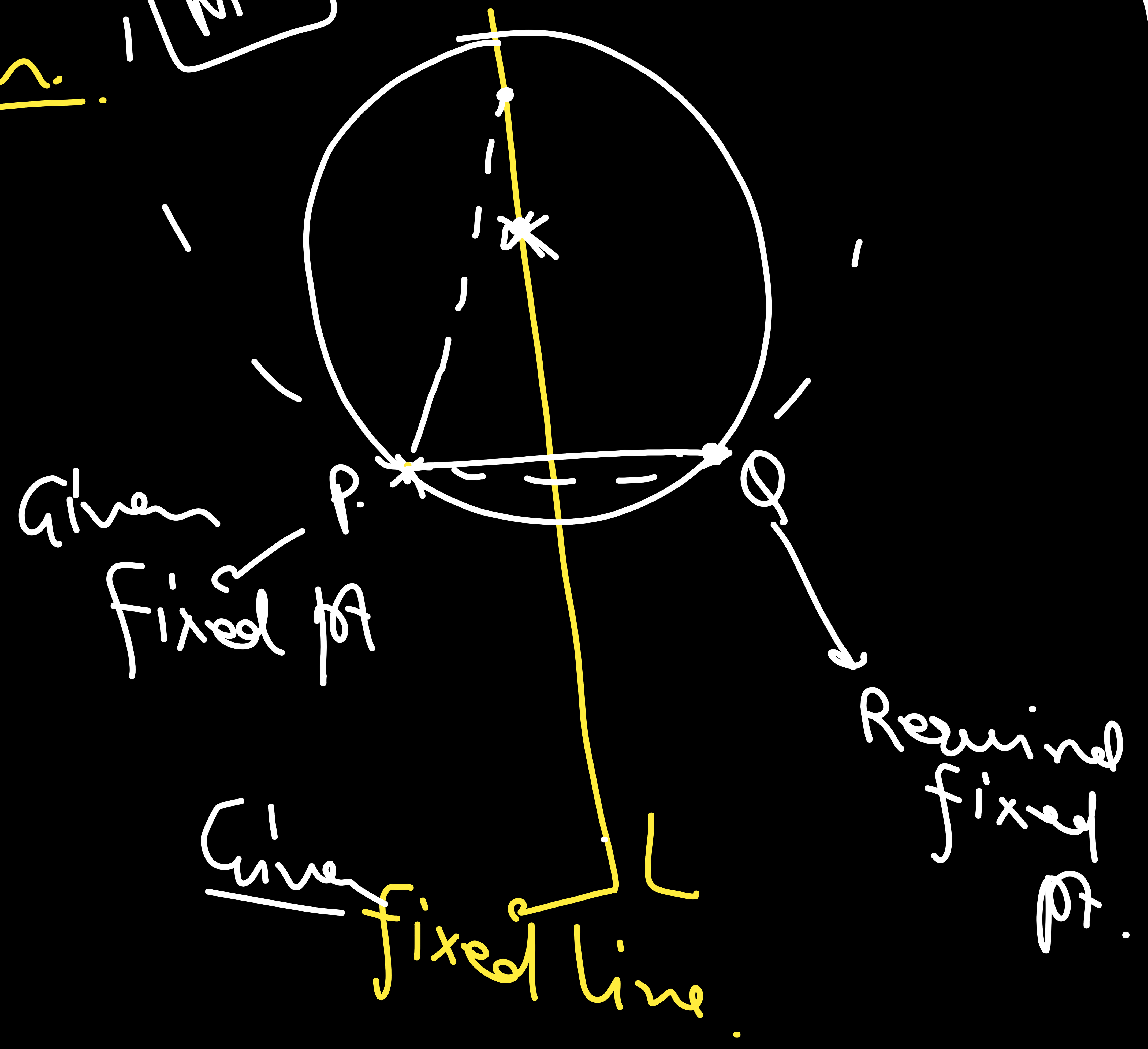
Que

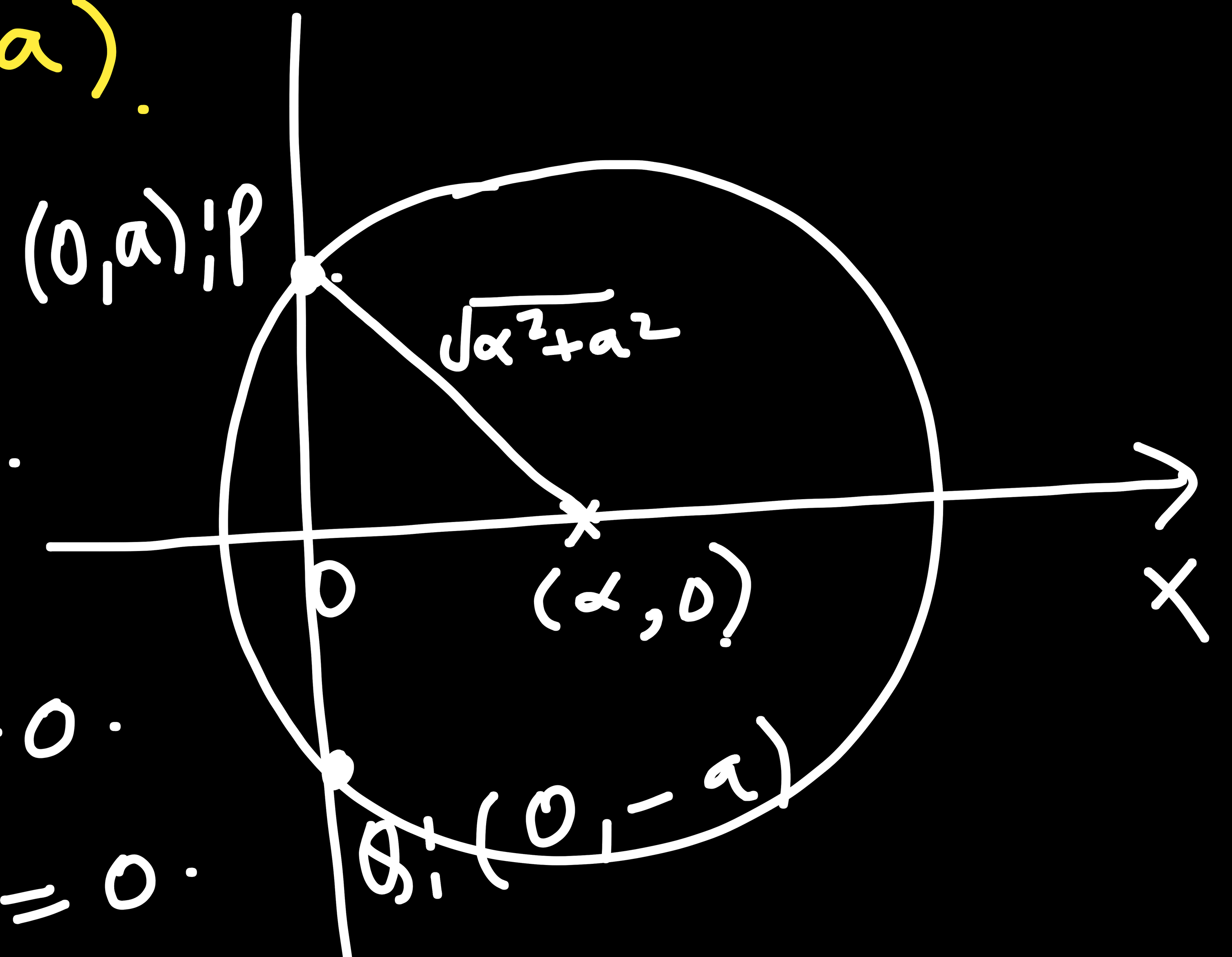
Prove that all the circles having their centres on a fixed line and passing through a fixed point also pass through another fixed point. Given fixed point does not lie on fixed line.

Soln. M-1



M-2

Let fixed line is X-axis & pt lies on Y axis. at P: (0, a).



Eqn of circle

$(x-\alpha)^2+y^2=\alpha^2+a^2.$

$\Rightarrow x^2+y^2-2x\alpha-a^2=0.$

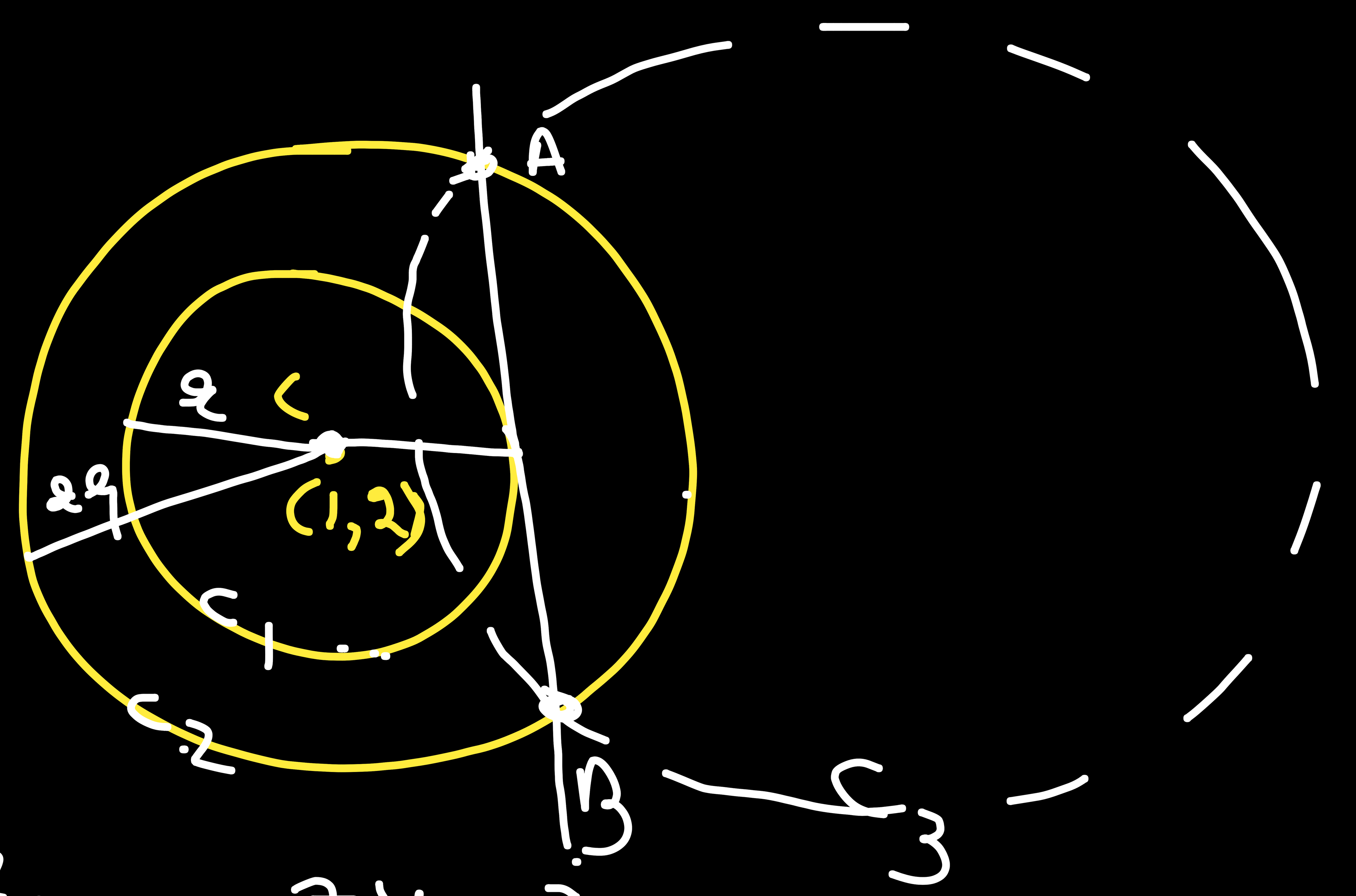
$\underbrace{x^2+y^2-a^2}_{S}-\underbrace{2\alpha}_{\lambda}\underbrace{x}_{L}=0.$

$S=0$ & $L=0$ $\rightarrow x^2+y^2=a^2$, $x=0$ $\Rightarrow y=\pm a$. is

$\Rightarrow$ other fixed pt $(0,-a)$

Two concentric circles have centres at (1, 2) and radii r and 2r. The circle $x^2+y^2+2x-7y-2=0$ cuts these circles such that the common chord with one of the circles is a tangent to the other. Find the equation of both the circles.

Sol



$C_3 \equiv x^2+y^2+2x-7y-2=0$

$C_1 \equiv (x-1)^2+(y-2)^2 = r^2$

$C_2 \equiv (x-1)^2+(y-2)^2 = 4r^2 \Rightarrow x^2+y^2-2x-4y+5-4r^2=0.$

$AB \equiv C_2 - C_3 = 0.$

$4x-3y-7+4r^2=0.$

↳ It is tangent to C_1

$\Rightarrow p_1 = r_1$

$\frac{|4-6-7+4r^2|}{5} = r \Rightarrow 5r = |4r^2-9|$

$4r^2 \pm 5r - 9 = 0.$

$4r^2+5r-9=0 \quad\quad 4r^2-5r-9=0$

$r = 1, \cancel{-\frac{9}{4}}. \quad\quad r = \cancel{-1}, \frac{9}{4}$

$\Rightarrow r = 1, \frac{9}{4}.$

Eqn of Circles & $(x-1)^2+(y-2)^2=1$

$(x-1)^2+(y-2)^2=4$

OR $(x-1)^2+(y-2)^2=\left(\frac{9}{4}\right)^2$

& $(x-1)^2+(y-2)^2=4\left(\frac{9}{4}\right)^2$

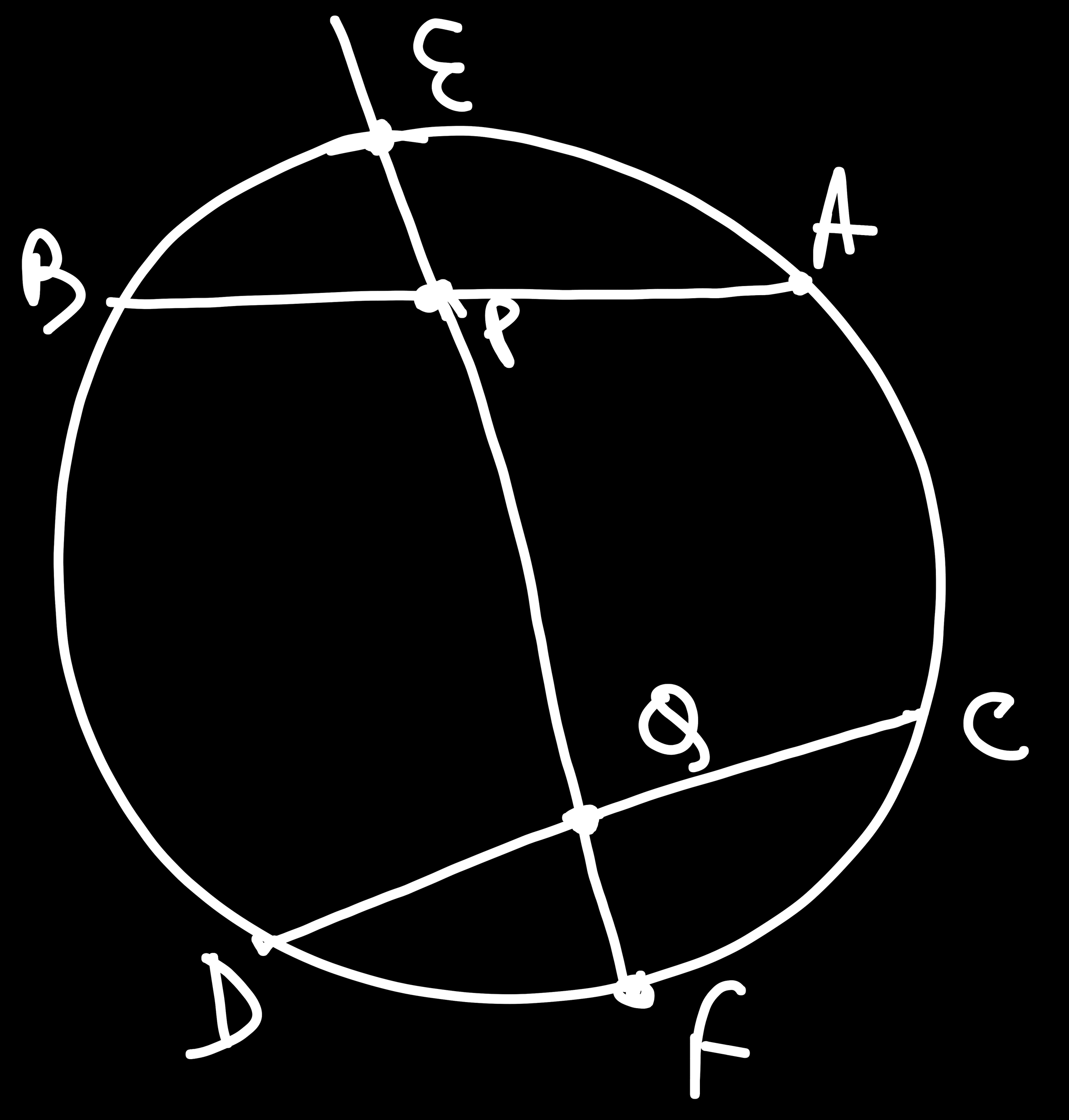
$r=1, \frac{9}{4}$

Que

AB and CD are two equal chords of a circle having mid points P and Q respectively. The line joining PQ intersects the circle at E and F respectively (where E is nearer to AB than CD).

Find $\frac{EP}{QF}$ if radius of circle is 2.

Sol



$$EP \cdot PF = PB \cdot PA = \left(\frac{AB}{2}\right)^2.$$

$$QF \cdot QE = DQ \cdot QC = \left(\frac{CD}{2}\right)^2 = \left(\frac{AB}{2}\right)^2.$$

$$EP \cdot PF = QF \cdot QE.$$

$$EP \cdot (PQ + \cancel{QF}) = QF \cdot (PQ + \cancel{EP})$$

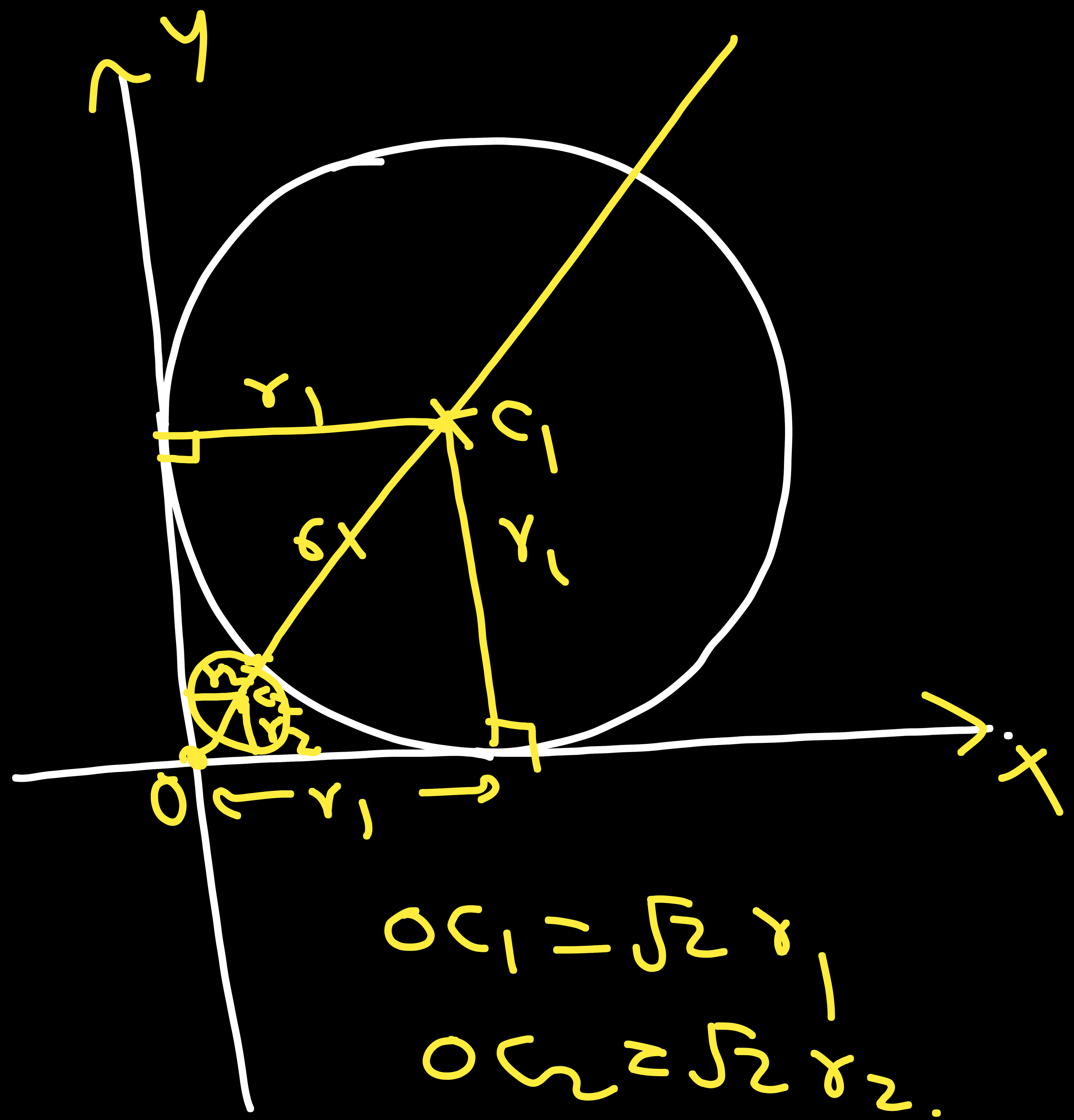
$$\Rightarrow EP \cdot \cancel{PQ} = QF \cdot \cancel{PQ}$$

$$EP = QF \Rightarrow \boxed{\frac{EP}{QF} = 1}$$

One

From a point P, two perpendicular tangents are drawn to a circle C_1. Now a circle C_2 is drawn so that it touches C_1 and also the perpendicular tangents from P. If r_1 and r_2 are the radii of C_1 and C_2 respectively, show that $\sqrt{r_1} - \sqrt{r_2} = \sqrt{2r_2}$ $(r_1 > r_2)$.

Sol



$OC_1 = \sqrt{2}\, r_1$

$OC_2 = \sqrt{2}\, r_2$.

$OC_1 = OC_1$

$\sqrt{2}\, r_1 = \sqrt{2}\, r_2 + r_2 + r_1$

$r_1(\sqrt{2}-1) = r_2(\sqrt{2}+1)$

$\frac{r_1}{r_2} = \frac{\sqrt{2}+1}{\sqrt{2}-1} = (\sqrt{2}+1)^2$.

$\sqrt{\frac{r_1}{r_2}} = \sqrt{2}+1$

$\sqrt{r_1} = \sqrt{2r_2} + \sqrt{r_2}$

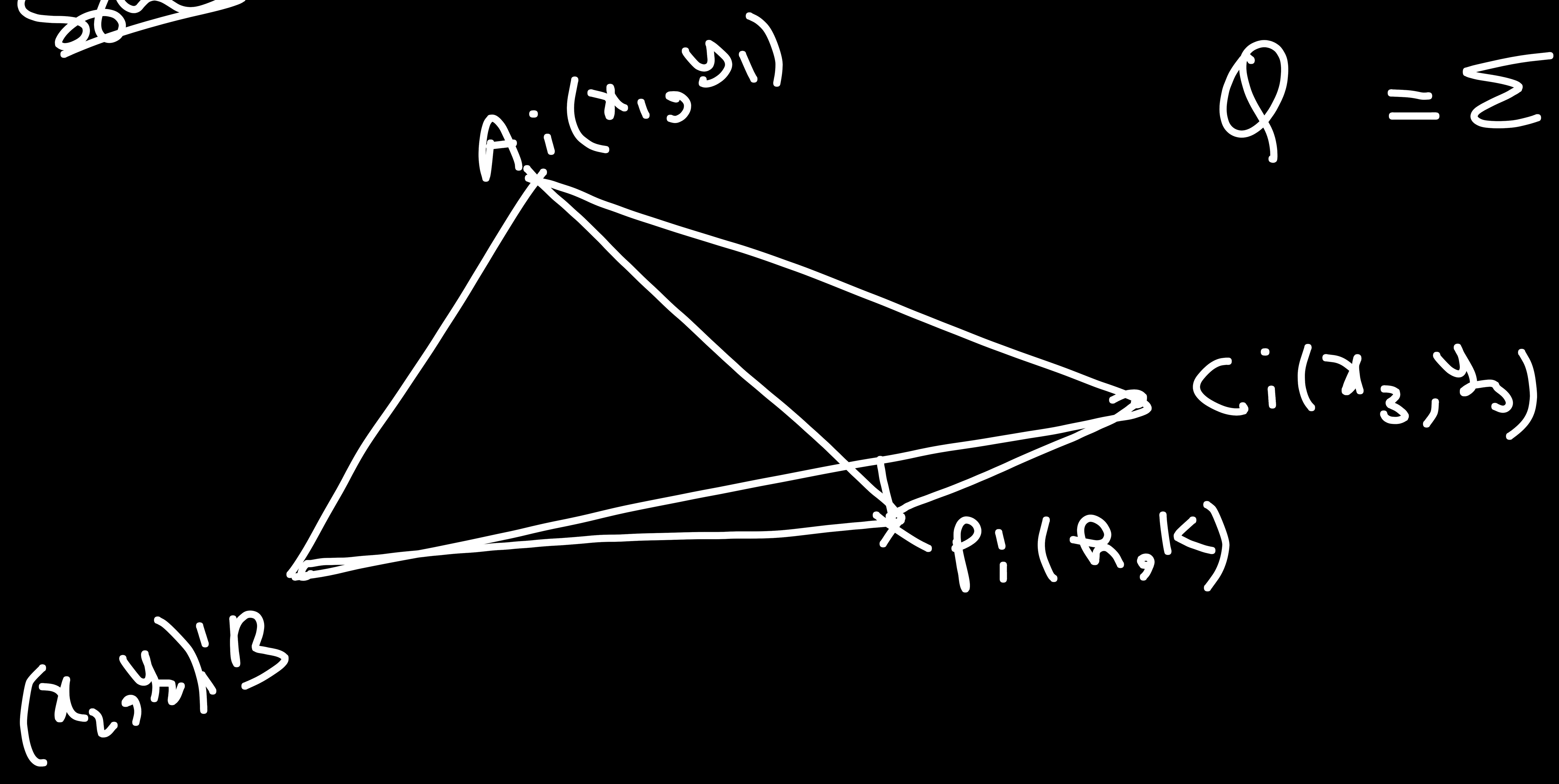
$\sqrt{r_1} - \sqrt{r_2} = \sqrt{2r_2}$.

Hence.

Que

Let ABC be a triangle. Find a point P in the plane of triangle ABC such that $aAP^2 + bBP^2 + cCP^2$ is minimum

Soln



$$Q = \Sigma a \cdot AP^2 = \Sigma a \cdot \left((x_1 - h)^2 + (y_1 - k)^2\right)$$

$$= \Sigma a\left(x_1^2 - 2hx_1 + h^2 + y_1^2 - 2y_1k + k^2\right)$$

$$= \Sigma a(x_1^2 + y_1^2) + \underbrace{h^2(\Sigma a) - 2h(\Sigma ax_1)}_{Q(h)} + \underbrace{k^2(\Sigma a) - 2k(\Sigma ay_1)}$$

$Q(h)$ which will be min. at $h = -\frac{B}{2A} = \frac{2\Sigma ax_1}{2 \cdot \Sigma a}$

$$h = \frac{\Sigma ax_1}{\Sigma a}.$$

$\Rightarrow$ for minimum value of Q, $h = \frac{\Sigma ax_1}{\Sigma a}$ & $k = \frac{\Sigma ay_1}{\Sigma a}$

$\Rightarrow$ Which are the coordinates of incentre of $\Delta ABC \Rightarrow P \equiv I$

Que

Consider a family of circles passing through two fixed points A(3, 7) and B(6, 5). Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

Sol

$$S \equiv (x-3)(x-6) + (y-7)(y-5) + \lambda \begin{vmatrix} x & y & 1 \\ 3 & 7 & 1 \\ 6 & 5 & 1 \end{vmatrix} = 0.$$

$$K_1 \equiv x^2 + y^2 - 4x - 6y - 3 = 0.$$

$$S \equiv x^2 + y^2 - 9x - 12y + 53 + \lambda(2x + 3y - 27) = 0.$$

For common Chord

$$K_1 - S = 0. \Rightarrow \underbrace{5x + 6y - 56}_{L_1} \underbrace{- \lambda}_{+\lambda} \underbrace{(2x + 3y - 27)}_{L_2} = 0.$$

which will pass through a fixed pt

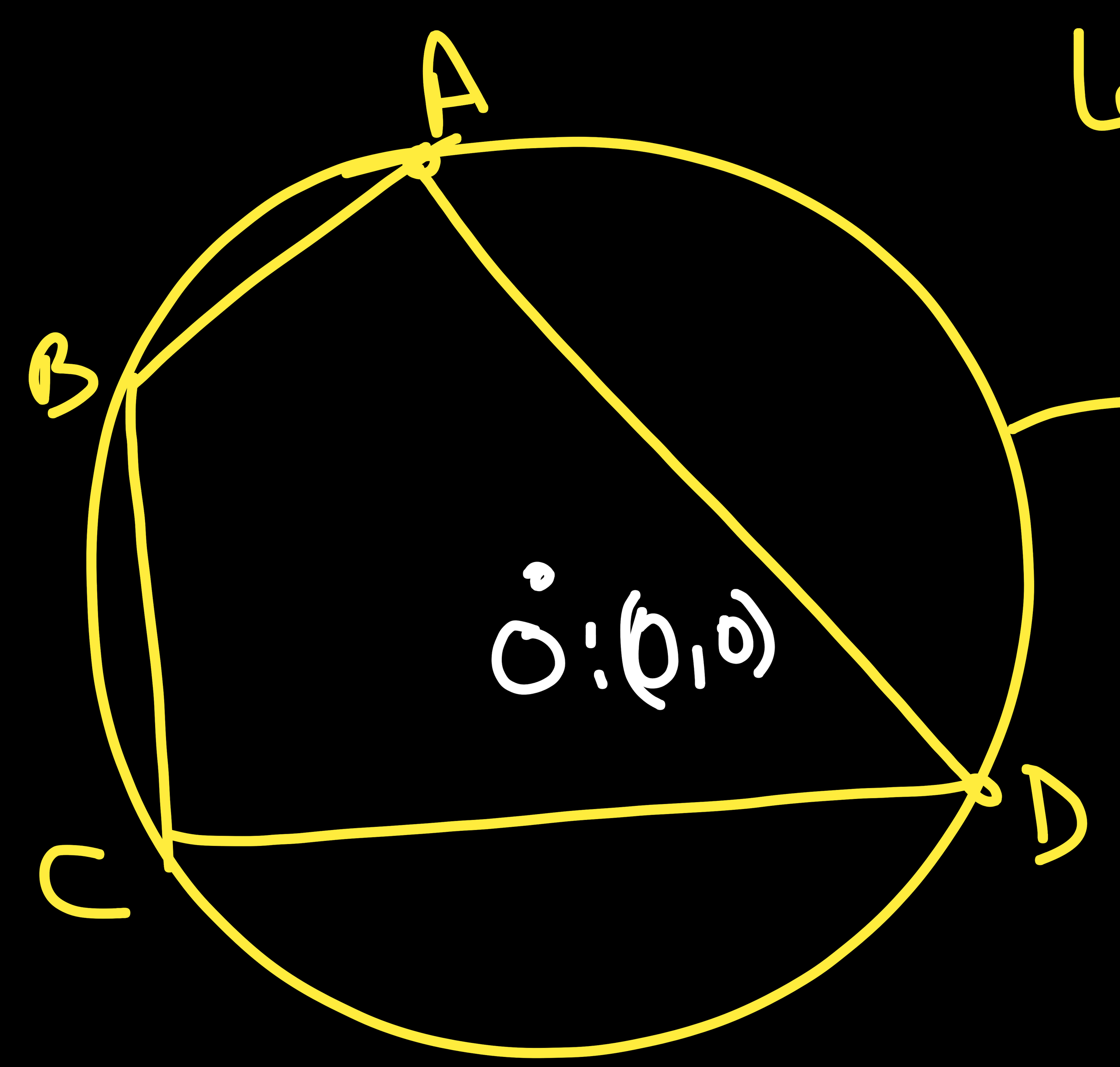
i.e. pt of 'x' of $L_1 = 0$ & $L_2 = 0$

$$\left.\begin{aligned} 5x + 6y &= 56 \\ 2x + 3y &= 27 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= 2 \\ y &= \tfrac{23}{3} \end{aligned}\right\} \Rightarrow \text{F.P. is } \left(2, \tfrac{23}{3}\right).$$

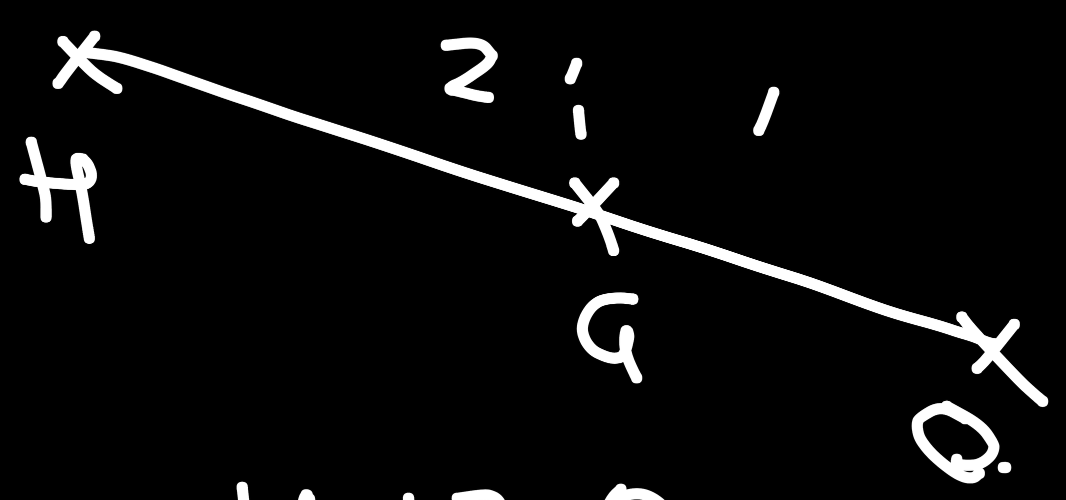
Ans

Que If ABCD is a cyclic quadrilateral then find the curve on which orthocentre of Δ's ABC, BCD, CDA & DAB lie.

Sol



Let A, B, C, D lies on $x^2+y^2=r^2$ and pts A, B, C, D are $(r\cos(\theta_i), r\sin(\theta_i))$, $i=1,2,3,4$.



$$G = \frac{H+2\cdot O}{1+2}$$

$$H \equiv 3G - O.$$

for ΔABC

$$G_1 : \left(\frac{r(\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3))}{3}, \frac{r(s(\theta_1)+s(\theta_2)+s(\theta_3))}{3}\right)$$

$$H_1 \equiv \left(r(c(\theta_1)+c(\theta_2)+c(\theta_3)), r(s(\theta_1)+s(\theta_2)+s(\theta_3))\right)$$

Now consider a circle

$$\left(x - r\left(\Sigma \cos(\theta_i)\right)\right)^2 + \left(y - r\left(\Sigma \sin(\theta_i)\right)\right)^2 = r^2 \quad \text{—} (I)$$

4-terms

Clearly H_1 lies on it.

$|||^{ly}$ H_2, H_3, H_4 will also satisfy.

Hence orthocentres of ΔABC, ΔABD, ΔACD, ΔBCD lie on a circle (I).

Que

Show that the equation $x^2 + y^2 - 2x - 2ay - 8 = 0$ represents for different values of 'a', a system of circles passing through two fixed points A, B on the x- axis. Also find the equation of that circle of the system, the tangents to which at A, B meet on the line $x + 2y + 5 = 0$.

Soln

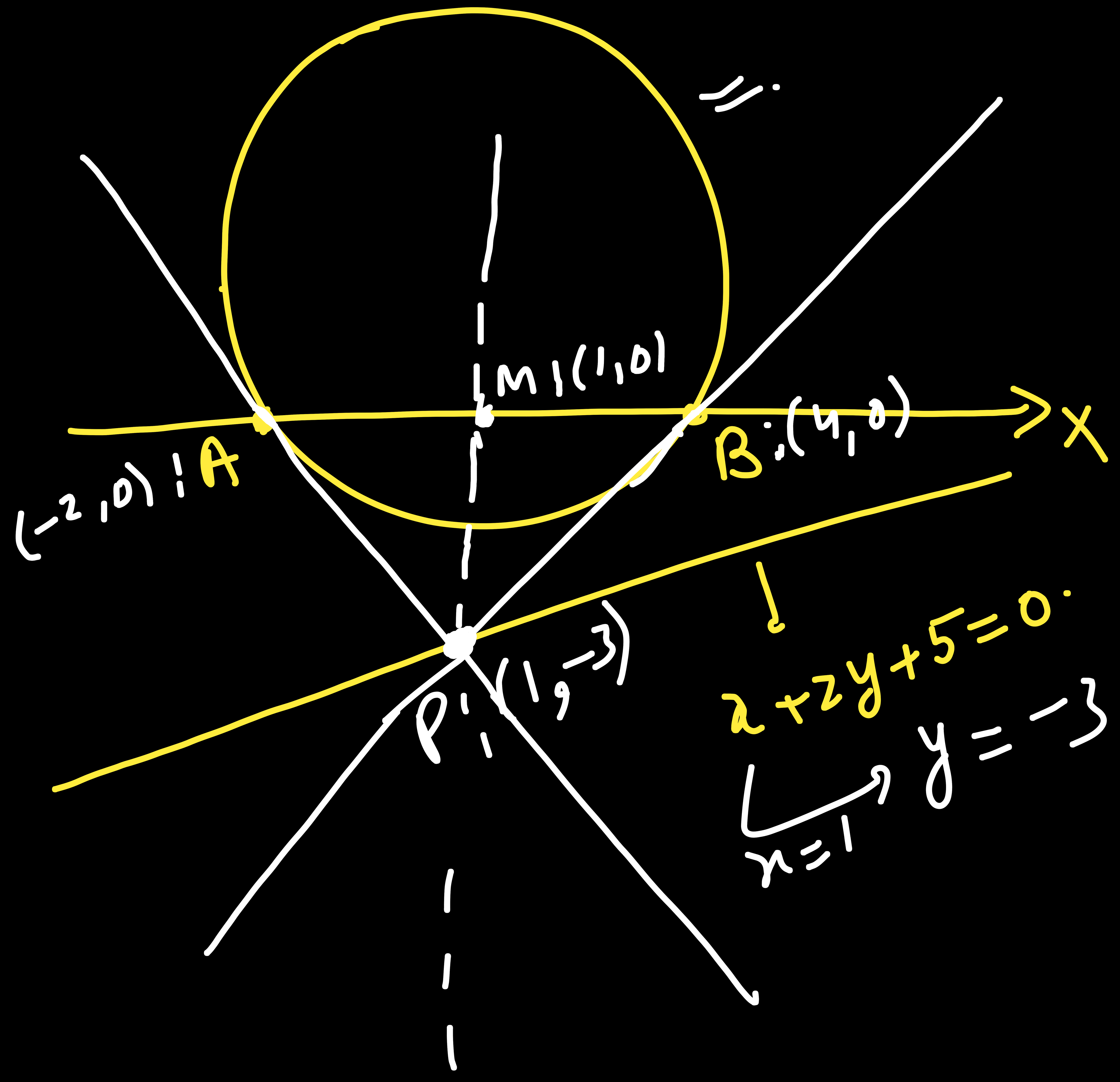
$$x^2+y^2-2x-2ay-8=0$$

$$\underbrace{x^2+y^2-2x-8}_{S} \underbrace{-2a}_{\lambda} \underbrace{(y)}_{L} = 0 \quad \text{—(1)}$$

Clearly (1) will pass through points of intersection of $S=0$ & $L=0$.

$$x^2+y^2-2x-8=0$$
$$y=0 \rightarrow x^2-2x-8=0$$
$$\hookrightarrow x=4,-2.$$

$A:(-2,0)$ & $B:(4,0)$



$$x^2+y^2-2x-8-2ay=0$$

$P_1(1,-3)$ $\xrightarrow{\text{Chord of Contact}}$ $T=0 \Rightarrow$ $x.1+y(-3)-(x+1)-8-a(y-3)=0.$

$\cancel{x}-3y-\cancel{x}-1-8-ay+3a=0$

$3a-9=0 \Rightarrow a=3$

$T=0$

$\downarrow$

x-axis

$\downarrow$

$y=0$

$\Downarrow$

Const term $=0.$

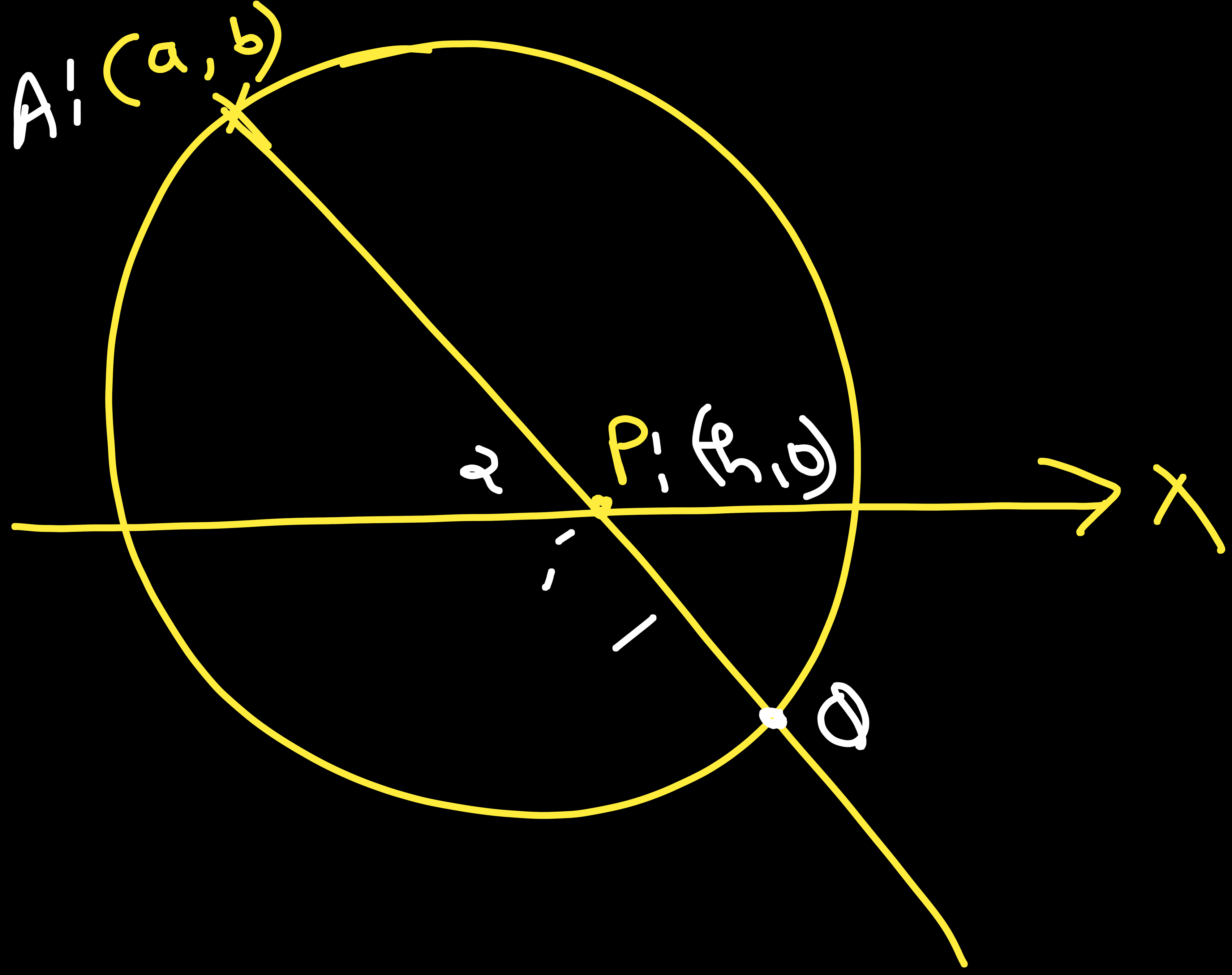
coff of $x=0.$

Que

If 2 distinct chords of the circle $x^2 + y^2 - ax - by = 0$ drawn from the point (a, b) is divided by the x-axis in the ratio 2 : 1 then prove that $a^2 > 3b^2$.

Sol

$x^2+y^2-ax-by=0$



$Q:\left(\frac{3h-a}{3-1}, \frac{3.0-b}{3-1}\right)$

$=\left(\frac{3h-a}{2}, -\frac{b}{2}\right)$

$\downarrow$ circle.

$\left(\frac{3h-a}{2}\right)^2+\frac{b^2}{4}-a\left(\frac{3h-a}{2}\right)+\frac{b^2}{2}=0.$

$9h^2+a^2-6ah+b^2-6ah+2a^2+2b^2=0.$

$9h^2-12ah+3a^2+3b^2=0$

$3h^2-4ah+a^2+b^2=0.$

$D>0 \Rightarrow 16a^2-\cancel{4}.3.(a^2+b^2)>0.$

$\boxed{a^2>3b^2}$ H.P.

Que

If $al^2 - bm^2 + 2dl + 1 = 0$ a, b, d are fixed real numbers such that $a + b = d^2$, then prove that the line $lx + my + 1 = 0$ touches a fixed circle. Find its equation.

Sol

$al^2 - bm^2 + 2dl + 1 = 0$ $\Rightarrow$ $al^2 + 2dl + 1 = bm^2$

$a + b = d^2$

$+bl^2$: $(a+b)l^2 + 2dl + 1 = b(l^2 + m^2)$

$d^2l^2 + 2dl + 1 = b(l^2 + m^2)$

$(dl+1)^2 = b(l^2+m^2)$

$\sqrt{(\ )}$: $\dfrac{|dl+1|}{\sqrt{l^2+m^2}} = \sqrt{b}$

$lx + my + 1 = 0$

$p = r$

$\dfrac{|\_l + \_m + 1|}{\sqrt{l^2+m^2}} = r$

Comp

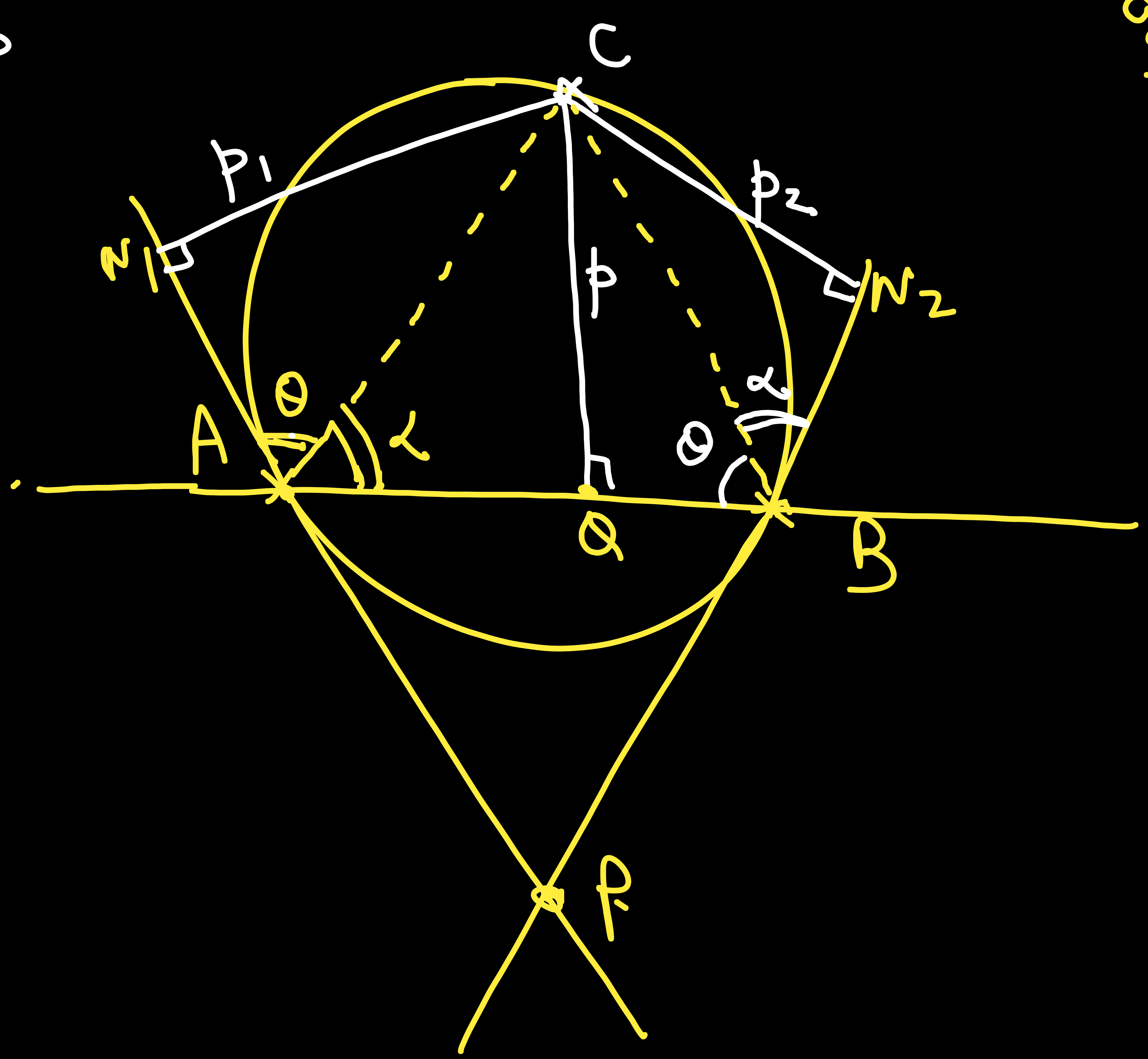
Center of circle $\equiv (d, 0)$

radius $= \sqrt{b}$

R. Circle $\boxed{(x-d)^2 + y^2 = b}$

Que Let tangents are drawn to a circle of radius r at pts A, B and point C also lies on the circle if length of ⊥'r from C to tangents is p_1 & p_2 and that to AB is p then find $\frac{p_1 p_2}{p^2}$.

Sol



In ΔCAQ

$p = AC \cdot \sin(\alpha)$

In ΔCQB.

$p = BC \sin(\theta)$

$p^2 = AC\, BC \cdot \sin(\alpha) \cdot \sin(\theta)$

$= (AC \sin(\theta)) \cdot (BC \sin(\alpha))$

from ΔCN_1A from ΔCN_2B

$= p_1 \cdot p_2$

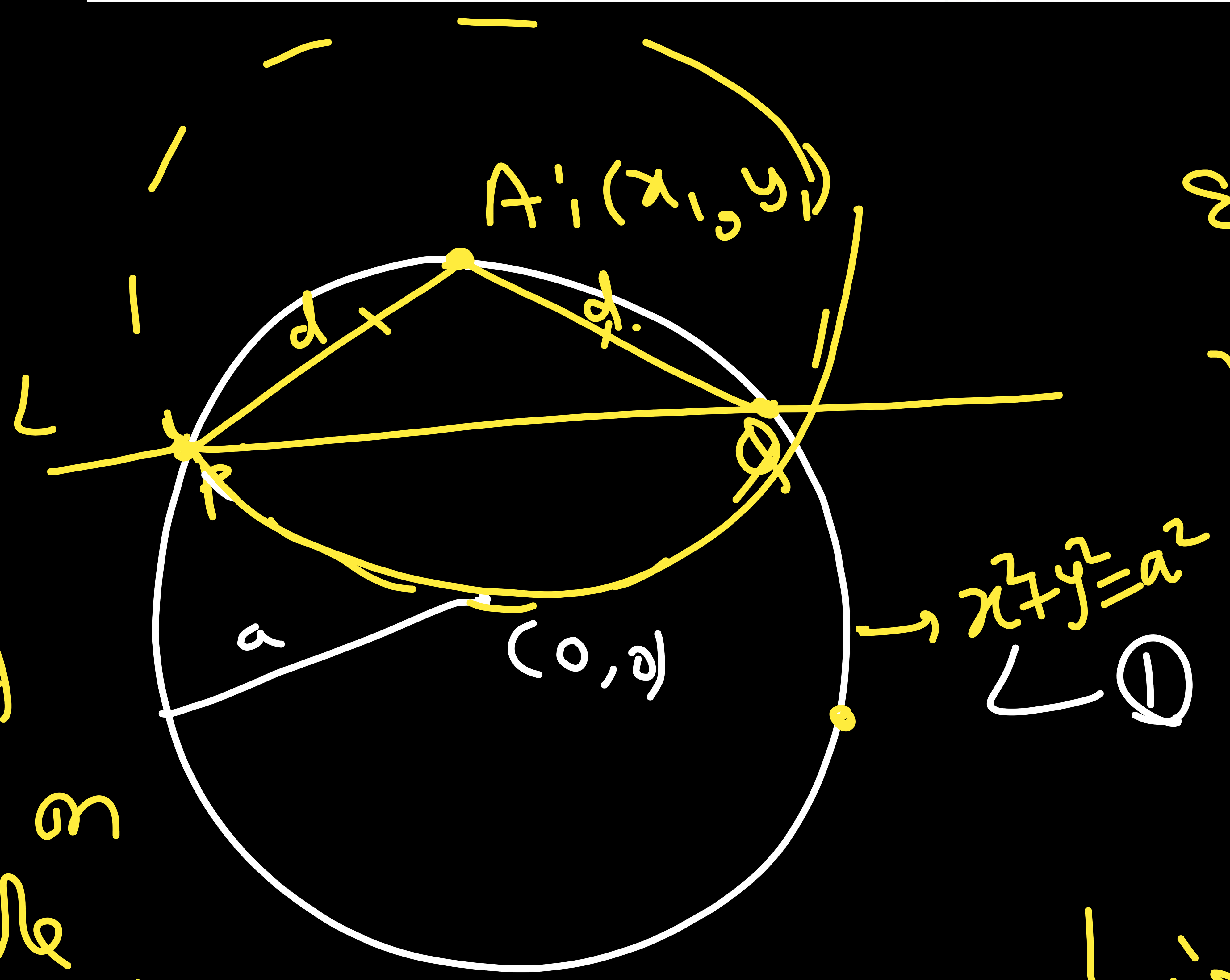
$\Rightarrow p^2 = p_1 \cdot p_2 \Rightarrow \boxed{\frac{p_1 p_2}{p^2} = 1}$ Ans

Que

Show that the equation of a straight line meeting the circle $x^2 + y^2 = a^2$ in 2 points at equal distance 'd' from the point (x_1, y_1) on its circumference is :

$$x x_1 + y y_1 - a^2 + \frac{d^2}{2} = 0$$

Sol



As A lies on circle $\Rightarrow x_1^2 + y_1^2 = a^2$

eqn of circle with centre A and radius 'd' will be

$$(x-x_1)^2 + (y-y_1)^2 = d^2.$$

$$x^2+y^2-2xx_1-2yy_1+x_1^2+y_1^2-d^2=0.$$

$$x^2+y^2-2xx_1-2yy_1+a^2-d^2=0. \quad ②$$

L is common chord of ① & ② hence it's eqn can be given as $S_1 - S_2 = 0$

$$x^2+y^2-a^2=0$$

$$x^2+y^2-2xx_1-2yy_1+a^2-d^2=0$$

$$-$$

$$2xx_1+2yy_1-2a^2+d^2=0$$

$$\boxed{xx_1+yy_1-a^2+\frac{d^2}{2}=0.}$$

H.P.

One

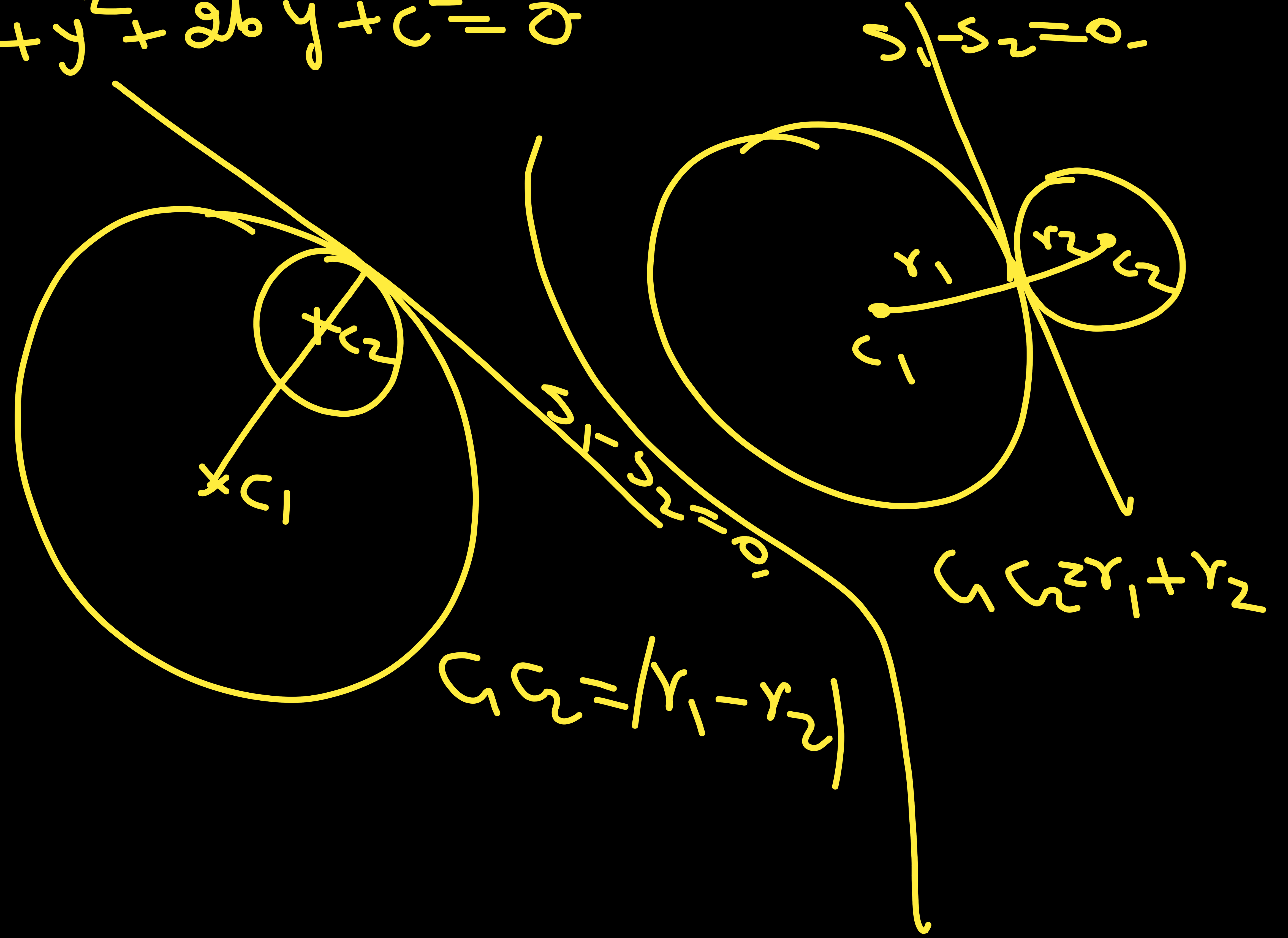
Prove that the circle $x^2+y^2+2ax+c^2=0$ and $x^2+y^2+2by+c^2=0$ touche each other if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

Sol

$x^2+y^2+2ax+c^2=0 \rightarrow C_1 : (-a, 0)$

$r_1=\sqrt{a^2-c^2}$.

$x^2+y^2+2by+c^2=0$



$S_1-S_2=0 \Rightarrow ax-by=0$.

It must be tangent to both the circles.

$p_1=r_1$

$\frac{|a^2-0|}{\sqrt{a^2+b^2}}=\sqrt{a^2-c^2}$.

$(\ )^2 \Rightarrow a^4=(a^2-c^2)(a^2+b^2)=\cancel{a^4}-c^2a^2+a^2b^2-b^2c^2$.

$\div a^2b^2c^2$

$0=-\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}$

$\Rightarrow \boxed{\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}}$ H.P.

Que

Prove that the 2 circles which passes through the 2 points (0, a) and (0, – a) and touche the straight line y = mx + d will cut orthogonally if $d^2 = a^2(2 + m^2)$

Sol

A: $(0, a)$
B: $(0, -a)$

$\to x^2 + y^2 - a^2 + 2\lambda x = 0$

$x^2 + y^2 - 2\lambda x - a^2 = 0$

$\hookrightarrow C : (\lambda, 0)$

$r = \sqrt{\lambda^2 + a^2}$

$y = mx + d$

$\to p = r \Rightarrow \dfrac{|m\lambda + d|}{\sqrt{1+m^2}} = \sqrt{\lambda^2 + a^2}$

$\cancel{m^2\lambda^2} + d^2 + 2dm\lambda = \lambda^2 + a^2 + \cancel{m^2\lambda^2} + m^2a^2$

$\lambda^2 - 2dm\lambda + \underbrace{a^2(1+m^2) - d^2} = 0 \to \lambda_1, \lambda_2$

Circle$_1 \to x^2 + y^2 - 2\lambda_1 x - a^2 = 0$

Circle$_2 \to x^2 + y^2 - 2\lambda_2 x - a^2 = 0$

$g_2 = -\lambda_2,\ f_2 = 0,\ c_2 = -a^2$

$2(g_1g_2 + f_1f_2) = c_1 + c_2$

$2(\lambda_1\lambda_2 + 0) = -a^2 - a^2$

$\cancel{2}\left(a^2(1+m^2) - d^2\right) = -\cancel{2}a^2$

$\boxed{a^2(2+m^2) = d^2}$

H.P.