Que Evaluate $\displaystyle\int_{1/3}^{3} \frac{1}{x} \ln\left(\left|\frac{x+x^2-1}{x-x^2+1}\right|\right) dx$

Soln $\displaystyle I = \int_{1/3}^{3} \frac{1}{x} \ln\left|\frac{x+x^2-1}{x-x^2+1}\right| \cdot dx$

put $x = \frac{1}{t}$

$dx = -\frac{dt}{t^2}$

$$= \int_{3}^{1/3} t \ln\left|\frac{\left(\frac{t+1-t^2}{\cancel{t^2}}\right)}{\left(\frac{t-1+t^2}{\cancel{t^2}}\right)}\right| \cdot \left(-\frac{dt}{t^2}\right)$$

$$= -\int_{1/3}^{3} \frac{1}{t} \ln\left|\frac{t-1+t^2}{t+1-t^2}\right| \cdot dt = -I \Rightarrow \boxed{I = 0} \text{ Ans}$$

Que Let $f(x)$ is continuous and positive function satisfying $\int_0^1 f(x).dx = 1$, $\int_0^1 x.f(x).dx = 2$ and $\int_0^1 x^2 f(x).dx = 4$, find $f(x)$.

Sol

Given $f(x) > 0$

$$\int_0^1 f(x).dx = 1 \quad \text{—— (1)}$$

$$\int_0^1 x.f(x).dx = 2 \quad \text{—— (2)}$$

$$\int_0^1 x^2 f(x).dx = 4 \quad \text{—— (3)}$$

Consider $4.(1) - 4.(2) + (3)$

$$\int_0^1 f(x)(4 - 4x + x^2).dx = 4.1 - 4.2 + 4$$

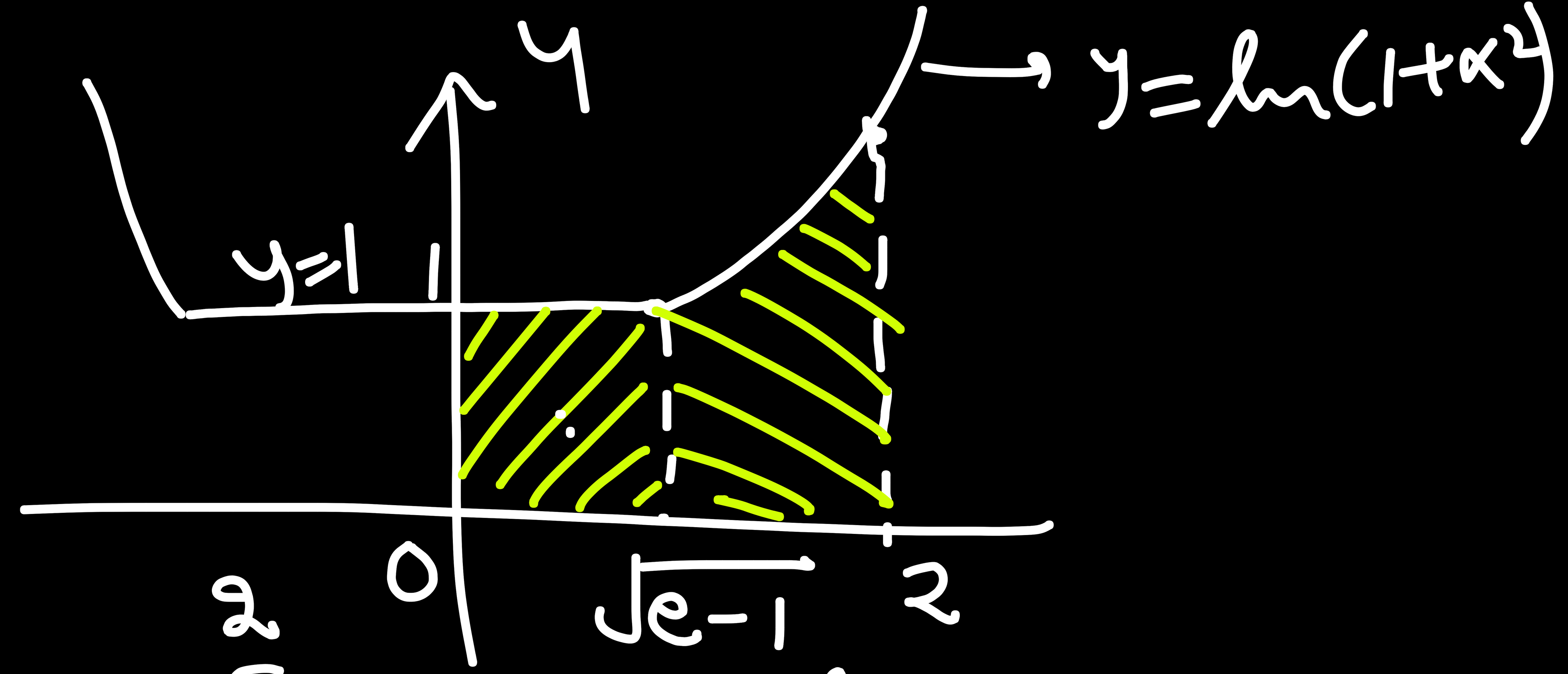
$$\int_0^1 (x-2)^2.f(x).dx = 0.$$

which is not possible as $(x-2)^2.f(x) > 0$ hence algebraic area must have some positive value.

$\Rightarrow$ There is no such function $f(x)$.

Que. Evaluate $\int_0^2 \max.\{\ln(1+x^2), 1\}\cdot dx$

Sol. $\int_0^2 \max.\{\ln(1+x^2), 1\}\cdot dx$

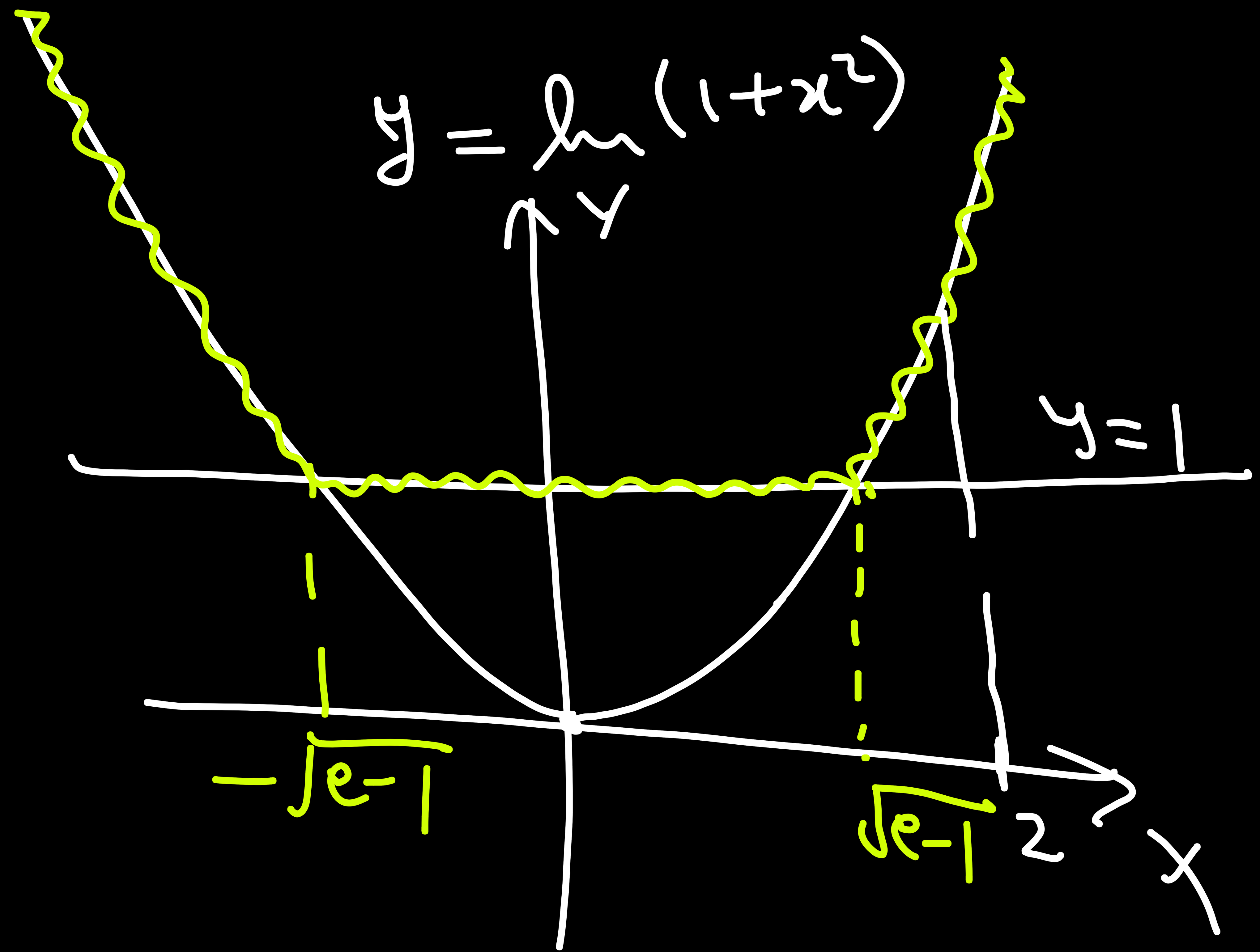


$= 1\cdot\sqrt{e-1} + \int_{\sqrt{e-1}}^{2} 1\cdot\ln(1+x^2)\cdot dx$

$= \sqrt{e-1} + \left[x\cdot\ln(1+x^2)\right]_{\sqrt{e-1}}^{2} - \int_{\sqrt{e-1}}^{2} \frac{2x^2}{1+x^2}\cdot dx$

$= \cancel{\sqrt{e-1}} + 2\ln(5) - \cancel{\sqrt{e-1}} - 2\int_{\sqrt{e-1}}^{2} \frac{x^2+1-1}{x^2+1}\,dx$

$= 2\ln(5) - 2\left(x - \tan^{-1}(x)\right)_{\sqrt{e-1}}^{2}$



$\ln(1+x^2) = 1$

$1+x^2 = e$

$x^2 = e-1$

$x = \pm\sqrt{e-1}$

$$= 2\ln(5) - 2\left(x - \tan^{-1}(x)\right)\Big|_{\sqrt{e-1}}^{2}$$

$$= 2\ln(5) - 2\left(2 - \tan^{-1}(2) - \sqrt{e-1} + \tan^{-1}\left(\sqrt{e-1}\right)\right)$$

$\boxed{\text{Que}}$ Evaluate $\displaystyle\int_0^{\infty} \frac{x^4+1}{x^2(x^2+1)^2}\cdot dx$

Sol

$$I = \int_0^{\infty} \frac{x^4+1}{x^2(x^2+1)^2}\cdot dx$$

$$x=\tan(\theta) \Rightarrow dx = \sec^2(\theta)\cdot d\theta$$

$$I = \int_0^{\pi/2} \frac{(\tan^4(\theta)+1)\cdot \sec^2(\theta)\cdot d\theta}{\tan^2(\theta)\cdot \sec^4(\theta)}$$

$$= \int_0^{\pi/2} \frac{s^4+c^4}{\cancel{c^4}}\cdot \frac{\cancel{c^2\cdot c^2}}{s^2}\, d\theta = \int_0^{\pi/2} \left(\frac{s^4+c^4}{s^2}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{(s^2+c^2)^2 - 2s^2c^2}{s^2}\cdot d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sin^2(\theta)}\, d\theta - \int_0^{\pi/2} 2\cos^2(\theta)\cdot d\theta$$

$$= \left[-\cot(\theta)\right]_0^{\pi/2} - \text{finite}$$

↳ D.N.E

$$= \text{D.N.E.}$$

Que Evaluate $\int_1^{\infty} \frac{x^4+1}{x^2(x^2+1)^2}\,dx$

Sol

$$\int_1^{\infty} \frac{x^4+2x^2+1-2x^2}{x^2(x^2+1)^2}\,dx$$

$$= \int_1^{\infty} \frac{1}{x^2}\,dx - 2\int_1^{\infty} \frac{dx}{(x^2+1)^2} \qquad x=\tan(\theta)$$

$$= \left[-\frac{1}{x}\right]_1^{\infty} - 2\int_{\pi/4}^{\pi/2} \frac{\sec^2(\theta)\,d\theta}{\sec^4(\theta)}$$

$$= 1 - \int_{\pi/4}^{\pi/2} (1+\cos(2\theta))\,d\theta$$

$$= 1 - \left[x + \frac{\sin(2\theta)}{2}\right]_{\pi/4}^{\pi/2} = 1 - \left(\frac{\pi}{4} + 0 - \frac{1}{2}\right) = \frac{3}{2} - \frac{\pi}{4} = \left(\frac{6-\pi}{4}\right) \text{Ans}$$

Que If $\int_0^{\pi} \theta^3 \cdot \ln(\sin(\theta))\, d\theta = \lambda \frac{\pi}{4} \int_0^{\pi} \theta^2 \ln(\sqrt{2}\sin(\theta)) \cdot d\theta$, then find '$\lambda$'.

Sol $\quad L.H.S = I = \int_0^{\pi} \theta^3 \cdot \ln(\sin(\theta)) \cdot d\theta = \int_0^{\pi} (\pi - \theta)^3 \cdot \ln(\sin(\theta)) \cdot d\theta$

$$I = \pi^3 \int_0^{\pi} \ln(\sin(\theta))\, d\theta - \underbrace{\int_0^{\pi} \theta^3 \ln(\sin(\theta))\, d\theta}_{= I} + 3\pi \int_0^{\pi} \theta^2 \ln(\sin(\theta))\, d\theta - 3\pi^2 \int_0^{\pi} \theta \ln(\sin(\theta))\, d\theta$$

$$2I = \pi^3 \cdot (-\pi \ln(2)) + 3\pi \int_0^{\pi} \theta^2 \ln(\sin(\theta)) \cdot d\theta - 3\pi^2 I_1 \text{ (say)}$$

$$I_1 = \int_0^{\pi} \theta \ln(\sin(\theta)) \cdot d\theta = \int_0^{\pi} (\pi - \theta) \ln \sin(\theta)\, d\theta = \pi(-\pi \ln(2)) - I_1$$

$$I_1 = -\frac{\pi^2}{2} \ln(2)$$

$$2I = -\pi^4 \ln(2) + 3\pi \int_0^{\pi} \theta^2 \ln(\sin(\theta)) \cdot d\theta - 3\pi^2 \left(-\frac{\pi^2}{2} \ln(2)\right)$$

$$= \frac{\pi^4}{2} \ln(2) + 3\pi \int_0^{\pi} \theta^2 \ln(\sin(\theta)) \cdot d\theta$$

$$I = \frac{\pi^4}{4}\ln(2) + \frac{3\pi}{2}\int_0^{\pi}\theta^2 \ln(\sin(\theta))\,d\theta.$$

$$R.H.S. = \lambda \cdot \frac{\pi}{4}\int_0^{\pi}\theta^2 \ln(\sqrt{2}\sin(\theta)).\,d\theta$$

$$= \frac{\lambda\pi}{4}\left(\int_0^{\pi}\theta^2 . \ln(\sqrt{2})\,d\theta + \int_0^{\pi}\theta^2 \ln(\sin(\theta))\,d\theta\right)$$

$$\left[\frac{\theta^3}{3}\cdot\frac{1}{2}\ln(2)\right]_0^{\pi}$$

$$= \frac{\lambda\pi}{4}\cdot\frac{\pi^3}{6}\cdot\ln(2) + \frac{\lambda\pi}{4}\int_0^{\pi}\theta^2 \ln(\sin(\theta))\,d\theta$$

Comparing with I we will get $\boxed{\lambda = 6}$ Ans

Que

Evaluate $\displaystyle\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)}{e^x+1} \cdot dx$

Soln

$x = \tan(\theta) \Rightarrow \theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

$\hookrightarrow 2\theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

$$\cos^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{2t(\theta)}{1+t^2(\theta)}\right)$$
$$= \cos^{-1}(\sin(2\theta))$$
$$= \frac{\pi}{2} - \sin^{-1}(\sin(2\theta))$$
$$= \frac{\pi}{2} - 2\theta$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}(\tan(2\theta)) = 2\theta$$

$+$

$\pi/2$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{\pi/2}{e^x+1} \cdot dx$$

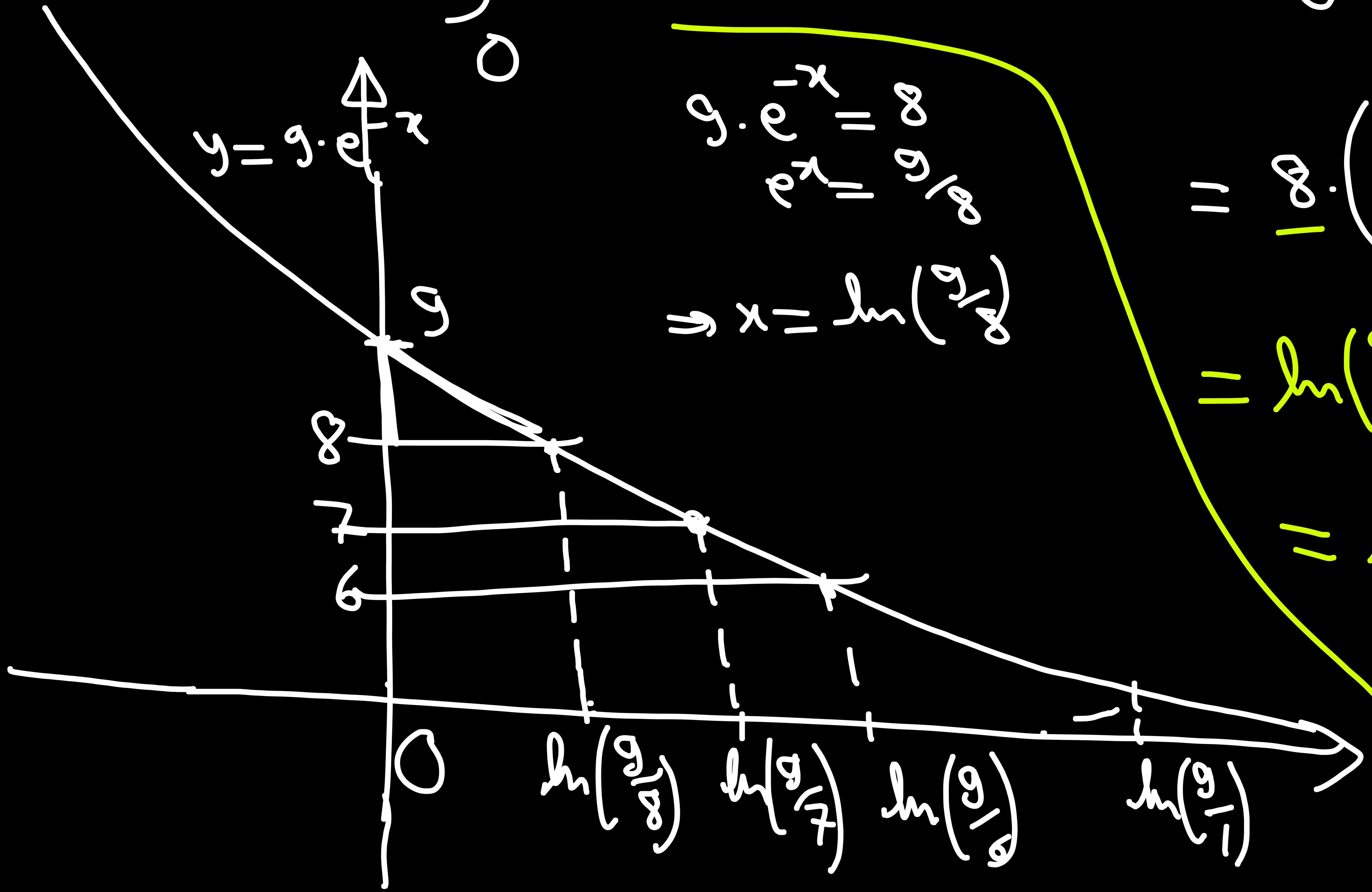
$$= \frac{\pi}{2} \int_{0}^{1/\sqrt{3}} \left(\frac{1}{e^x+1} + \frac{1}{e^{-x}+1}\right) dx$$

$$= \frac{\pi}{2} \int_{0}^{1/\sqrt{3}} \frac{1+e^x}{1+e^x} dx = \frac{\pi}{2} \cdot \frac{1}{\sqrt{3}} = \frac{\pi}{2\sqrt{3}}$$

Que Evaluate $\int_0^\infty \left[\frac{9}{e^x}\right] dx$, where $[\cdot]$ denotes G.I.F.

Sol

$$\int_0^\infty [9e^{-x}]\,dx = \int_0^{\ln(9/8)} 8\,dx + \int_{\ln(9/8)}^{\ln(9/7)} 7\,dx + \int_{\ln(9/7)}^{\ln(9/6)} 6\,dx + \dots + \int_{\ln(9/2)}^{\ln(9/1)} 1\,dx + \int_{\ln(9)}^{\infty} 0\,dx$$



$9\cdot e^{-x} = 8$

$e^{-x} = 9/8$

$\Rightarrow x = \ln(9/8)$

$$= 8\left(\ln\left(\frac{9}{8}\right) - 0\right) + 7\left(\ln\left(\frac{9}{7}\right) - \ln\left(\frac{9}{8}\right)\right) + 6\left(\ln\left(\frac{9}{6}\right) - \ln\left(\frac{9}{7}\right)\right) + \dots + 1\cdot\left(\ln(9) - \ln\left(\frac{9}{2}\right)\right)$$

$$= \ln\left(\frac{9}{8}\right) + \ln\left(\frac{9}{7}\right) + \ln\left(\frac{9}{6}\right) + \dots + \ln\left(\frac{9}{2}\right) + \ln(9)$$

$$= \ln\left(\frac{9^8}{\lfloor 8}\right) = \ln\left(\frac{9^9}{\lfloor 9}\right)$$

Ans

Que If $\displaystyle\int_0^\infty \frac{dx}{1+x^{2021}} = \int_0^1 \frac{dx}{(1-x^{2021})^\lambda}$, then find '$\lambda$'.

Sol

$$\text{L.H.S.} = \int_0^\infty \frac{dx}{1+x^{2021}} = \int_0^\infty \frac{dx}{1+\left(x^{\frac{2021}{2}}\right)^2}$$

$x^{\frac{2021}{2}} = \tan(\theta)$

$x = \tan^{\frac{2}{2021}}(\theta)$

$dx = \frac{2}{2021}\tan^{\frac{2}{2021}-1}(\theta)\cdot\sec^2(\theta)\,d\theta$

$$= \frac{2}{2021}\int_0^{\pi/2} \frac{\tan^{-\frac{2019}{2021}}(\theta)\cdot\cancel{\sec^2(\theta)}\,d\theta}{\cancel{\sec^2(\theta)}}$$

$$= \frac{2}{2021}\int_0^{\pi/2} \tan^{-\frac{2019}{2021}}(\theta)\cdot d\theta.$$

$$R.H.S. = \int_0^1 \frac{dx}{\left(1 - x^{2021}\right)^{\lambda}}$$

$$x^{\frac{2021}{2}} = \sin(\theta)$$

$$x = \sin^{\frac{2}{2021}}(\theta)$$

$$dx = \frac{2}{2021} \sin^{\frac{2}{2021}-1}(\theta) \cdot \cos(\theta)\, d\theta$$

$$= \frac{2}{2021} \int_0^{\pi/2} \frac{s^{-\frac{2019}{2021}} \cdot c}{\left(c^2\right)^{\lambda}} \cdot d\theta$$

$$= \frac{2}{2021} \int_0^{\pi/2} \frac{s^{-\frac{2019}{2021}}}{c^{2\lambda - 1}} \cdot d\theta$$

Compare

$$2\lambda - 1 = -\frac{2019}{2021}$$

$$2\lambda = \frac{2}{2021}$$

$$\boxed{\lambda = \frac{1}{2021}}$$

Ans

$$L.H.S. = \frac{2}{2021} \int_0^{\pi/2} \tan^{-\frac{2019}{2021}}(\theta)\, d\theta = \frac{2}{2021} \int_0^{\pi/2} \frac{\sin^{-\frac{2019}{2021}}(\theta)}{\cos^{-\frac{2019}{2021}}(\theta)}\, d\theta$$

Que If $I(u) = \int_0^{\pi} \ln(1-2u\cos(x)+u^2)\,.dx$, then find $\dfrac{I(u)+I(-u)}{I(u^2)}$.

Sol

$$I(u) = \int_0^{\pi} \ln(1-2u\cos(x)+u^2)\,.dx$$

$u \to -u$

$$I(-u) = \int_0^{\pi} \ln(1+2u\cos(x)+u^2)\,.dx = \int_0^{\pi} \ln(1-2u\cos(x)+u^2)\,.dx.$$

$a+b-x$

$\hookrightarrow x \to \pi - x$

$$= I(u).$$

$+$

$$I(u)+I(-u) = \int_0^{\pi} \ln\big((1+2u\cos(x)+u^2)(1-2u\cos(x)+u^2)\big)\,.dx$$

$$= \int_0^{\pi} \ln\big((1+u^2)^2 - 4u^2\cos^2(x)\big)\,dx = \int_0^{\pi} \ln(1+u^4+\underbrace{2u^2-4u^2\cos^2(x)}\,)\,.dx$$

$\hookrightarrow = 2u^2(1-2\cos^2(x))$

$= 2u^2(-\cos(2x))$

$$I(u)+I(-u) = \int_0^{\pi} \ln\left(1+u^4-2u^2\cos(2x)\right)\cdot dx$$

$\downarrow$ 0 – 2a

$$= 2\int_0^{\pi/2} \ln\left(1+u^4-2u^2\cos(2x)\right)\cdot dx$$

$2x = t$

$dx = \frac{dt}{2}$

$$= \cancel{2}\int_0^{\pi} \ln\left(1+u^4-2u^2\cos(t)\right)\cdot \frac{dt}{\cancel{2}} = \int_0^{\pi} \ln\left(1+u^4-2u^2\cos(t)\right)dt$$

$$= I(u^2)$$

$$\Rightarrow I(u)+I(-u) = I(u^2)$$

$$\Rightarrow \frac{I(u)+I(-u)}{I(u^2)} = \boxed{1}$$

Que Evaluate $\int_0^\infty \frac{dx}{1+x^4}$.

Sol

$$I = \int_0^\infty \frac{dx}{1+x^4} \xrightarrow[dx = -\frac{dt}{t^2}]{x=\frac{1}{t}} \int_\infty^0 \frac{-\frac{dt}{t^2}}{\left(\frac{1+t^4}{t^4}\right)}$$

$$I = \int_0^\infty \frac{t^2\,dt}{1+t^4} = \int_0^\infty \frac{x^2\,dx}{1+x^4}$$

$+$

$$2I = \int_0^\infty \frac{1+x^2}{1+x^4}dx = \int_0^\infty \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \int_0^\infty \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)\Bigg|_0^\infty = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{\sqrt{2}}$$

$$2I = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \boxed{I = \frac{\pi}{2\sqrt{2}}}$$

$\boxed{\text{Que}}$ Evaluate $\int_0^1 \cot^{-1}(1-x+x^2)\cdot dx$

Sol

$$I = \int_0^1 \cot^{-1}(1-x+x^2)\cdot dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)\cdot dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{(1-x)+x}{1-x\cdot(1-x)}\right) dx$$

$$= \int_0^1 \left(\tan^{-1}(1-x)+\tan^{-1}(x)\right) dx$$

$$= \int_0^1 \underbrace{\tan^{-1}(1-x)}_{a+b-x} dx + \int_0^1 \tan^{-1}(x)\cdot dx = 2\int_0^1 \tan^{-1}(x)\, dx$$

$$I = 2\int_0^1 1\cdot\tan^{-1}(x)\, dx$$

$$= 2\left(\left[x\cdot\tan^{-1}(x)\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx\right)$$

$$= 2\left(1\cdot\frac{\pi}{4} - 0\right) - \left[\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{2} - \ln(2)$$

Ans

$$\frac{1}{m+n+1}\left(1+t+t^2+\cdots+t^{m+n}\right) = \sum_{r=0}^{m+n} {}^{m+n}C_r \cdot t^r \cdot \int_0^1 (1-x)^{m+n-r} \cdot x^r \cdot dx$$

$$= \left(\int_0^1 (1-x)^{m+n} \cdot dx\right) + t\left(\quad \right) + t^2\left(\quad \right)$$

$$+ \cdots + t^m\left({}^{m+n}C_m \cdot \int_0^1 (1-x)^n \cdot x^m \cdot dx\right) + \cdots$$

Comparing coeff of t^m on both sides

$$\frac{1}{m+n+1} = {}^{m+n}C_m \cdot \int_0^1 (1-x)^n \cdot x^m \cdot dx$$

$$\int_0^1 (1-x)^n \cdot x^m \, dx = \frac{1}{(m+n+1)\cdot {}^{m+n}C_m} = \boxed{\frac{\underline{|n}\;\underline{|m}}{\underline{|m+n+1}}}$$

Ans

$\boxed{\text{Que}}$ If $\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\cdot\frac{\ln(x)}{x}\cdot dx = \lambda\cdot\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\cdot\frac{dx}{x}$ then find 'λ'.

given 'a' is positive real constant.

$\underline{\text{Sol}}$

$$L.H.S = I = \int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\cdot\frac{\ln(x)}{x}\cdot dx$$

$$\frac{x}{a} = \frac{a}{t}$$

$$dx = -\frac{a^2}{t^2}\cdot dt$$

$$= \int_\infty^0 f\left(\frac{t}{a}+\frac{a}{t}\right)\cdot\frac{\ln\left(\frac{a^2}{t}\right)}{\left(\frac{\cancel{a^2}}{t}\right)}\cdot\left(-\frac{\cancel{a^2}}{t^2}\cdot dt\right)$$

$$= \int_0^\infty f\left(\frac{t}{a}+\frac{a}{t}\right)\cdot\frac{\ln\left(\frac{a^2}{t}\right)}{t}\cdot dt = \int_0^\infty f\left(\frac{t}{a}+\frac{a}{t}\right)\cdot\frac{\ln(a^2)}{t}dt - \underbrace{\int_0^\infty f\left(\frac{t}{a}+\frac{a}{t}\right)\cdot\frac{\ln(t)}{t}dt}_{I}$$

$$\cancel{2}I = \cancel{2}\ln(a)\int_0^\infty f\left(\frac{t}{a}+\frac{a}{t}\right)\cdot\frac{dt}{t} \Rightarrow I = \ln(a)\int_0^\infty f\left(\frac{t}{a}+\frac{a}{t}\right)\cdot\frac{dt}{t}$$

Comparing

$$\lambda = \frac{\ln(a)}{A}$$