Que

For a certain curve $y = f(x)$ satisfying $\frac{d^2y}{dx^2} = 6x - 4$, $f(x)$ has a local minimum value 5 when $x = 1$. Find the equation of the curve and also the global maximum and global minimum values of $f(x)$ given that $0 \le x \le 2$.

Sol

$$\frac{d^2y}{dx^2} = 6x - 4 \xrightarrow{\int} \frac{dy}{dx} = 3x^2 - 4x + c$$

$$y'(1) = 0$$

& $y(1) = 5$

$x = 1 \Rightarrow 0 = 3 - 4 + c$

$c = 1$

$$\frac{dy}{dx} = 3x^2 - 4x + 1 = (3x-1)(x-1).$$

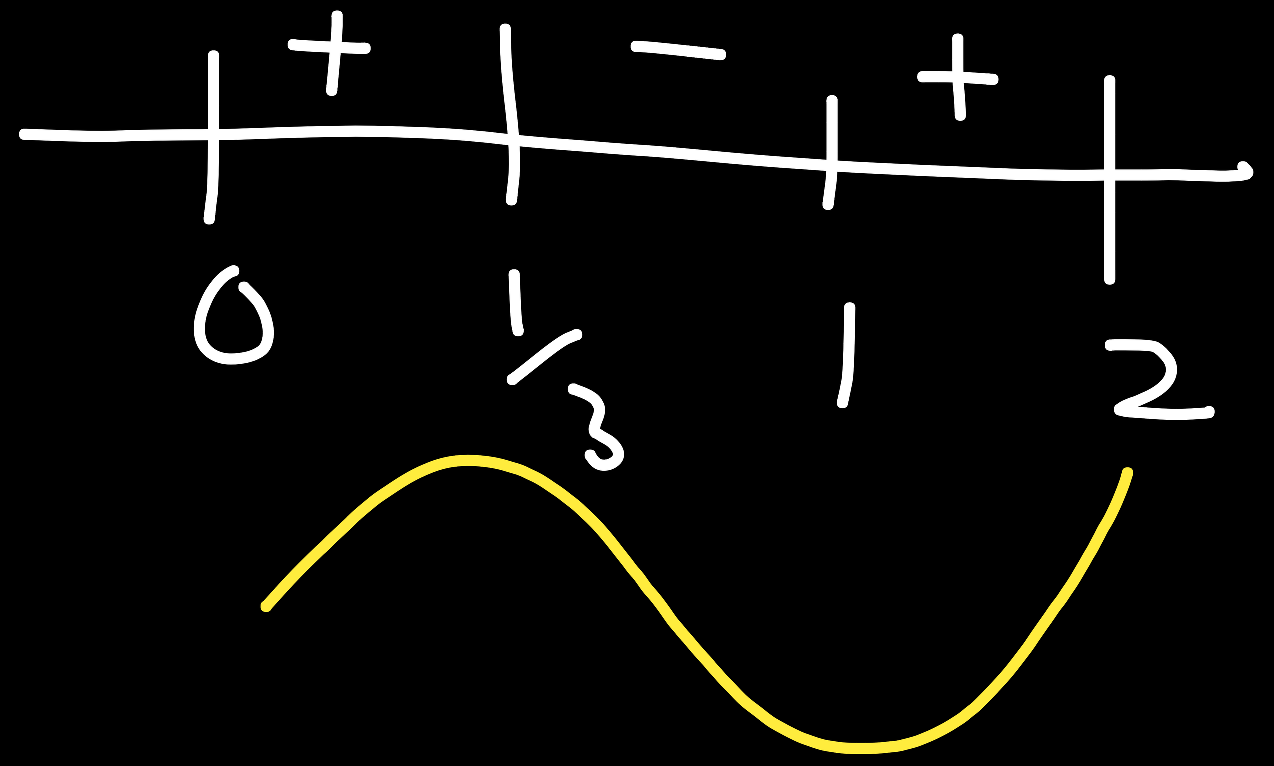
$$\xrightarrow{\int} y = x^3 - 2x^2 + x + c_1$$

$x = 1,\ y = 5$

$5 = 1 - 2 + 1 + c_1 \Rightarrow c_1 = 5$

$$\boxed{f(x) = x^3 - 2x^2 + x + 5}$$

$f'(x) = 0 \Rightarrow x = 1/3,\ 1$



$f(1) = 5$

$f(0) = 5$

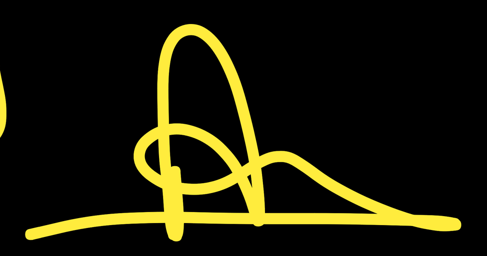
Global min. value $= \boxed{5}$

$$f(x) = x^3 - 2x^2 + x + 5$$

$$f\left(\tfrac{1}{3}\right) = \underbrace{\frac{1}{27} - \frac{2}{9} + \frac{1}{3}} + 5 = \frac{1-6+9}{27} + 5 = 5 + \frac{4}{27}$$

$$f(2) = 8 - 8 + 2 + 5 = 7$$

Global maximum value $= \boxed{7}$



One

$$\frac{dy}{dx}=\frac{(x-1)^2+(y-2)^2\tan^{-1}\left(\frac{y-2}{x-1}\right)}{(xy-2x-y+2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$$

$$\frac{dy}{dx}=\frac{(x-1)^2+(y-2)^2\cdot\tan^{-1}\left(\frac{y-2}{x-1}\right)}{(x-1)\cdot(y-2)\tan^{-1}\left(\frac{y-2}{x-1}\right)}$$

$$\frac{d(y-2)}{d(x-1)}=\frac{\cancel{(x-1)^2}}{\cancel{(x-1)^2}}\left(\frac{1+\left(\frac{y-2}{x-1}\right)^2\cdot\tan^{-1}\left(\frac{y-2}{x-1}\right)}{\left(\frac{y-2}{x-1}\right)\tan^{-1}\left(\frac{y-2}{x-1}\right)}\right)$$

$y-2=Y$

$x-1=X$

$$\frac{dY}{dX}=\frac{1+\left(\frac{Y}{X}\right)^2\tan^{-1}\left(\frac{Y}{X}\right)}{\frac{Y}{X}\tan^{-1}\left(\frac{Y}{X}\right)}\qquad \frac{Y}{X}=t$$

$$\frac{dX}{X}=\frac{dt}{\frac{1+t^2\tan^{-1}(t)}{t\tan^{-1}(t)}-t}=\frac{t\tan^{-1}(t)\,dt}{}$$

$$\int\frac{dX}{X}=\int \underset{\text{II}}{t}\,\underset{\text{I}}{\tan^{-1}(t)}\,dt$$

$$\Rightarrow \ln(X)=\frac{t^2}{2}\tan^{-1}(t)-\int\frac{t^2}{2(1+t^2)}\cdot dt \qquad 1+t^2-1$$

$$\ln(X)=\frac{t^2}{2}\tan^{-1}(t)-\frac{1}{2}\left(t-\tan^{-1}(t)\right)+C$$

$$\boxed{2\ln(x-1)=\left(\frac{y-2}{x-1}\right)^2\tan^{-1}\left(\frac{y-2}{x-1}\right)-\frac{y-2}{x-1}+\tan^{-1}\left(\frac{y-2}{x-1}\right)+C}$$

Q.

If y_1, y_2 are two different solutions of the equation $\frac{dy}{dx}+P(x).y=Q(x)$ then

(a) Prove that $y = y_1 + c(y_2 - y_1)$ is the general solution of the same equation where c is any constant.

(b) If $\alpha y_1 + \beta y_2$ be a solution of the given equation then find the relation between α and β.

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

$$\frac{dy_1}{dx} + P(x)\cdot y_1 = Q(x)$$

$$\frac{dy_2}{dx} + P(x)\, y_2 = Q(x)$$

$$\Rightarrow \frac{d}{dx}(y-y_1) + P(x)(y-y_1) = 0$$

$$\Rightarrow \frac{d}{dx}(y_2-y_1) + P(x)(y_2-y_1) = 0$$

$$\frac{d(y-y_1)}{y-y_1} = \frac{d(y_2-y_1)}{y_2-y_1}$$

$$\Rightarrow \ln(y-y_1) = \ln(y_2-y_1) + \ln C = \ln(c(y_2-y_1))$$

$$\Rightarrow y - y_1 = c(y_2-y_1) \Rightarrow \boxed{y = y_1 + c\cdot(y_2-y_1)}$$

Proved (a)

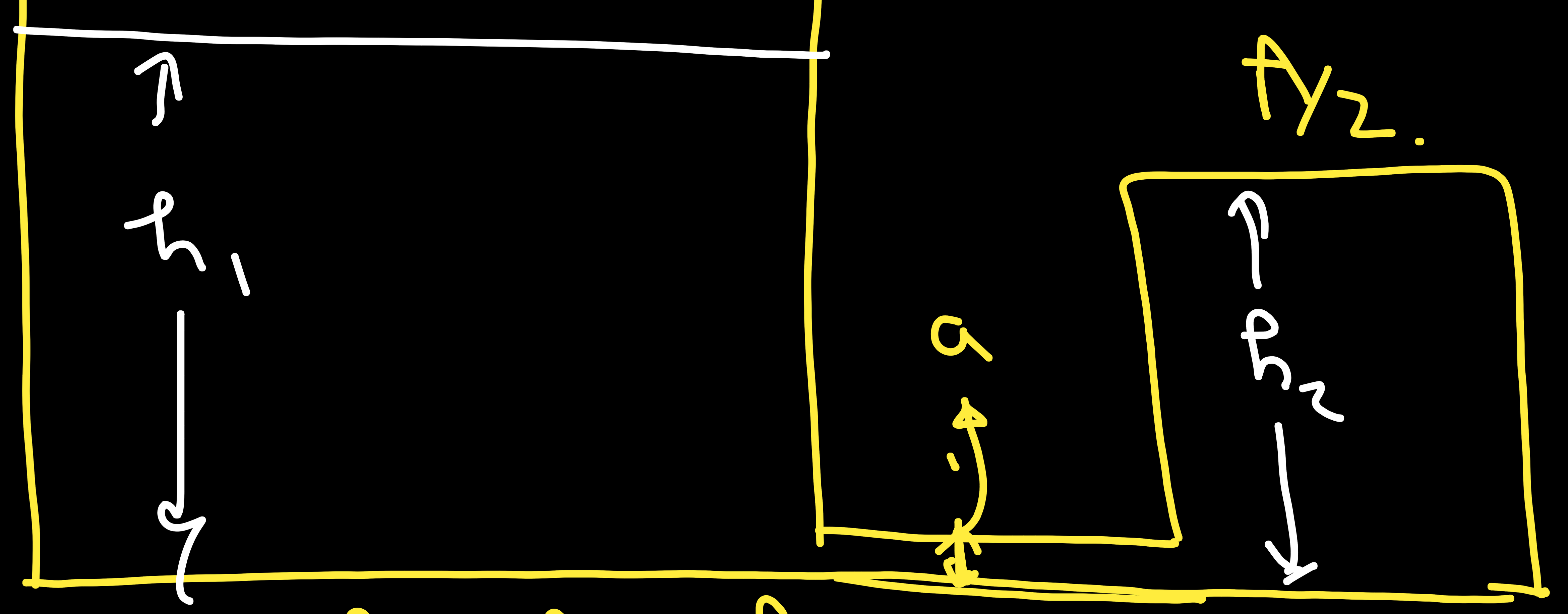
$$y = \underbrace{(1-c)}_{\alpha}\cdot y_1 + \underbrace{c}_{\beta}\cdot y_2$$

$$\boxed{\alpha + \beta = 1}$$

Ans (ii)

One

Two cylindrical tanks in which initially one is filled with water to the height of 1 m and other is empty, are connected by a pipe at the bottom. Water is allowed to flow from filled tank to the empty tank through the pipe. The rate of flow of water through the pipe at any time is $a\sqrt{2g(h_1 - h_2)}$, where 'h_1' and 'h_2' are the heights of water level (above pipe) in the tanks at that time and 'g' is acceleration due to gravity. If the cross sectional area of the filled and empty tanks be A and A/2 and that of the pipe be 'a', then find the time when the level of water in both tanks will be same (neglect the volume of the water in pipe).



when $h_1 = h_2 = h_f$

$2 = 2h_f + h_f \Rightarrow h_f = 2/3.$

$$\frac{dV}{dt} = a\sqrt{2g(h_1-h_2)}$$

$$\cancel{A}\cdot 1 = \cancel{A}\cdot h_1 + \frac{\cancel{A}}{2}\cdot h_2$$

$$2 = 2h_1 + h_2$$

$$h_1 - h_2 = h_1 - (2-2h_1)$$

$$= 3h_1 - 2.$$

$t \longrightarrow t+dt$

Water gone = Water less.

$$a\sqrt{2g(h_1-h_2)}\cdot dt = -A\cdot dh_1$$

$$a\sqrt{2g(3h_1-2)}\,dt = -A\,dh_1$$

$$a\sqrt{2g}\,\sqrt{3h_1-2}\,dt = -A\,dh_1$$

$$\frac{a\sqrt{2g}}{A}\int_0^t dt = -\int_1^{2/3}\frac{dh_1}{\sqrt{3h_1-2}}$$

$$a\cdot\frac{\sqrt{2g}}{A}\cdot t = -\frac{2}{3}\left[\sqrt{3h_1-2}\right]_1^{2/3} = -\frac{2}{3}\left(0-1\right) = 2/3$$

$$\boxed{t = \frac{2A}{3a\sqrt{2g}}}$$

One

Let f : (0, ∞) → (0, ∞) be a differentiable function satisfying, $x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dt$ $\forall x \in R^+$ and f (1) = 1. Determine f (x).

$f: (0,\infty) \longrightarrow (0,\infty)$

$$x \int_0^x (1-t) f(t)\cdot dt = \int_0^x t\, f(t)\cdot dt$$

$$\xrightarrow{\frac{d}{dx}} 1\cdot\int_0^x (1-t) f(t)\cdot dt + x\cdot(1-x) f(x) = \cancel{x f(x)}$$

$$\Rightarrow \int_0^x (1-t) f(t)\, dt = x^2 f(x) \xrightarrow{\frac{d}{dx}} (1-x) f(x) = x^2 f'(x) + 2x f(x)$$

$$(1-3x) f(x) = x^2 f'(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1-3x}{x^2} = \frac{1}{x^2} - \frac{3}{x} \xrightarrow{\int} \ln(f(x)) = -\frac{1}{x} + 3\ln(x) + C.$$

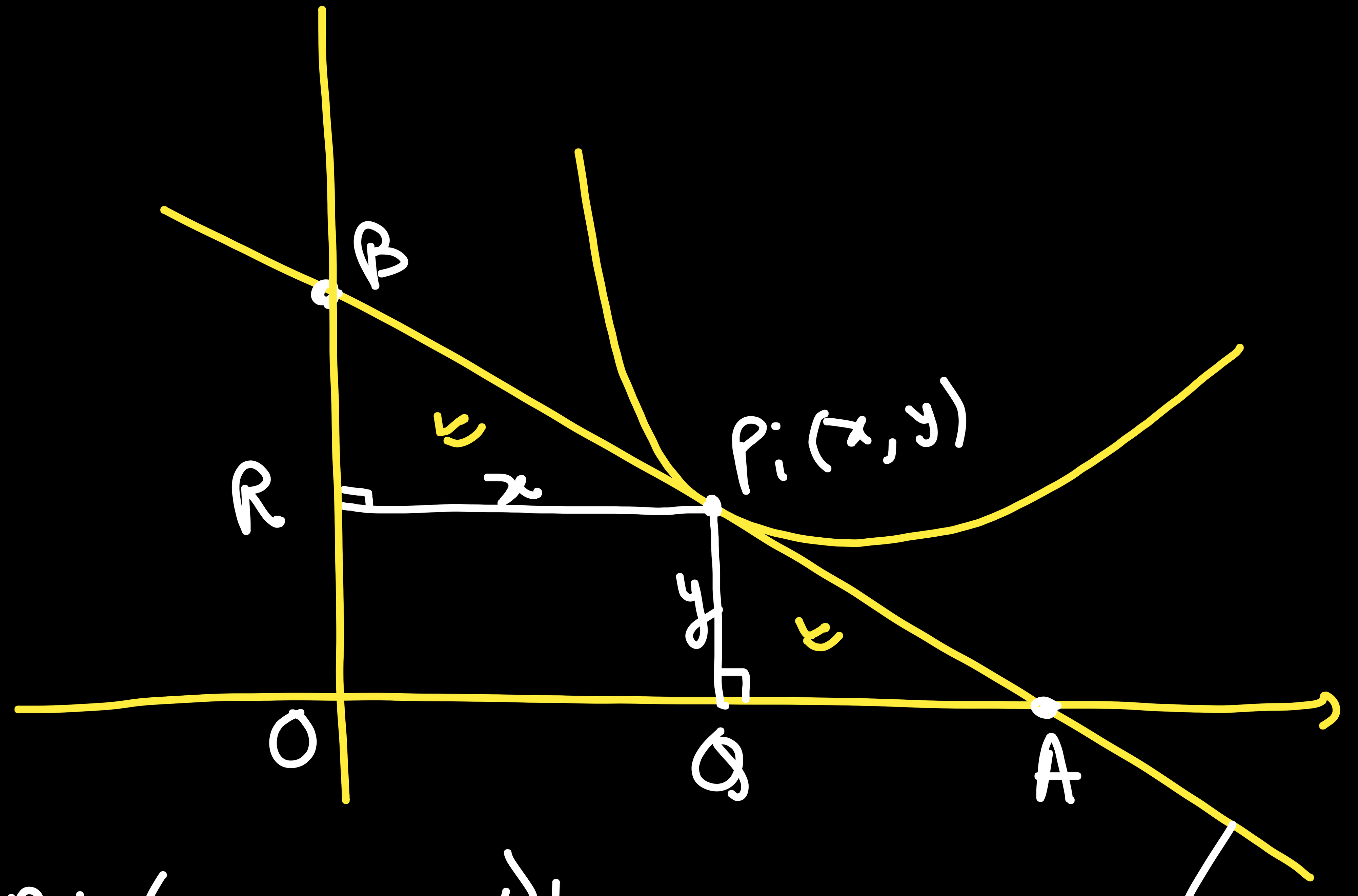
$$\xrightarrow[f(1)=1]{x=1} 0 = -1 + 0 + C \Rightarrow C = 1$$

$$\ln(f(x)) = -\frac{1}{x} + 3\ln(x) + 1$$

$$f(x) = e^{1-\frac{1}{x}} \cdot e^{3\ln(x)} = x^3 \cdot e^{1-\frac{1}{x}}.$$

Que

The tangent to a curve, together with the coordinate axes, the ordinate and the abscissa of the point of contact, makes two triangles of equal areas. Find the curve.



$$Y - y = y'(X - x)$$

$B;\ (0,\ y - xy')$

$$RB = OB - OR = (y - xy') - y = -xy'$$

$A;\ \left(x - \dfrac{y}{y'},\ 0\right)$

$$QA = OA - OQ = x - \frac{y}{y'} - x = -\frac{y}{y'}$$

Given: $\text{Area}(\Delta PBR) = \text{Ar}(\Delta PQA)$

$$\Rightarrow \left|\frac{1}{2}\cdot x\cdot(-xy')\right| = \left|\frac{1}{2}\cdot y\left(-\frac{y}{y'}\right)\right|$$

$$x^2 y' = \frac{y^2}{y'}$$

$$(y')^2 = \left(\frac{y}{x}\right)^2 \Rightarrow y' = \pm\, y/x$$

$$\int \frac{dy}{y} \pm \frac{dx}{x} = 0$$

$$\ln(y) \pm \ln(x) = K$$

$$\Rightarrow xy = c_1 \quad \text{or} \quad \frac{y}{x} = K$$

$$\boxed{xy = c_1} \quad \text{or} \quad \boxed{y = Kx}$$

Que

The line y = k intersects the curve y = f (x) atleast at two points. If $\int_2^x f(t)dt = \frac{x^2}{2} + \int_x^2 t^2 f(t)dt$ find $k \in R$.

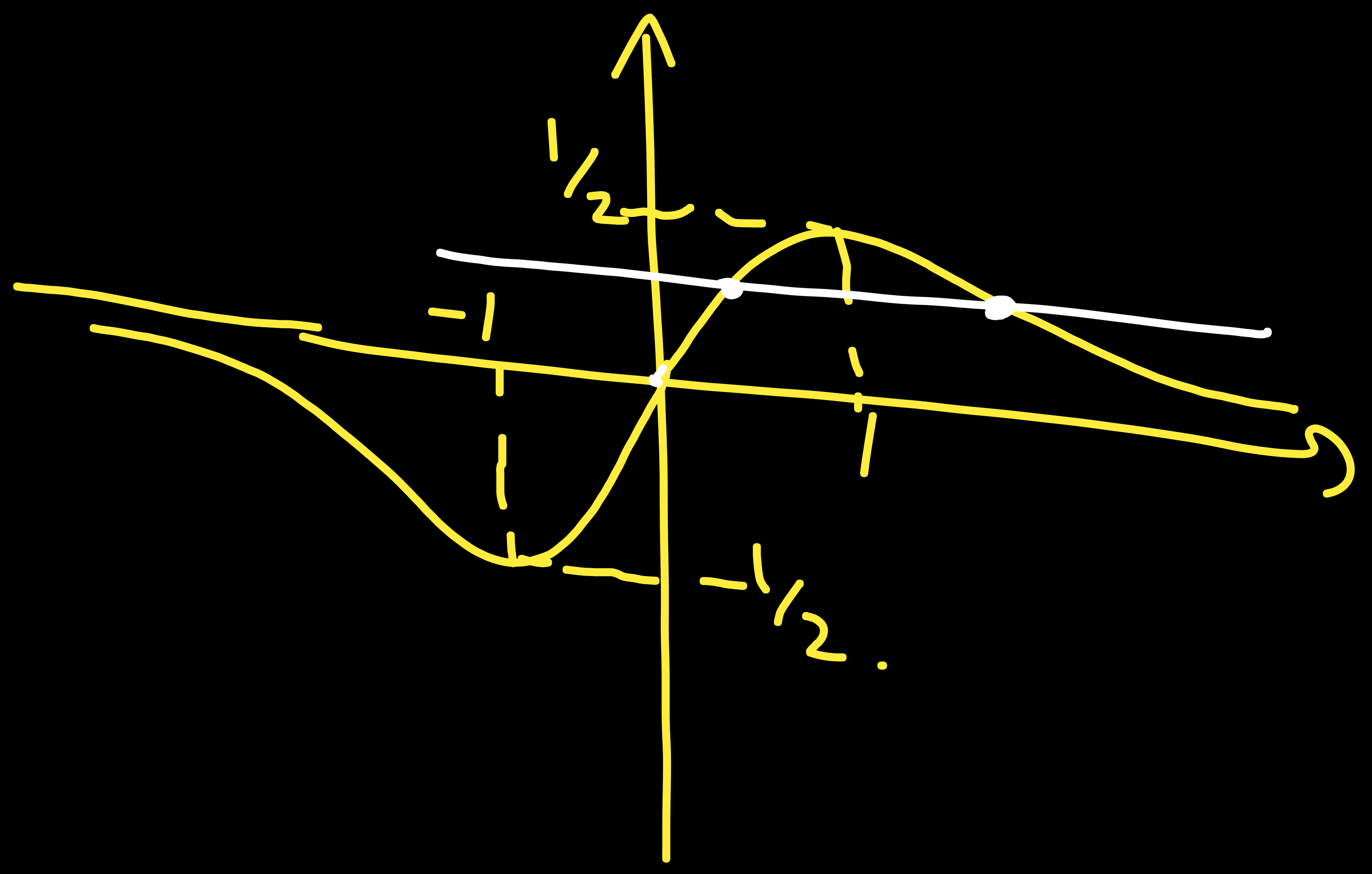
$$\int_2^x f(t)\,dt = \frac{x^2}{2} + \int_x^2 t^2 f(t)\cdot dt$$

$\frac{d}{dx}$

$$f(x) = x - x^2 f(x) \Rightarrow f(x) = \frac{x}{x^2+1}$$

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}.$$



$$\boxed{k \in \left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}}$$

Que

If $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ $\forall$ x, y $\in$ R and f′(1) exists, and area under the curve f(x) bounded by x-axis x = 0 and x = 1 is $\frac{1}{3}$, then find $\lim_{n\to\infty}\sum_{r=1}^{n} e^{r/n} f\left(\frac{\sqrt{r}}{n}\right)$.

$$f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}, \quad f'(1) \to \text{exists}$$

$$x = y = 1$$

$$f(1) = \frac{f(1)}{f(1)} \Rightarrow f^2(1) = \cancel{f(x)}$$

$$f(1) = 1$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x) \lim_{h\to 0} \frac{\frac{f(x+h)}{f(x)} - 1}{h}$$

$$= f(x) \lim_{h\to 0} \frac{f\left(\frac{x+h}{x}\right) - 1}{h}$$

$$= f(x) \cdot \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - 1}{h}$$

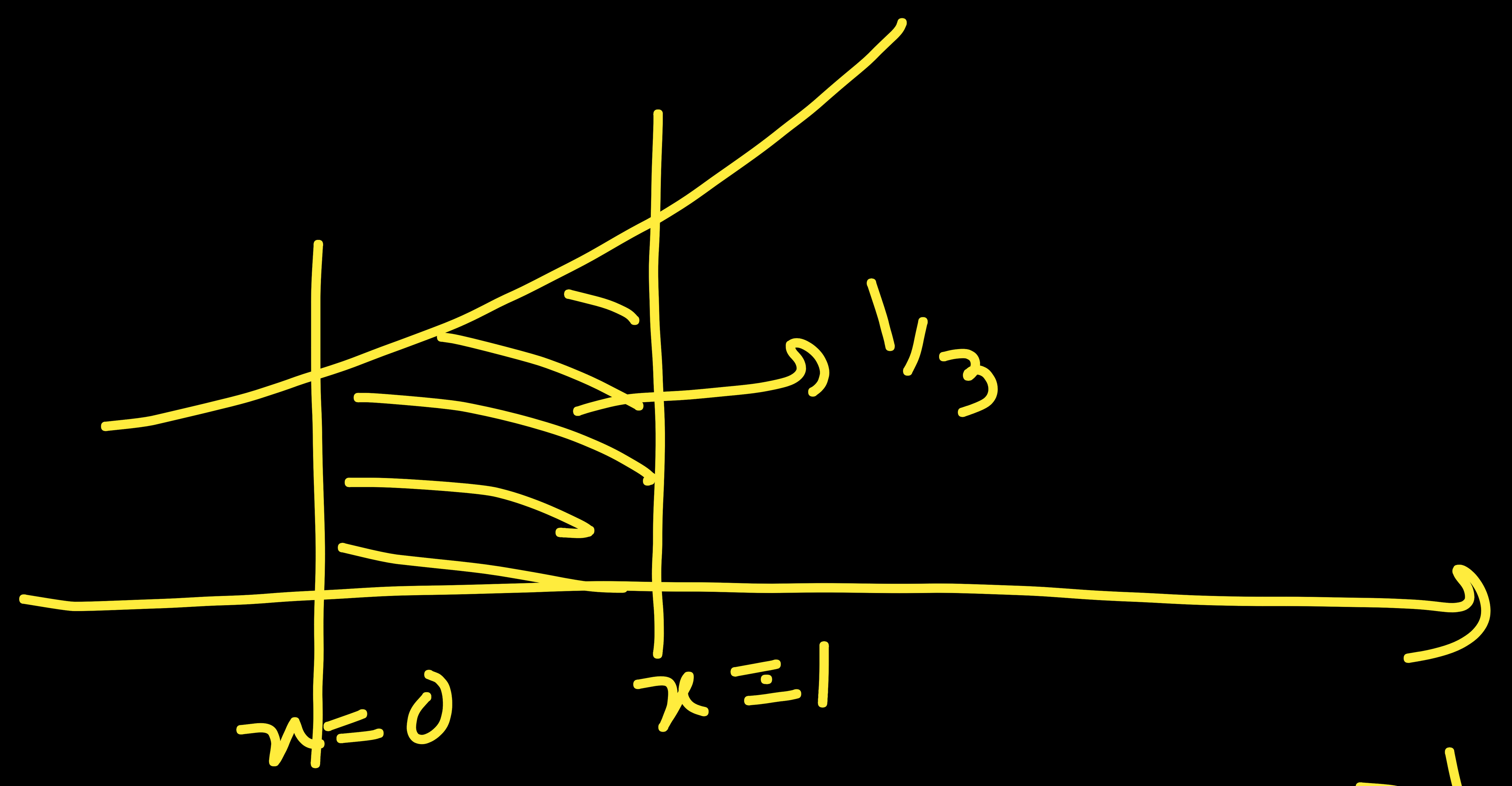
$$= \frac{f(x)}{x} \cdot \boxed{\lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{\left(\frac{h}{x}\right)}} \to f'(1) = K \text{ (say)}$$

$$f'(x) = K \cdot \frac{f(x)}{x} \Rightarrow \frac{f'(x)}{f(x)} = \frac{K}{x}$$

$$\int \quad \ln f(x) = K \ln(x) + c_1$$

$$f(x) = c \cdot x^K \xrightarrow{f(1)=1} c = 1$$

$$\boxed{f(x) = x^k}$$



$$\frac{1}{3} = \int_0^1 x^k\,dx = \left.\frac{x^{k+1}}{k+1}\right]_0^1 = \frac{1}{k+1}$$

$$\Rightarrow \boxed{k=2}$$

$$\Rightarrow f(x) = x^2.$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} e^{r/n}\cdot f\left(\frac{\sqrt{r}}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} e^{r/n}\cdot \frac{r}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{r}{n}\, e^{r/n}$$

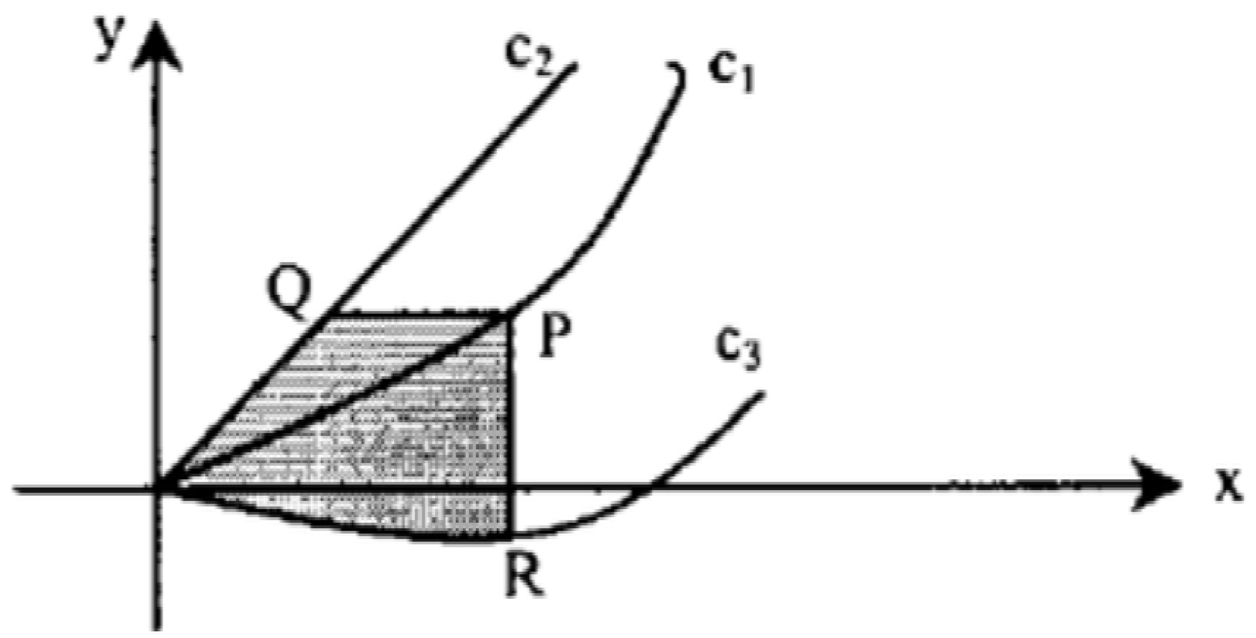
$$= \int_0^1 \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^x}\,dx = \left[x e^x\right]_0^1 - \int_0^1 e^x\,dx$$

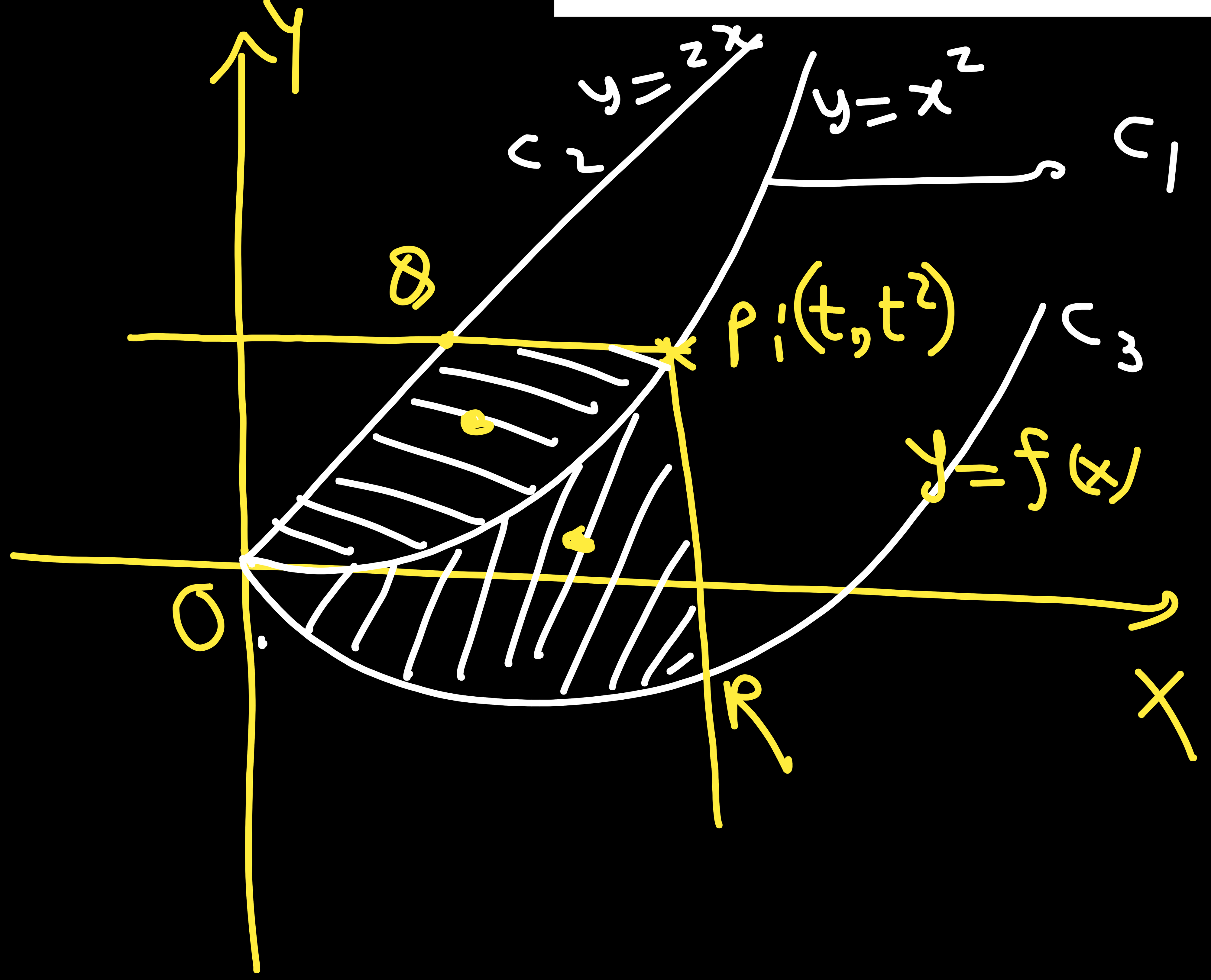
$$= (e-0) - (e-1) = \boxed{1}$$

Ans

Que

Let c_1, c_2 and c_3 be the graphs of the function $y = x^2$, $y = 2x$, $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on c_1, let the lines through P, parallel to the axes meet c_2 and c_3 at Q and R respectively (see figure). If for every position P(on c_1) the area of the shaded region OPQ and ORP are equal, determine the function f(x).





$$\int_0^{t^2}\left(\sqrt{y}-\frac{y}{2}\right)dy=\int_0^{t}\left(x^2-f(x)\right)dx$$

$\frac{d}{dt}$

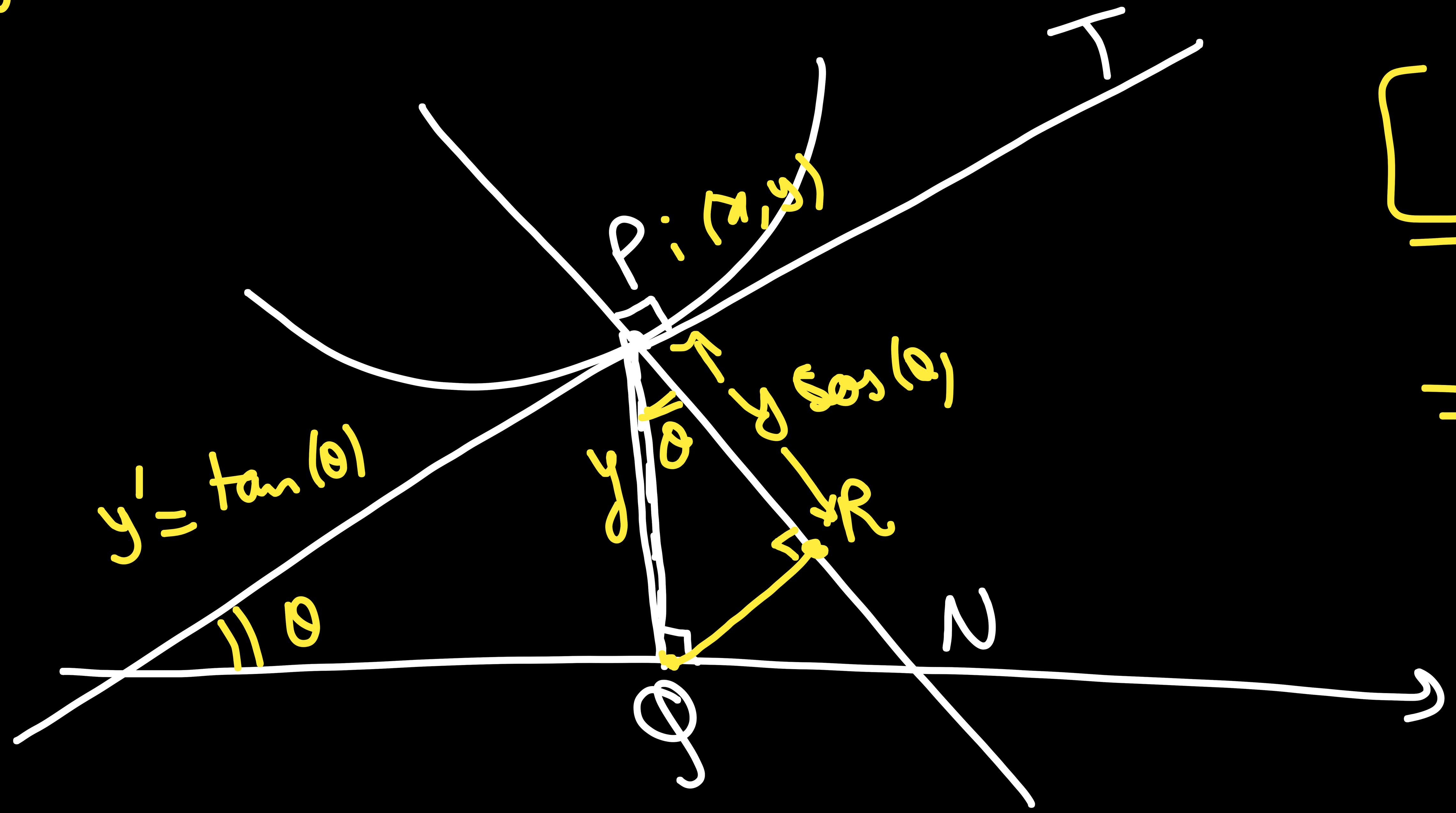
$$2t\left(t-\frac{t^2}{2}\right)=t^2-f(t)$$

$$f(t)=t^2-2t^2+t^3=t^3-t^2.$$

$$\boxed{f(x)=x^3-x^2}$$ Ans

Que Find the equation of the curve such that the projection of it's ordinate upon the normal is equal to it's abscissa.

Sol



$PR = x$

$\Rightarrow |y\cos(\theta)| = |x|$

$\Rightarrow \dfrac{y^2}{x^2} = 1 + \tan^2(\theta) = 1 + (y')^2$

$(y')^2 = \left(\dfrac{y^2}{x^2}\right) - 1 \Rightarrow y' = \pm\sqrt{\left(\dfrac{y}{x}\right)^2 - 1}$

$y/x = t$

$\dfrac{dx}{x} = \dfrac{dt}{\pm\sqrt{t^2-1} - t}$

$$\frac{dx}{x} = \frac{dt}{\pm\sqrt{t^2-1} - t} = \frac{\left(\pm\sqrt{t^2-1} + t\right)}{\cancel{t^2} - 1 - \cancel{t^2}} \cdot dt$$

$$\int \quad \ln(x) = -\int \left(\pm\sqrt{t^2-1} + t\right) dt$$

$$= \pm\left(\frac{t}{2}\sqrt{t^2-1} - \frac{1}{2}\ln\left|t+\sqrt{t^2-1}\right|\right) - \frac{t^2}{2} + C$$

$$\Big| \; t = y/x$$

Que

$$\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$$

Sol

$$\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$$

$$\Rightarrow e^{-y}\cdot\frac{dy}{dx} = \frac{1}{x^2} - \frac{e^{-y}}{x}$$

$$\Rightarrow e^{-y}\cdot\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2}$$

$$e^{-y} = Y$$

$$\Rightarrow -e^{-y}\cdot y' = \frac{dY}{dx}$$

$$\Rightarrow -\frac{dY}{dx} + \frac{Y}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{dY}{dx} - \frac{1}{x}\cdot Y = -\frac{1}{x^2}$$

L.D. Eqn

$\hookrightarrow$ I.F. $= \exp\left(-\int \frac{1}{x}\,dx\right)$

$= 1/x$

$$Y\cdot\frac{1}{x} = -\int \frac{1}{x^3}\,dx$$

$$\frac{Y}{x} = \frac{1}{2x^2} + C$$

$$\Rightarrow \boxed{\frac{e^{-y}}{x} = \frac{1}{2x^2} + C}$$

Ans

Que $\dfrac{dy}{dx} = y + \displaystyle\int_0^1 y \cdot dx$ given $y = 1$, when $x = 0$.

Sol

$$\frac{dy}{dx} = y + \int_0^1 y \cdot dx = y + k.$$

$$\frac{dy}{y+k} = dx$$

$$\ln(y+k) = x + c.$$

$x = 0,\ y = 1 \Rightarrow \ln(1+k) = c$

$$\ln(y+k) = x + \ln(1+k) = \ln\left(e^x(1+k)\right).$$

$$y + k = e^x(1+k) \Rightarrow y = e^x(1+k) - k.$$

But $k = \displaystyle\int_0^1 y\, dx = \int_0^1 \left(e^x(1+k) - k\right) dx$

$$k = (1+k)(e-1) - k.$$

$$k(2 - e + 1) = e - 1 \Rightarrow k = \frac{e-1}{3-e} \Rightarrow \boxed{y = e^x\left(1 + \frac{e-1}{3-e}\right) - \frac{e-1}{3-e}}$$