Que

If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$ then the locus of the middle points of these chords is the curve

(a) $y^2(x + a) = x^3$ (b) $y^2(x - a) = x^3$ (c) $y^2(x + 2a) = 3x^3$ (d) $y^2(x - 2a) = 2x^3$

Que

The normal to the rectangular hyperbola $xy = c^2$ at the point 't_1' meets the curve again at the point 't_2'. Then the value of $t_1^3 t_2$ is

(a) 1 (b) -1 (c) c (d) $-c$

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The line $\ell x + my + n = 0$ will be a normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ if

(A) $\dfrac{a^2}{\ell^2} - \dfrac{b^2}{m^2} = \dfrac{(a^2+b^2)^2}{n^2}$ (B) $\dfrac{a^2}{\ell^2} - \dfrac{b^2}{m^2} = \dfrac{(a^2-b^2)^2}{n^2}$

(C) $\dfrac{a^2}{\ell^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2+b^2)^2}{n^2}$ (D) none of these

Que

The tangent at a point P on an ellipse intersect the major axis in T, and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Que

If a rectangular hyperbola has the equation $xy = c^2$, prove that the locus of the middle points of chords of length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$

Que

If α, β, γ and δ be the eccentric angles of the four points of intersection of the ellipse and any circle, prove that $\alpha + \beta + \gamma + \delta$ is an even multiple of π radians.

Que

With a given point and line as focus and directrix, a series of ellipses are described ; prove that the locus of the extremities of their minor axes is a parabola.

Qu

Given the base of a triangle and the sum of its sides, prove that the locus of the centre of its incircle is an ellipse.

Qu

Two tangents to the ellipse intersect at right angles ; Prove that the sum of the squares of the chords which the auxiliary circle intercepts on them is constant, and equal to the square on the line joining the foci.

Qu

If a number of ellipses be described having the same major axis, but a variable minor axis, prove that the tangents at the ends of their latera recta pass through one or other of two fixed points.

Qu

If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) $PF \cdot PG = b^2$ **(ii)** $PF \cdot Pg = a^2$

(iii) $PG \cdot Pg = SP \cdot S'P$ **(iv)** $CG \cdot CT = (CS)^2$ $= a^2e^2$.

Que

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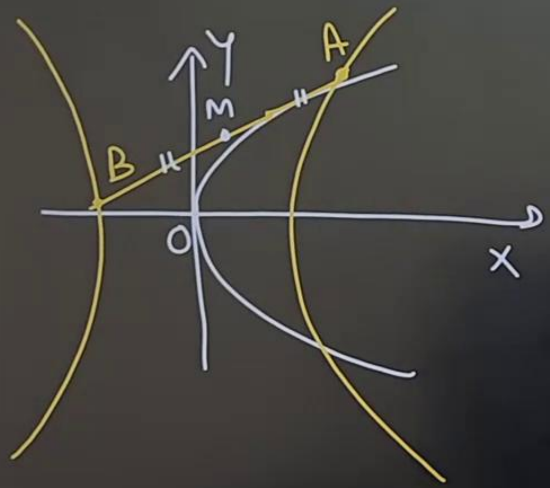
Sol



Let Mid pt is M: (h, k)

Eqn of chord $T = S_1$ (for M w.r.t. HB)

$$xh - yk - a^2 = h^2 - k^2 - a^2.$$

$$y = \left(\frac{h}{k}\right)x - \left(\frac{h^2-k^2}{k}\right)$$

$$c = \frac{a}{m} \Rightarrow -\left(\frac{h^2-k^2}{k}\right) = \frac{a.k}{h}$$

$$-x^3 + xy^2 = ay^2.$$

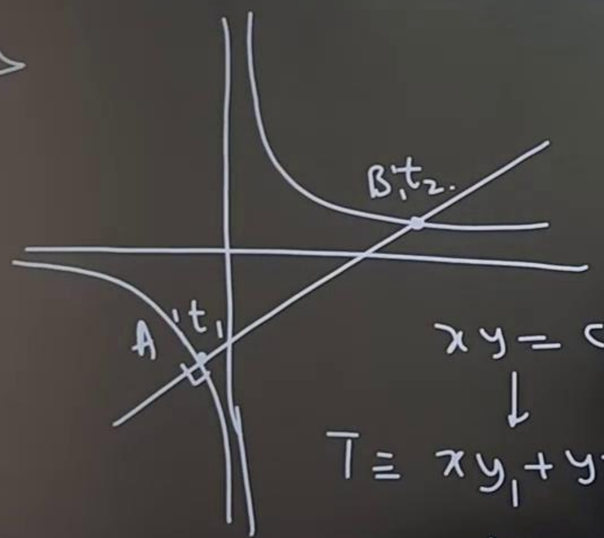
$$\boxed{(x-a)\,y^2 = x^3.}$$

(B) Ans

The normal to the rectangular hyperbola $xy = c^2$ at the point 't_1' meets the curve again at the point 't_2'. Then the value of $t_1^3 t_2$ is

(a) 1 (b) -1 (c) c (d) $-c$

Sol



$xy = c^2$

$\downarrow$

$T \equiv x y_1 + y x_1 = 2c^2$

$m_t = -\frac{y_1}{x_1}$

$m_n = \frac{x_1}{y_1} = \frac{c t_1}{\left(\frac{c}{t_1}\right)} = t_1^2$

$N_A \equiv y - \frac{c}{t_1} = t_1^2 (x - c t_1)$

$\left(ct, \frac{c}{t}\right) \longrightarrow Q(t) \longrightarrow t_1, t_2.$

$\frac{c}{t} - \frac{c}{t_1} = t_1^2 (ct - ct_1)$

$\hookrightarrow \frac{\cancel{t_1 - t}}{t t_1} = t_1^2 \cancel{(t - t_1)}$ -1.

$t t_1^3 = -1. \longrightarrow t_2.$

$\hookrightarrow t_2 t_1^3 = \boxed{-1}$

(B)

Que

The line $\ell x + my + n = 0$ will be a normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ if

(A) $\dfrac{a^2}{\ell^2} - \dfrac{b^2}{m^2} = \dfrac{(a^2+b^2)^2}{n^2}$

(B) $\dfrac{a^2}{\ell^2} - \dfrac{b^2}{m^2} = \dfrac{(a^2-b^2)^2}{n^2}$

(C) $\dfrac{a^2}{\ell^2} + \dfrac{b^2}{m^2} = \dfrac{(a^2+b^2)^2}{n^2}$

(D) none of these

Sol

$lx + my + n = 0$

$b^2x^2 - a^2y^2 = a^2b^2$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\Rightarrow N \equiv lx + my + n = 0$

$\Rightarrow N \equiv ax\cos(\theta) + by\cot(\theta) = a^2 + b^2.$

$\Downarrow$ Comparing coeff.

$$\frac{a\cos(\theta)}{l} = \frac{b\cot(\theta)}{m} = -\frac{(a^2+b^2)}{n}$$

$\Rightarrow \sec(\theta) = -\dfrac{na}{l(a^2+b^2)}$

$\Rightarrow \tan(\theta) = -\dfrac{nb}{m(a^2+b^2)}$

$\sec^2(\theta) - \tan^2(\theta) = 1 = \dfrac{n^2}{(a^2+b^2)^2}\left(\dfrac{a^2}{l^2} - \dfrac{b^2}{m^2}\right)$

$$\boxed{\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}.}$$

A

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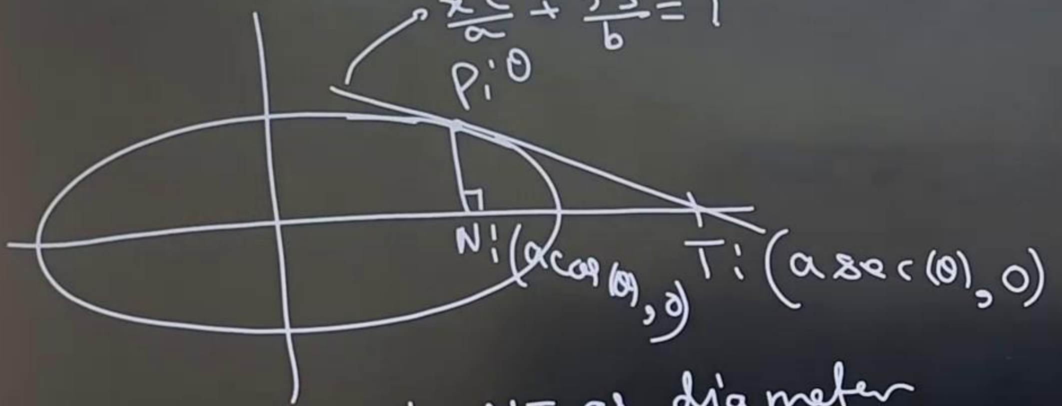
Que

The tangent at a point P on an ellipse intersect the major axis in T, and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Sol



Eqn of circle with NT as diameter

$$(x - a\cos(\theta))(x - a\sec(\theta)) + y^2 = 0$$

$$x^2 + y^2 - a(\cos(\theta) + \sec(\theta)) \cdot x + a^2 = 0$$

$$g_1 = -\frac{(\cos(\theta) + \sec(\theta))}{2}, \quad f_1 = 0, \quad c_1 = a^2.$$

$$x^2 + y^2 - a^2 = 0$$

$$g_2 = 0$$

$$f_2 = 0$$

$$c_2 = -a^2$$

$$\Rightarrow 2(g_1 g_2 + f_1 f_2) = c_1 + c_2$$

$$0 + 0 = a^2 - a^2$$

$$0 = 0$$

H.P.

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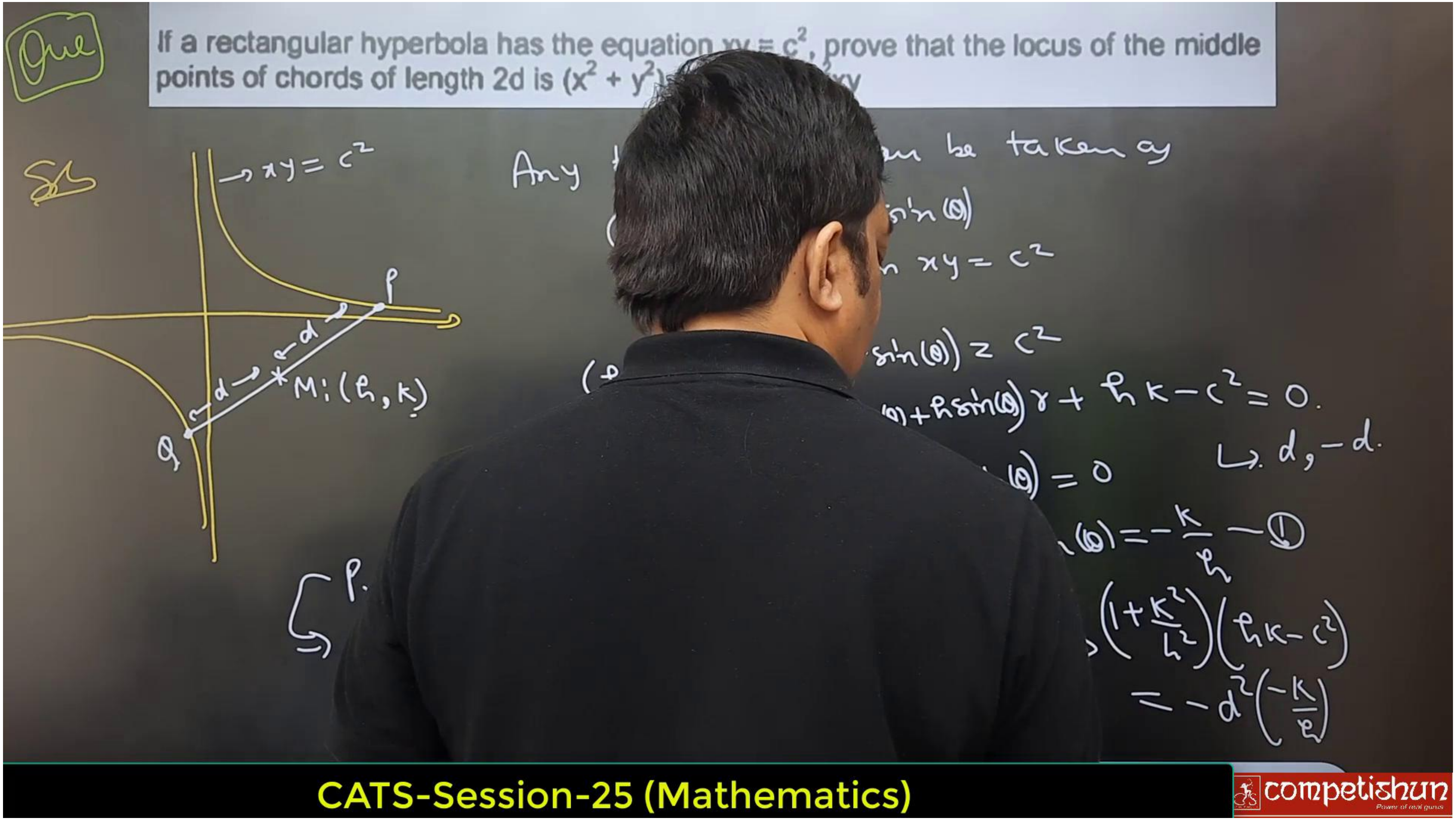

Que
If a rectangular hyperbola has the equation xy = c², prove that the locus of the middle
points of chords of length 2d is (x² + y²)
Sol
xy = c²
P
Q
d
M : (h, k)
Any
be taken as
sin(θ)
xy = c²
sin(θ) = c²
+ h sin(θ)) r + hk − c² = 0.
d, −d.
(θ) = 0
(θ) = −k/h — ①
(1 + k²/h²)(hk − c²)
= −d²(−k/h)
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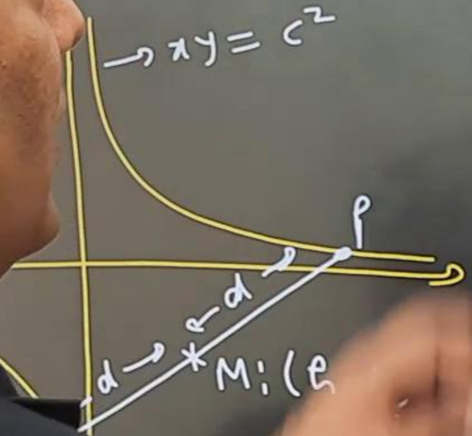
If a rectangular hyperbola has the equation $xy = c^2$, prove that the locus of the middle points of chords of length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$



Any point on PQ can be taken as

$$(h + r\cos(\theta),\ k + r\sin(\theta))$$

put it in $xy = c^2$

$$(h + r\cos(\theta))(k + r\sin(\theta)) = c^2$$

$$\sin(\theta)\cdot\cos(\theta)\cdot r^2 + (k\cos(\theta) + h\sin(\theta))\,r + hk - c^2 = 0.$$

$\hookrightarrow d, -d.$

$$\text{Sum of roots} = 0 \Rightarrow (k\cos(\theta) + h\sin(\theta)) = 0$$

$$\hookrightarrow \tan(\theta) = -\frac{k}{h} \quad \text{— (1)}$$

$$\text{Product of roots} = -d^2$$

$$\frac{hk - c^2}{\sin(\theta)\cos(\theta)} = -d^2 \Rightarrow \frac{\sec^2(\theta)(hk - c^2)}{\tan(\theta)} = -d^2 \Rightarrow \left(1 + \frac{k^2}{h^2}\right)(hk - c^2) = -d^2\left(-\frac{k}{h}\right)$$

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$$\left(\frac{h^2+k^2}{h^2}\right)(hk-c^2)=\frac{d^2k}{h}$$

$$\boxed{(x^2+y^2)(xy-c^2)=d^2xy}$$

H.P.

CATS-Session-25 (Mathematics)

Que

If α, β, γ and δ be the eccentric angles of the four points of intersection of the ellipse and any circle, prove that $\alpha+\beta+\gamma+\delta$ is an even multiple of π radians.

Sol let ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle is $x^2+y^2+2gx+2fy+c=0$.

For points of intersection put $(a\cos(\theta), b\sin(\theta))$ in circle.

$$a^2\cos^2(\theta)+b^2\sin^2(\theta)+2ga\cos(\theta)+2fb\sin(\theta)+c=0.$$

$\cos(\theta) = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

$\sin(\theta) = \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

& let $\tan\left(\frac{\theta}{2}\right)=t$

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\left(\frac{2t}{1+t^2}\right)^2+2ga\left(\frac{1-t^2}{1+t^2}\right)+2fb\frac{2t}{1+t^2}+c=0$$

$$a^2(1+t^4-2t^2)+b^2(4t^2)+2ga(1-t^4)+2fb\,2t(1+t^2)+c(1+t^4+2t^2)=0.$$

$$t^4(a^2-2ga+c)+t^3(4fb)+t^2(\quad)+t(\quad)+a^2+2ga+c=0.$$

Consider

$$\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{S_1-S_3}{1-S_2+S_4}.$$

$$\tan\left(\frac{\Sigma\alpha}{2}\right) = \frac{\frac{-4fb}{(\quad)}-\left(\frac{-4fb}{(\quad)}\right)}{1-S_2+S_4}=0.$$

$$\frac{\Sigma\alpha}{2}=n\pi \Rightarrow \Sigma\alpha=2n\pi$$

$$\alpha+\beta+\gamma+\delta=2n\pi$$

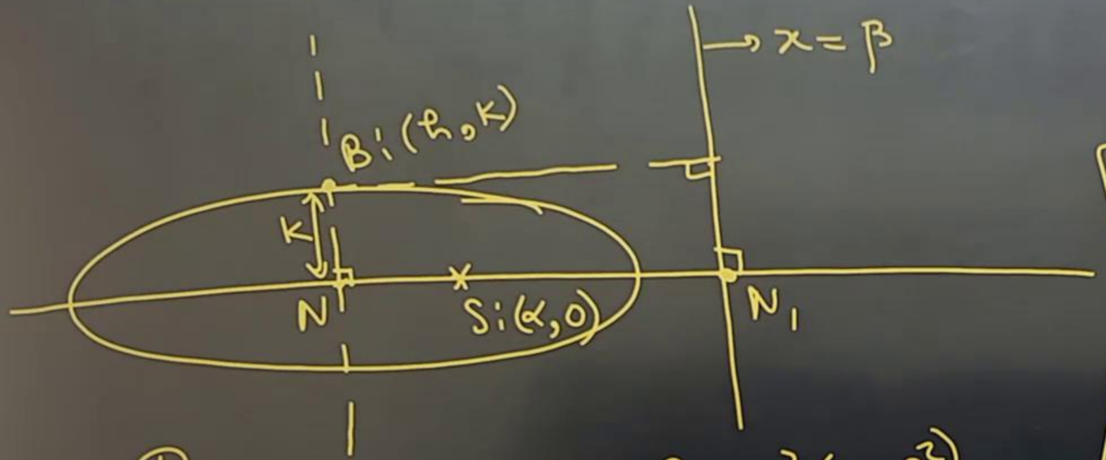
$= \text{Even multiple of } \pi$

$\Rightarrow$ H.P.

Que

With a given point and line as focus and directrix, a series of ellipses are described ; prove that the locus of the extremities of their minor axes is a parabola.

Sol



① $k = b \Rightarrow k^2 = b^2 = a^2(1-e^2)$

$$k^2 = a^2(1-e^2) \longleftarrow ①$$

② $\beta - h = \dfrac{a}{e}$

$\alpha - h = ae$

$\xrightarrow{\times} (\beta-h)(\alpha-h) = a^2$

$\xrightarrow{\div} \dfrac{\alpha-h}{\beta-h} = e^2 \Rightarrow 1-e^2 = 1-\dfrac{\alpha-h}{\beta-h} = \dfrac{\beta-\alpha}{\beta-h}$

$$k^2 = \cancel{(\beta-h)}(\alpha-h)\cdot\frac{(\beta-\alpha)}{\cancel{\beta-h}}$$

$$\Downarrow$$

$$y^2 = (\alpha-x)(\beta-\alpha)$$

$$\Downarrow$$

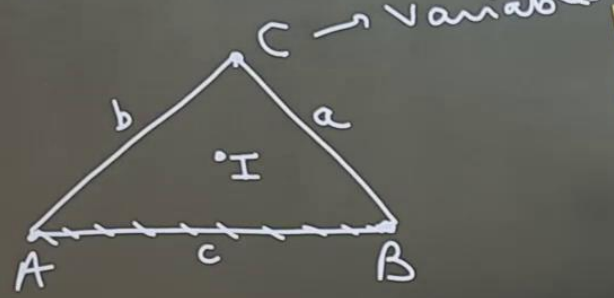
Parabola

$\boxed{\text{H.P.}}$

Qu

Given the base of a triangle and the sum of its sides, prove that the locus of the centre of its incircle is an ellipse.

Sol



given $a+b+c = \text{const.}$

AB is fixed

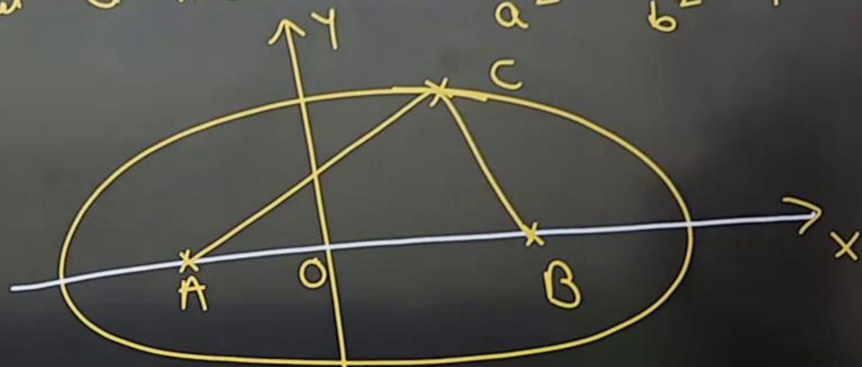
→ c is const.

→ $a+b \equiv \text{Const.}$

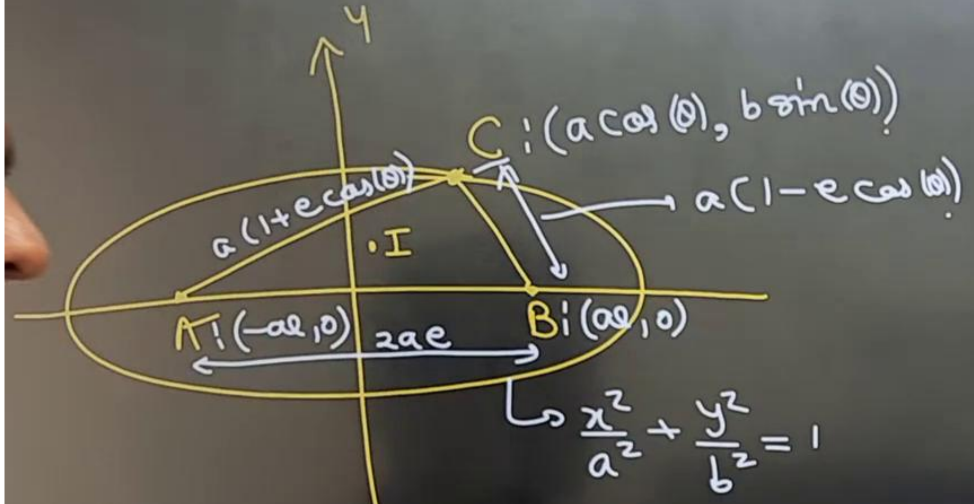
⇓

C will move on an ellipse.

Let C moves on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —— (1)



according to question C moves on Ellipse (1) and we need to prove that locus of incentre of ΔABC is an ellipse.



$I : (h, k)$

$: \left(\frac{\Sigma a x_1}{\Sigma a}, \frac{\Sigma a y_1}{\Sigma a}\right)$

$\underline{a+b}+c = 2a + 2ae = 2a(1+e)$

$PS_1 = a(1 \pm e\cos(\theta))$

$$h = \frac{(ae)(a(1+e\cos(\theta))) + (-ae)(a(1-e\cos(\theta))) + 2ae\cdot a\cos(\theta)}{2a(1+e)}$$

$$k = \frac{\cancel{2}\cancel{a}e\cdot b\sin(\theta) + 0 + 0}{\cancel{2}\cancel{a}(1+e)} = \frac{e\cdot b\cdot\sin(\theta)}{1+e}$$

$$h = \frac{a^{\cancel{2}}e}{\cancel{2}\cancel{a}(1+e)}\left(\cos(\theta)(\cancel{2}e+\cancel{2})\right) = \frac{ae\cdot\cos(\theta)}{1}$$

$$s^2(\theta) + c^2(\theta) = 1 = \frac{(1+e)^2\cdot k^2}{e^2b^2} + \frac{h^2}{a^2e^2}$$

$$\Rightarrow \boxed{\frac{x^2}{a^2e^2} + \frac{y^2}{\left(\frac{b^2e^2}{(1+e)^2}\right)} = 1}$$

H.P.

eccentricity

$$e_1^2 = 1 - \frac{b_1^2}{a_1^2} = 1 - \frac{\left(\frac{b^2 \cancel{e^2}}{(1+e)^2}\right)}{a^2 \cancel{e^2}}$$

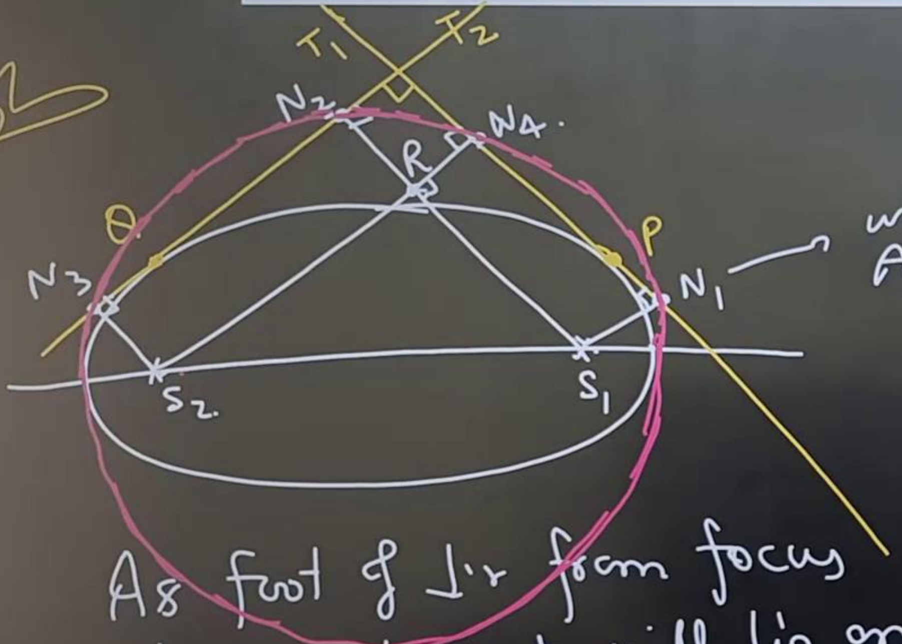
$$e_1^2 = 1 - \frac{1-e^2}{(1+e)^2} = \frac{\cancel{1} + e^2 + 2e - \cancel{1} + e^2}{(1+e)^2}.$$

$$e_1^2 = \frac{2e\cancel{(1+e)}}{(1+e)^{\cancel{2}}} = \frac{2e}{1+e}$$

$$e_1 = \sqrt{\frac{2e}{1+e}}$$

Que

Two tangents to the ellipse intersect at right angles ; Prove that the sum of the squares of the chords which the auxiliary circle intercepts on them is constant, and equal to the square on the line joining the foci.



will lie on Aux. circle

As foot of ⊥r from focus to any tangent will lie on Aux. circle $\Rightarrow N_1, N_2, N_3, N_4$ are on Aux. circle.

Required Sum

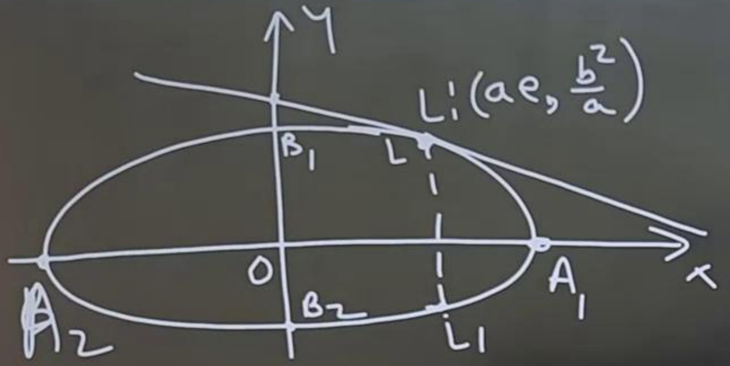
$$= N_1N_4^2 + N_2N_3^2.$$

$$= S_1R^2 + S_2R^2$$

$$= S_1S_2^2 = (2ae)^2 = 4a^2e^2.$$

H.P.

If a number of ellipses be described having the same major axis, but a variable minor axis, prove that the tangents at the ends of their latera recta pass through one or other of two fixed points.



Let ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ $a \to$ const

$b \to$ variable

$e \nearrow$

$$T_L \equiv \frac{x \cdot \cancel{a}e}{a^{\cancel{2}}} + \frac{y \cdot \cancel{b^2}/a}{\cancel{b^2}} = 1.$$

$$\frac{xe}{a} + \frac{y}{a} = 1$$

$$\underbrace{\left(\frac{y}{a} - 1\right)}_{L_1} + \underset{\lambda}{e} \cdot \underbrace{\left(\frac{x}{a}\right)}_{L_2} = 0$$

f.P. $\to$ $L_1 = 0$ & $L_2 = 0 \Rightarrow (0, a)$.

$\hookrightarrow$ f.P$_1$

$|||^{ly}$ other fixed pt will be $(0, -a)$.

Que

If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) $PF \cdot PG = b^2$ (ii) $PF \cdot Pg = a^2$

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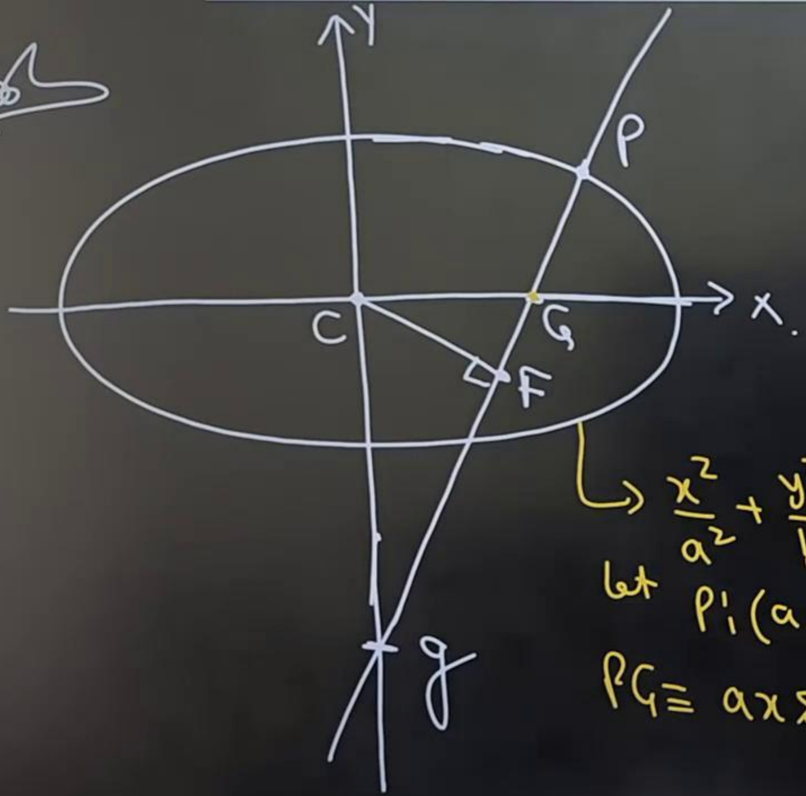
Sol



$\hookrightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

let $P: (a\cos(\theta), b\sin(\theta))$

$PG \equiv ax\sec(\theta) - by\csc(\theta) = a^2 - b^2$

$= a^2e^2$

$G: (ae^2\cos(\theta), 0)$

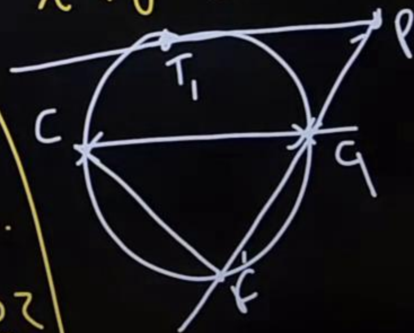
$g: (0, -a^2e^2 b\sin(\theta))$

Consider a circle through CFG.

$\hookrightarrow$ CG as diameter.

$C_1 \equiv (x-0)(x - ae^2\cos(\theta)) + (y-0)(y-0) = 0.$

$x^2 + y^2 - ae^2\cos(\theta)\, x = 0.$



$PF \cdot PG = PT_1^2.$

$= S_1$ (for C_1 w.r.t. P)

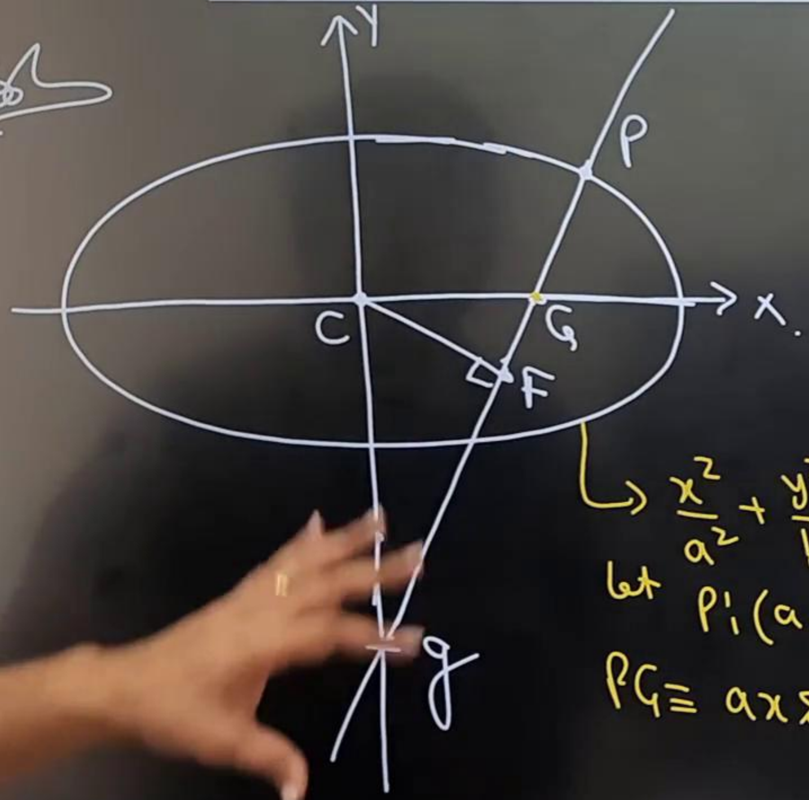
Que

If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) PF . PG = b^2 (ii) PF . Pg = a^2

(iii) PG . Pg = SP . S′ P (iv) CG . CT = $(CS)^2$

Sol



$\hookrightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

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$= a^2e^2$

$G: (ae^2\cos(\theta), 0)$

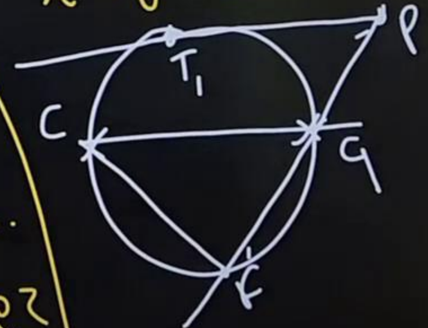
$g: \left(0, -\frac{a^2e^2}{b}\sin(\theta)\right)$

Consider a circle through CFG.

↳ CG as diameter.

$C_1 \equiv (x-0)(x-ae^2\cos(\theta)) + (y-0)(y-0) = 0$.

$x^2+y^2-ae^2\cos(\theta)\,x = 0$.



$PF \cdot PG = PT_1^2$.

$= S_1$ (for C_1 w.r.t. P)