Que Let a and b are positive real numbers such that $a-b=2$, then find the least possible value of 'L' such that

$$\sqrt{x^2+ax}-\sqrt{x^2+bx}<L,\quad x>0.$$

Soln $a, b \in R^+$
$a-b=2.$

Let $f(x)=\sqrt{x^2+ax}-\sqrt{x^2+bx}$

$$=\sqrt{x(x+a)}-\sqrt{x(x+b)}=\sqrt{x}\left(\sqrt{x+a}-\sqrt{x+b}\right)$$

$$=\frac{\sqrt{x}\,(a-b)}{\sqrt{x+a}+\sqrt{x+b}}=\frac{2\cdot\sqrt{x}}{\sqrt{x+a}+\sqrt{x+b}}$$

$$=\frac{2}{\sqrt{1+\frac{a}{x}}+\sqrt{1+\frac{b}{x}}}$$

$\uparrow$ ing, $D_f \equiv (0,\infty)$

$x\uparrow$

$\Rightarrow \frac{1}{x}\downarrow \Rightarrow \frac{a}{x}, \frac{b}{x}\downarrow$

$\Downarrow$

$\sqrt{1+\frac{a}{x}}, \sqrt{1+\frac{b}{x}}\downarrow$

$\Downarrow$

$\frac{1}{\sqrt{1+\frac{a}{x}}+\sqrt{1+\frac{b}{x}}}\uparrow \Leftarrow \sqrt{1+\frac{a}{x}}+\sqrt{1+\frac{b}{x}}\downarrow$

$$f(x) = \sqrt{x(x+a)} - \sqrt{x(x+b)}$$

$$= \frac{2}{\sqrt{1+\frac{a}{x}} + \sqrt{1+\frac{b}{x}}}$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^+} f(x) = 0. \\ \lim\limits_{x \to \infty} f(x) = 1 \end{array}\right\} \Rightarrow$$ As $f(x)$ is $\uparrow$ing & continuous fn.

$$R_{f(x)} \equiv (0, 1).$$

$$f(x) < 1 \Rightarrow \boxed{L_{min.} = 1}$$

Que Let $f(x) = x^2 + kx$, where $k \in R$, then find all possible values of 'k' such that $f(x) = 0$ and $f(f(x)) = 0$ has same solution set of real solution.

Sol $f(x) = x(x+k) \Rightarrow f(x) = 0 \Rightarrow x(x+k) = 0 \Rightarrow x \in \{-k, 0\}$

$f(f(x)) = f(x)(f(x)+k) \Rightarrow f(f(x)) = 0 \Rightarrow f(x)\cdot(f(x)+k) = 0$

$\Downarrow$

$f(x) = 0$ or $f(x) + k = 0$.

$\Downarrow$ $x \in \{-k, 0\}$

$\Downarrow$ This equation should not add any extra real solution.

C-1 $f(x) + k = 0 \rightarrow$ Imaginary Soln.

$\hookrightarrow x^2 + kx + k = 0 \Rightarrow D < 0 \Rightarrow k^2 - 4k < 0$

$\boxed{k \in (0, 4)}$

C-2 $f(x) + k = 0 \longrightarrow$ Soln is $x = 0$

$\hookrightarrow x^2 + kx + k = 0 \xrightarrow{x=0} k = 0 \Rightarrow x^2 = 0$

$\hookrightarrow x = 0, 0$

C-3 $f(x) + k = 0 \rightarrow x = -k$

$\hookrightarrow k^2 - k^2 + k = 0 \Rightarrow k = 0$

$\Rightarrow$ Required values of k are

$\boxed{k \in [0, 4)}$ Ans

Que Find the range of the function $f(x) = (\tan^{-1}(x))^2 + \frac{2}{\sqrt{1+x^2}}$

Sol

$$f(x) = (\tan^{-1}(x))^2 + \frac{2}{\sqrt{1+x^2}}, \quad D_f \equiv R$$

$\tan^{-1}(x) = t \Rightarrow t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$x = \tan(t)$

$$g(t) = t^2 + \frac{2}{\sqrt{1+\tan^2(t)}} = t^2 + 2\cdot|\cos(t)| = t^2 + 2\cos(t) \Rightarrow$$

Even function hence range for $t \in \left(-\frac{\pi}{2}, 0\right]$ & for $t \in \left[0, \frac{\pi}{2}\right)$ will be the same hence we will find range only for $t \in \left[0, \frac{\pi}{2}\right)$

$t \in \left[0, \frac{\pi}{2}\right)$

$g(t) = t^2 + 2\cos(t)$

$g'(t) = 2t - 2\sin(t) = 2(t - \sin(t)) > 0$

$\Rightarrow g(t)$ ↑ing fn. Range of $g(t) \equiv \left[g(0), \lim_{t \to \pi/2} g(t)\right)$

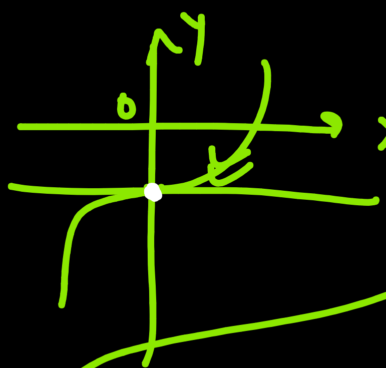
$$\equiv \left[2, \frac{\pi^2}{4}\right)$$

Ans

Que. Solve the equation $x^3 - [x] = 5$, where $[x]$ is G.I.F.

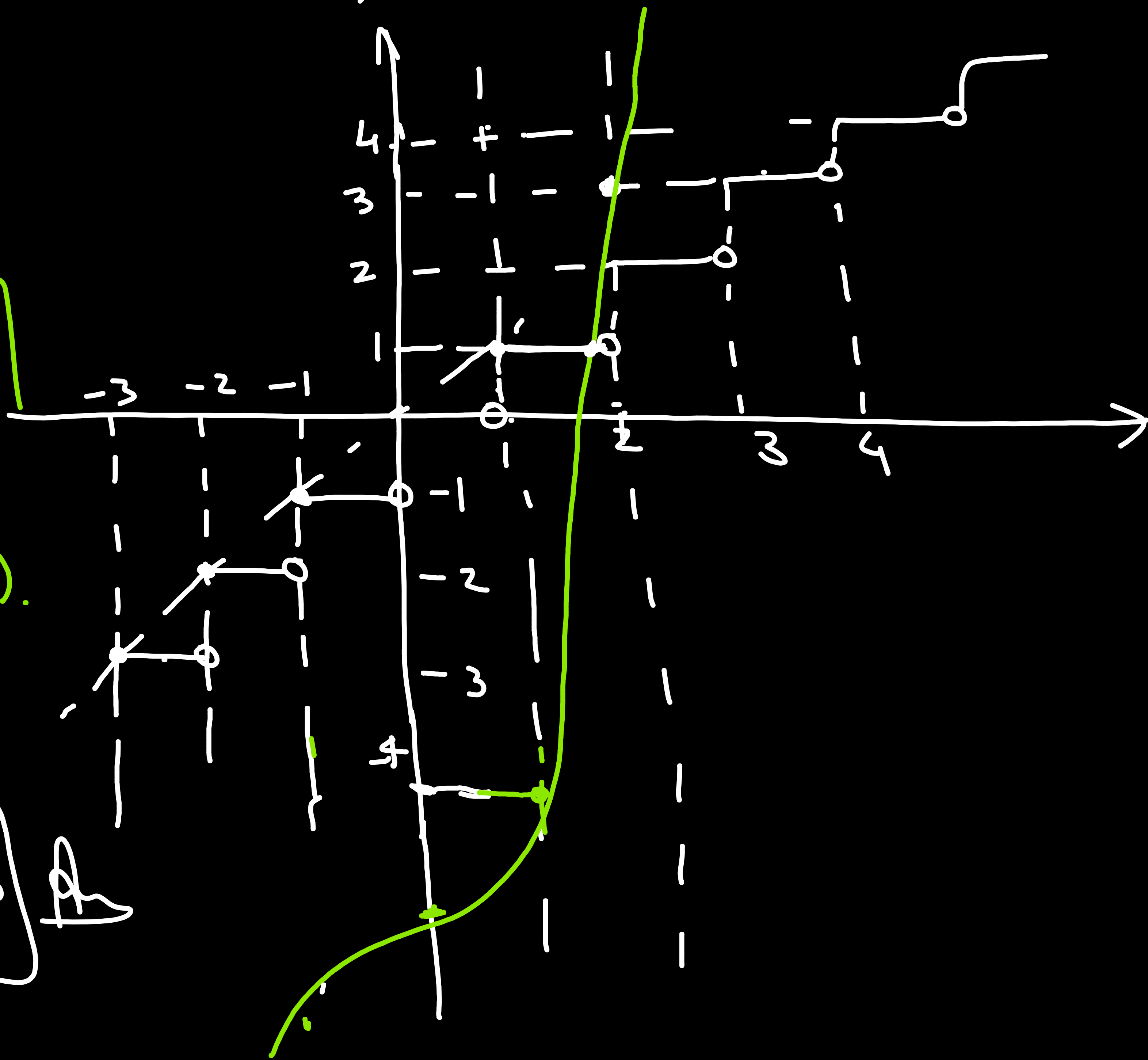
Sol^n $x^3 - [x] = 5$

$\Rightarrow \underbrace{x^3 - 5}_{y} = \underbrace{[x]}_{y}$



from graph

Solution will lie in $(1, 2)$.



$\Rightarrow x^3 - [x] = 5$ [x]=1

$x^3 = 6 \Rightarrow x = 6^{1/3}$

$\boxed{x = 6^{1/3}}$ Ans

Que. Let $f(x)$ is defined for $\forall x \in R$ and satisfy $f(x+y) = f(x) + 2y^2 + kxy$, $\forall x, y \in R$, where k is given constant, and if $f(1) = 2$ and $f(2) = 8$ find $f(x)$ and hence find $f(x+y) \cdot f\left(\frac{1}{x+y}\right)$.

Sol

$$f(x+y) = f(x) + 2y^2 + kxy$$

$x=0$

$$f(y) = f(0) + 2y^2 + 0$$

$$f(y) = c + 2y^2$$

put $y=1$ & $f(1)=2$

$$2 = c + 2 \Rightarrow c = 0.$$

$$\Rightarrow f(y) = 2y^2$$

$$f(x) = 2x^2$$

$$2(x+y)^2 = 2x^2 + 2y^2 + kxy \Rightarrow \boxed{k=4}$$

Now

$$f(x+y) \cdot f\left(\frac{1}{x+y}\right) = 2 \cdot \cancel{(x+y)^2} \cdot 2 \cdot \cancel{\left(\frac{1}{x+y}\right)^2}$$

$$= \boxed{4}$$

Ans

Que. The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right]+\left[\frac{4}{x}\right]=5$ (where [.] denotes G.I.F.) belongs to the interval $\left(a, \frac{b}{c}\right]$ where $a, b, c \in N$ and b, c are coprime. Find $\Sigma a + \Pi a$.

Soln.: Clearly $x>0$.

$\frac{1}{x}=t>0$

$\left[\frac{3}{x}\right]+\left[\frac{4}{x}\right]=5$

$[3t]+[4t]=5$

1 & 4 $\Rightarrow$ $3t \in [1,2)$ & $4t \in [4,5)$

$t \in \left[\frac{1}{3}, \frac{2}{3}\right)$ & $t \in \left[1, \frac{5}{4}\right]$

$\cap \rightarrow \phi$

OR

2 & 3 $\Rightarrow 3t \in [2,3)$ & $4t \in [3,4)$

$t \in \left[\frac{2}{3}, 1\right)$ & $t \in \left[\frac{3}{4}, 1\right)$

$\downarrow \cap$

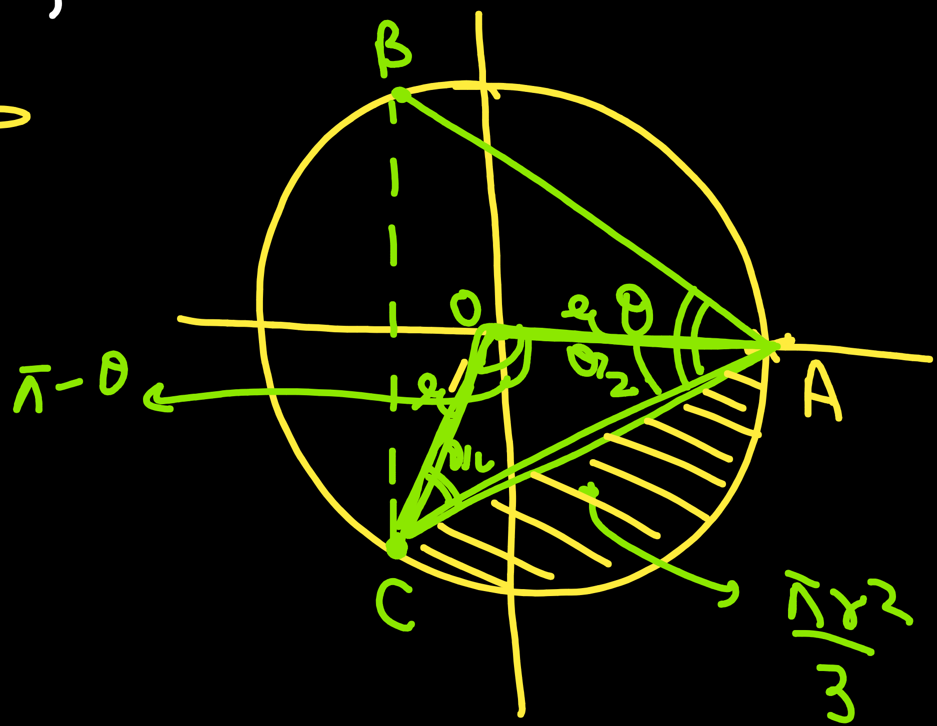
$t \in \left[\frac{3}{4}, 1\right) \Rightarrow \frac{1}{x} \in \left[\frac{3}{4}, 1\right) \Rightarrow x \in \left(1, \frac{4}{3}\right]$

Compare $\rightarrow a=1, b=4$ & $c=3$.

$\Sigma a + \Pi a = 8+12 = \boxed{20}$

Que Let A is a point on the circumference of a circle. chords AB and AC divides the area of the circle into three equal parts. If the $\angle BAC$ is the root of the equation $f(x)=0$, then find $f(x)$.

Sol



$$\text{Shaded Area} = \frac{\pi \cancel{r^2}}{3} = \frac{1}{2}\cdot\cancel{r^2}\cdot(\pi-\theta) - \frac{1}{2}\cdot\cancel{r}\cdot\cancel{r}\cdot\sin(\pi-\theta)$$

$$\frac{2\pi}{3} = \pi - \theta - \sin(\theta)$$

$$\sin(\theta) + \theta - \frac{\pi}{3} = 0.$$

$$f(x) = \sin(x) + x - \frac{\pi}{3}.$$

Que Find all functions $f : N \to N$ for which

$$f^3(1) + f^3(2) + f^3(3) + \cdots + f^3(n) = \left(f(1) + f(2) + f(3) + \cdots + f(n)\right)^2, \ \forall n \in N.$$

Sol

$$f^3(1) + f^3(2) + \cdots + f^3(n) = \left(f(1) + f(2) + \cdots + f(n)\right)^2 \quad \text{—— (1)}$$

$n \to n+1$

$$f^3(1) + f^3(2) + \cdots + f^3(n) + f^3(n+1) = \left(f(1) + f(2) + \cdots + f(n) + f(n+1)\right)^2.$$

$$= \left(f(1) + f(2) + \cdots + f(n)\right)^2 + f^2(n+1)$$

$$+ 2f(n+1)\cdot\left(f(1) + \cdots + f(n)\right) \quad \text{—— (2)}$$

Now (2) $-$ (1) $\Rightarrow$

$$f^{\cancel{3}\,2}(n+1) = 2\cancel{f(n+1)}\cdot\left(f(1) + f(2) + \cdots + f(n)\right) + f^{\cancel{2}}(n+1)$$

$n \to n+1$

$$f^2(n+1) = 2\left(f(1) + f(2) + \cdots + f(n)\right) + f(n+1)$$

$$f^2(n+2) = 2\left(f(1) + f(2) + \cdots + f(n+1)\right) + f(n+2)$$

$$f^2(n+2) - f^2(n+1) = \cancel{2}f(n+1) + f(n+2) - \cancel{f(n+1)} = f(n+1) + f(n+2)$$

$$f^2(n+2) - f^2(n+1) = f(n+2) + f(n+1)$$

$$\Rightarrow f(n+2) - f(n+1) = 1 \Rightarrow f(1), f(2), f(3), \text{---} \quad \underline{\underline{A.P.}}$$

with $\underline{\text{common diff}} = 1$.

$\underline{\underline{\text{Now}}}$ $f^3(1) + f^3(2) \cdots + f^3(n) = (f(1) + f(2) + \text{---} + f(n))^2$

$n=1$

$$f^3(1) = (f(1))^2 \Rightarrow f(1) = 1.$$

$n=2$

$$f^3(1) + f^3(2) = (f(1) + f(2))^2$$

$$1 + f^3(2) = (1 + f(2))^2$$

$$f^3(2) + \cancel{1} = \cancel{1} + f^2(2) + 2f(2)$$

$$\hookrightarrow f^2(2) = f(2) + 2 \Rightarrow f(2) = 2, \cancel{-1}$$

$$\boxed{f(n) = n}, \quad \forall n \in N.$$

~~Ans~~

Que find value of K if $f : [a, \infty) \longrightarrow \left[0, \frac{\pi}{2}\right)$, where $f(x) = \tan^{-1}(x^2 + x + K)$ is an <u>invertible</u> function, given 'a' takes minimum possible value.

Sol $\Rightarrow$ $f(x) = \tan^{-1}(x^2 + x + K)$.

for $f(x)$ to be one-one
$\Downarrow$
$x^2 + x + K$ must be one-one in domain.

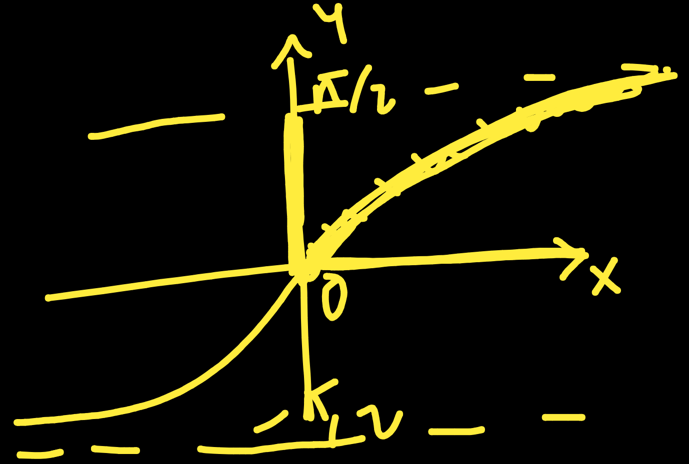
for function to be one-one $\boxed{a \geq -\frac{1}{2}}$

acc. to question $a_{min} = -\frac{1}{2}$.

$\Rightarrow f : \left[-\frac{1}{2}, \infty\right) \longrightarrow \left[0, \frac{\pi}{2}\right)$.

As $f(x)$ is invertible hence it must be onto fn.

$\Rightarrow R_{f(x)} \equiv \left[0, \frac{\pi}{2}\right) \Rightarrow x^2 + x + K$ must take <u>all values</u> in the interval $[0, \infty)$.



Now in domain $\left[-\frac{1}{2}, \infty\right)$

range of x^2+x+K will be $\left[K-\frac{1}{4}, \infty\right)$

$$\Rightarrow K-\frac{1}{4}=0$$

$$\boxed{K=\frac{1}{4}}$$

Ans