$p_1 = \frac{2a\Delta}{a^2+b^2+c^2}, \quad p_2 = \frac{2b\Delta}{a^2+b^2+c^2}$$

$$\& \quad p_3 = \frac{2c\Delta}{a^2+b^2+c^2}.$$

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{b} = b_1\hat{i} + b_2\hat{j} + a_3\hat{k}$$

$$\bar{a}\cdot\bar{b} = \Sigma a_1 b_1 = ab\cos(\theta)$$

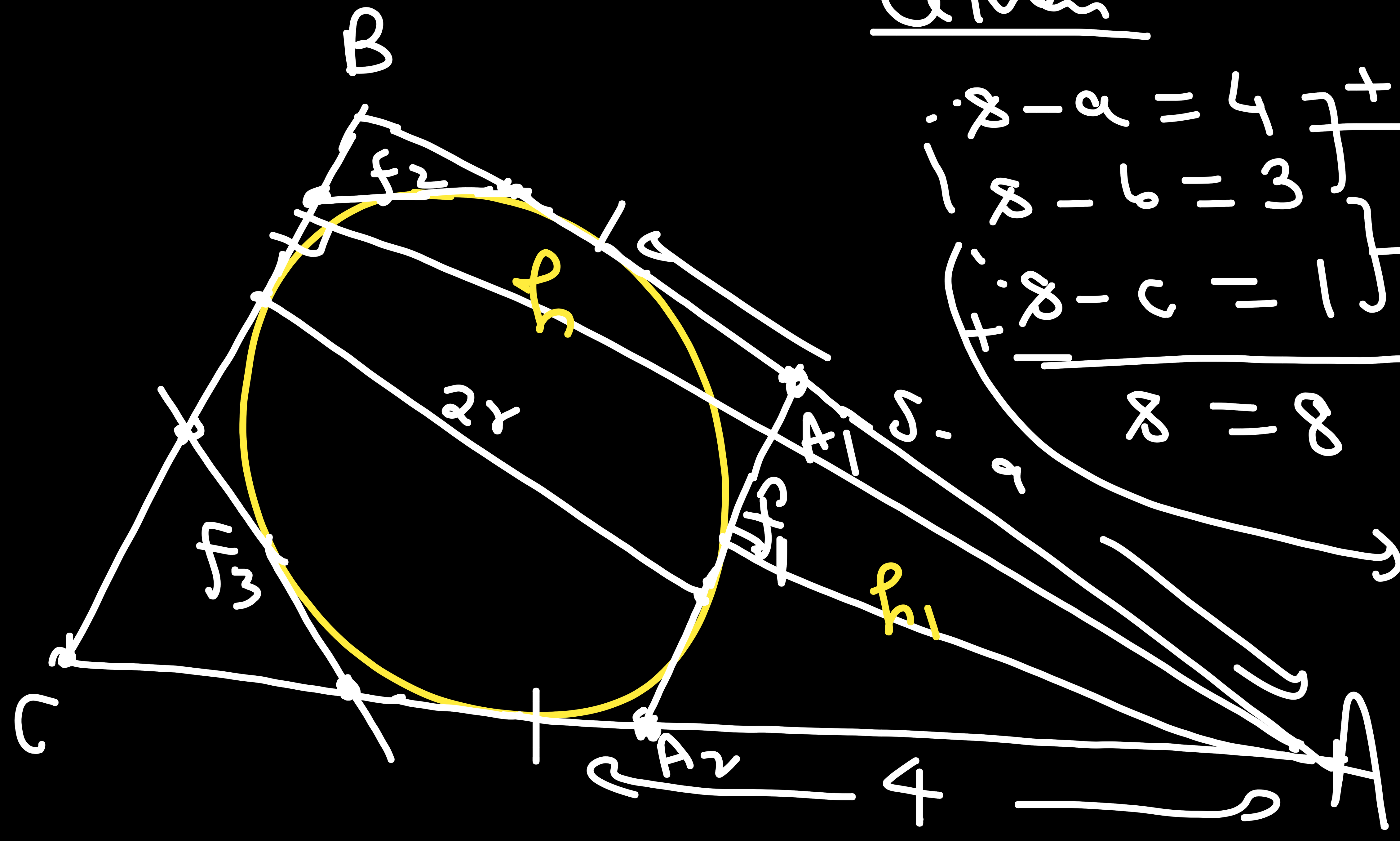
$$\Downarrow$$

$$(\Sigma a_1 b_1)^2 = a^2 b^2 \cos^2(\theta) \leq a^2 \cdot b^2$$

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2) \cdot (b_1^2 + b_2^2 + b_3^2).$$

One The lenght of the tangents drawn from A, B, C to the incircle of ΔABC are 4, 3, 1 respectively. If lengths of part of the tangents within the triangle which are drawn parallel to the 3-sides BC, CA and AB of the ΔABC to the circle, are f_1, f_2, f_3 respectively then find $f_1+f_2-f_3$.

Soln:



Given

$\because s-a=4$
$\quad s-b=3$
$\because s-c=1$

$s=8$

$\rightarrow c=7$
$\rightarrow a=4$
$\rightarrow b=5$

Clearly $\Delta AA_1A_2 \sim \Delta ABC$.

hence $\dfrac{f_1}{BC}=\dfrac{h_1}{h}=\dfrac{h-2r}{h}=1-\dfrac{2r}{h}$

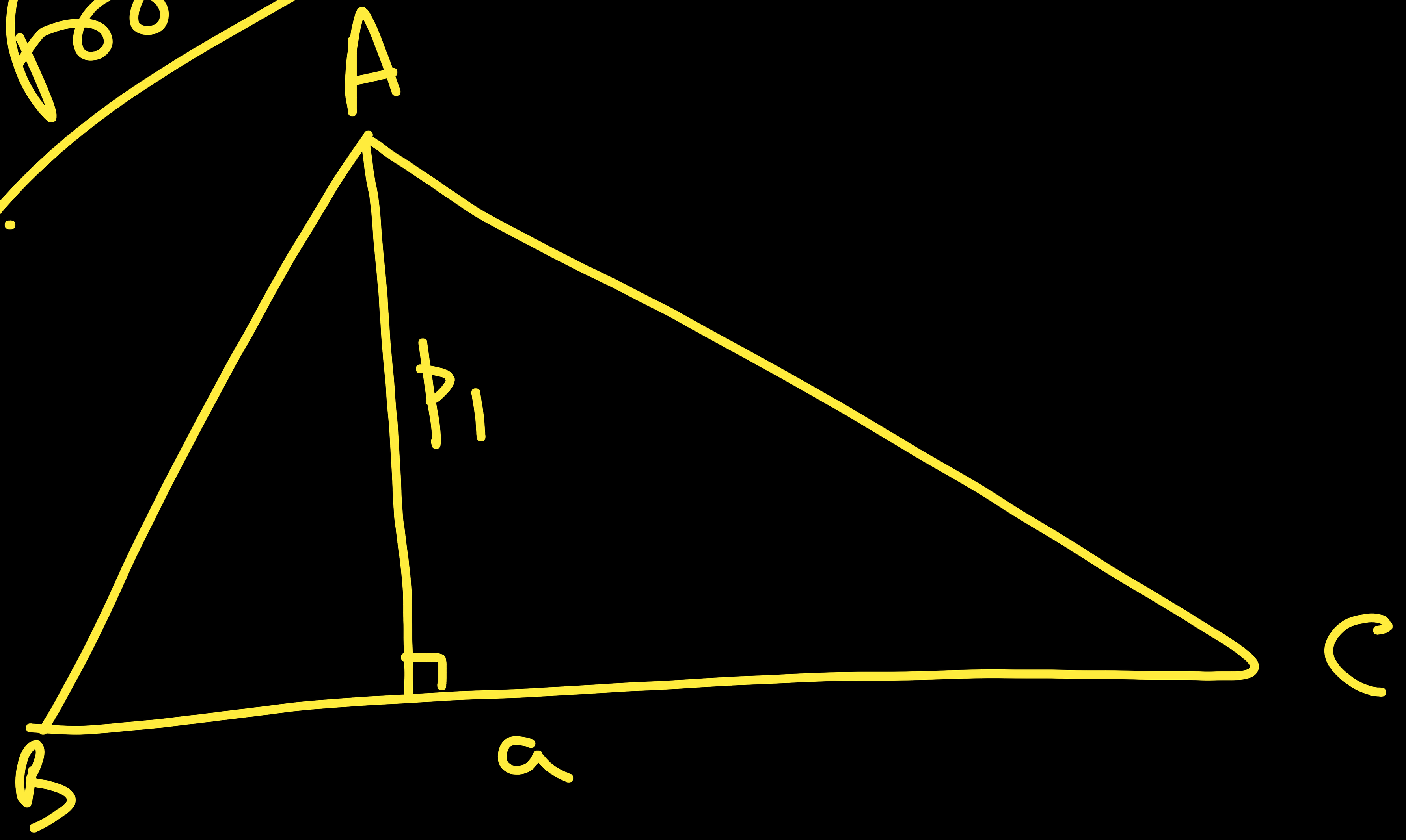
$f_1=a\cdot\left(1-\dfrac{2r}{h}\right)=a\cdot\left(1-\dfrac{2\cdot(\Delta/s)}{\left(\frac{2\Delta}{a}\right)}\right)=a\left(1-\dfrac{a}{s}\right)$

$f_1=\dfrac{a(s-a)}{s}=\dfrac{4\times4}{8}=2.$

$f_2=\dfrac{b(s-b)}{s}=\dfrac{5\times3}{8}=\dfrac{15}{8}$

$f_3=\dfrac{c(s-c)}{s}=\dfrac{7\times1}{8}=\dfrac{7}{8}$

formula



$$\Delta = \frac{1}{2} \cdot p_1 \cdot a$$

$$p_1 = \frac{2\Delta}{a}$$

$$f_1 = 2$$
$$f_2 = \frac{15}{8}$$
$$f_3 = \frac{7}{8}$$

$$f_1 + f_2 - f_3$$

$$= 2 + \frac{15}{8} - \frac{7}{8}$$

$$= \boxed{3}$$ Ans

Que Find 'k' if $\sum_{n=1}^{10} \sum_{m=1}^{10} \tan^{-1}\left(\frac{m}{n}\right) = k \cdot \pi .$

Soln:

$$\text{L.H.S.} = \sum_{n=1}^{10} \sum_{m=1}^{10} \tan^{-1}\left(\frac{m}{n}\right)$$

$$= (\quad) + \Big(\underbrace{\tan^{-1}\left(\tfrac{3}{4}\right) + \tan^{-1}\left(\tfrac{4}{3}\right)}_{=\frac{\pi}{2}}\Big) + \text{—} \text{———} + (\text{—})$$

$$= 50 \cdot \frac{\pi}{2}$$

$$= 25\pi$$

$$\boxed{k = 25}$$

Que Find the integral values of K for which the system of equations $\cos^{-1}(x) + (\sin^{-1}(y))^2 = \frac{K\pi^2}{4}$ & $(\sin^{-1}(y))^2 \cdot \cos^{-1}(x) = \frac{\pi^4}{16}$, has real solutions.

Soln:

$\frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{2\pi^2}{4}$

$\cos^{-1}(x) + (\sin^{-1}(y))^2 = \frac{K\pi^2}{4}$

$\cos^{-1}(x) \cdot (\sin^{-1}(y))^2 = \frac{\pi^4}{16}$

Let $\cos^{-1}(x) = t$

$\cos^{-1}(x) \in [0, \pi]$

$t + (\sin^{-1}(y))^2 = \frac{K\pi^2}{4}$

$t \cdot (\sin^{-1}(y))^2 = \frac{\pi^4}{16}$

$y = 1$

$t_{min} = \frac{\left(\frac{\pi^4}{16}\right)}{\left(\frac{\pi}{2}\right)^2} = \frac{\pi^2}{4}$

$t \in \left[\frac{\pi^2}{4}, \pi\right]$

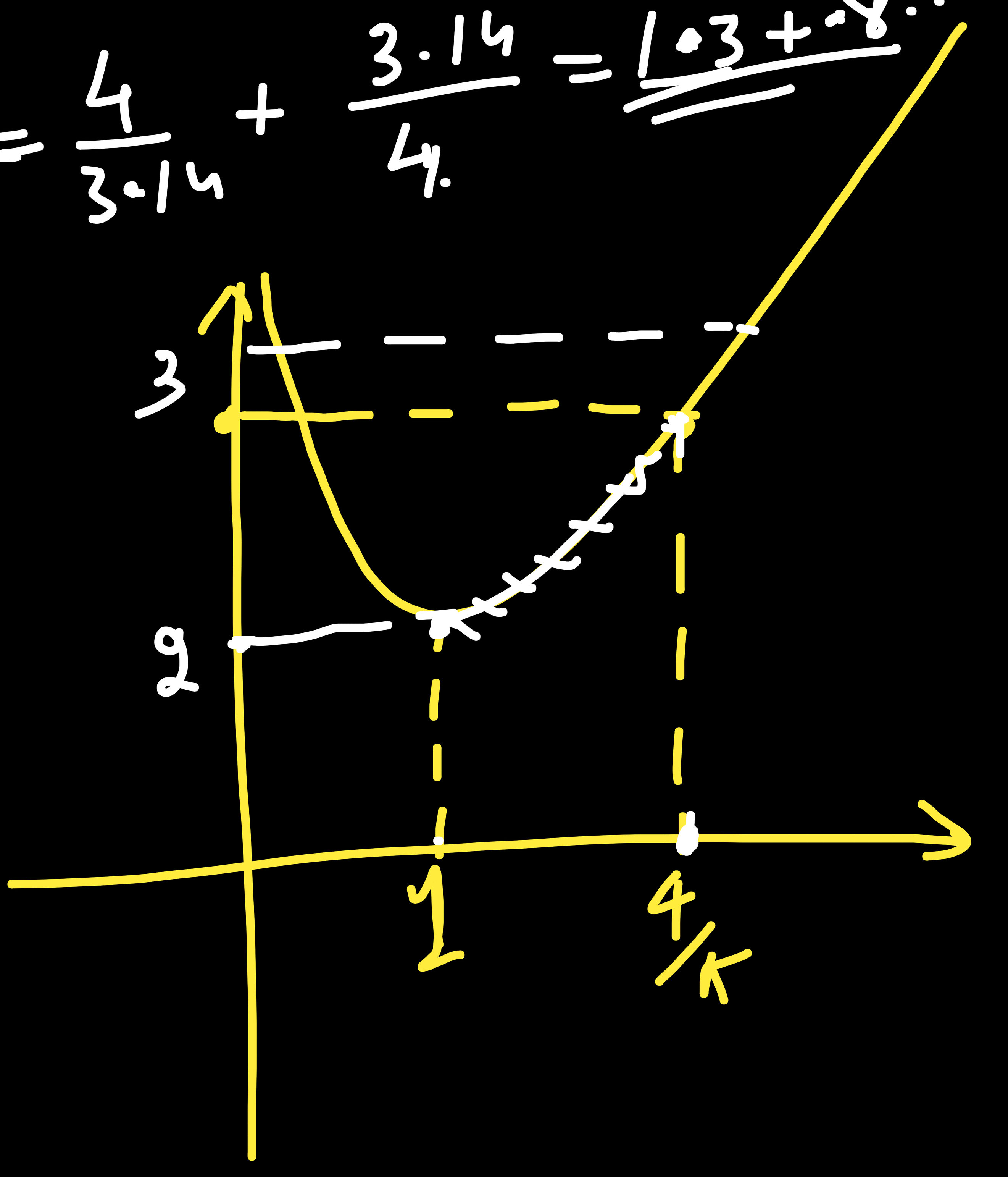
$t + \frac{\left(\frac{\pi^4}{16}\right)}{t} = K \cdot \frac{\pi^2}{4}$.

$K = \boxed{\frac{4}{\pi^2} \cdot t} + \frac{\pi^2}{4t} = z + \frac{1}{z}$

$\frac{4t}{\pi^2}$

$z \in \left[1, \frac{4}{\pi}\right]$

$z + \frac{1}{z}\Big)_{z=\frac{4}{\pi}} = \frac{4}{\pi} + \frac{\pi}{4} = \frac{4}{3.14} + \frac{3.14}{4} = 1.3 + .8..$



$\boxed{K = 2}$

$K = 2$

$\cos^{-1}(x) = \frac{\pi^2}{4} \Rightarrow x = \cos\left(\frac{\pi^2}{4}\right).$

& $\sin^{-1}(y) = \pm\frac{\pi}{2}$

$y = \pm 1$

Que Find positive integral solutions x and y of the equation $\rightarrow$

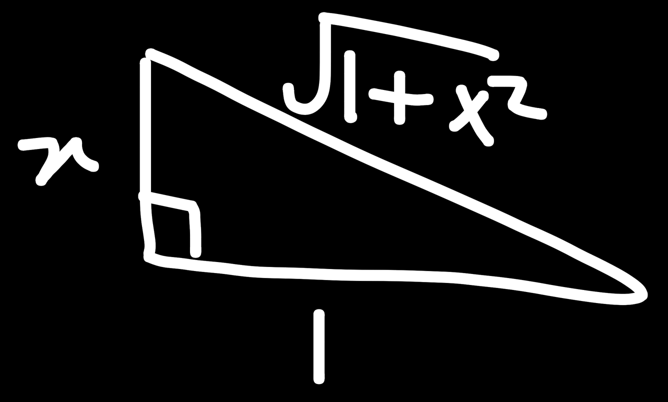
$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\tan^{-1}(3).$$

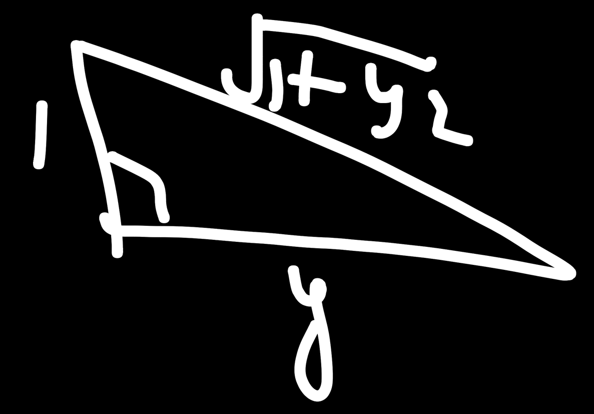
Soln

$$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\tan^{-1}(3).$$

$$\tan^{-1}(x)+\tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}(3)$$

$$\tan^{-1}(x)=\tan^{-1}(3)-\tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}\left(\frac{3-\frac{1}{y}}{1+\frac{3}{y}}\right)=\tan^{-1}\left(\frac{3y-1}{y+3}\right)$$





$$x=\frac{3y-1}{y+3}=\frac{3(y+3)-10}{y+3}=3-\frac{10}{y+3}$$

$y+3=5, 10$

$y=2, 7$

$y=2 \Rightarrow x=1$ or $y=7$ & $x=2$

$(x,y)=(2,7)$ & $(1,2)$

Ans

Que If a and b are positive quantities, such that

$a_1 = \frac{a+b}{2}$, $b_1 = \sqrt{a_1 \cdot b}$, $a_2 = \frac{a_1+b_1}{2}$, $b_2 = \sqrt{a_2 b_1}$ and so on...

then find $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ in terms of a and b given $(b>a)$.

Soln. Let $a = b\cos(\theta) \Rightarrow \cos(\theta) = \frac{a}{b}$

$$a_1 = \frac{a+b}{2} = \frac{b(1+\cos(\theta))}{2} = b\cos^2\left(\frac{\theta}{2}\right)$$

$$b_1 = \sqrt{a_1 \cdot b} = \sqrt{b\cos^2\left(\frac{\theta}{2}\right)\cdot b} = b\cos\left(\frac{\theta}{2}\right)$$

$$a_2 = \frac{a_1+b_1}{2} = \frac{b\cos^2\left(\frac{\theta}{2}\right)+b\cos\left(\frac{\theta}{2}\right)}{2}$$

$$= \frac{b\cos\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right)+1\right)}{2}$$

$$a_2 = b\cos\left(\frac{\theta}{2}\right)\cdot\cos^2\left(\frac{\theta}{4}\right).$$

$$b_2 = \sqrt{a_2 b_1} = \sqrt{b\cos\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{4}\right)\cdot b\cos\left(\frac{\theta}{2}\right)}$$

$$b_2 = b\cos\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{4}\right)$$

$$b_n = b\cdot\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{2^3}\right)\cdots\cos\left(\frac{\theta}{2^n}\right).$$

$$b_n = b\cdot\frac{\sin(\theta)}{2^n\cdot\sin\left(\frac{\theta}{2^n}\right)} \Rightarrow \lim_{n\to\infty} b_n = \frac{b\sin(\theta)}{\theta\cdot\frac{\sin\left(\frac{\theta}{2^n}\right)}{\left(\frac{\theta}{2^n}\right)}}$$

$$1$$

$$b_n = \frac{b\cdot\sin(\theta)}{\theta} = \frac{b\cdot\sqrt{1-\frac{a^2}{b^2}}}{\cos^{-1}\left(\frac{a}{b}\right)}$$

$$b_n = \frac{\sqrt{b^2 - a^2}}{\cos^{-1}\left(\frac{a}{b}\right)} \cdot \text{Ans}$$

$$a_1 = b\cos^2\left(\frac{\theta}{2}\right)$$

$$a_2 = b\cos\left(\frac{\theta}{2}\right)\cdot\cos^2\left(\frac{\theta}{4}\right)$$

$$a_3 = b\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cdot\cos^2\left(\frac{\theta}{2^3}\right)$$

|

|

|

$$a_n = b\cos\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{4}\right) - - - \cos\left(\frac{\theta}{2^{n-1}}\right)\cdot\underline{\cos^2}\left(\frac{\theta}{2^n}\right).$$

$$a_n = b_n \cdot \cos\left(\frac{\theta}{2^n}\right)$$

$n\to\infty$

$$\underset{n\to\infty}{\text{lt}}\, a_n = \underset{n\to\infty}{\text{lt}}\, b_n \cdot \cos(0) = \underset{n\to\infty}{\text{lt}}\, b_n = \frac{\sqrt{b^2 - a^2}}{\cos^{-1}\left(\frac{a}{b}\right)} \cdot \text{Ans}$$