Que

Draw the graph of $y = g(x)$, where

$$g(x) = \max.\{f(t)\ ;\ x+1 \le t \le x+2\}, \quad x < 0$$
$$= 1 - x, \quad x \ge 0,$$

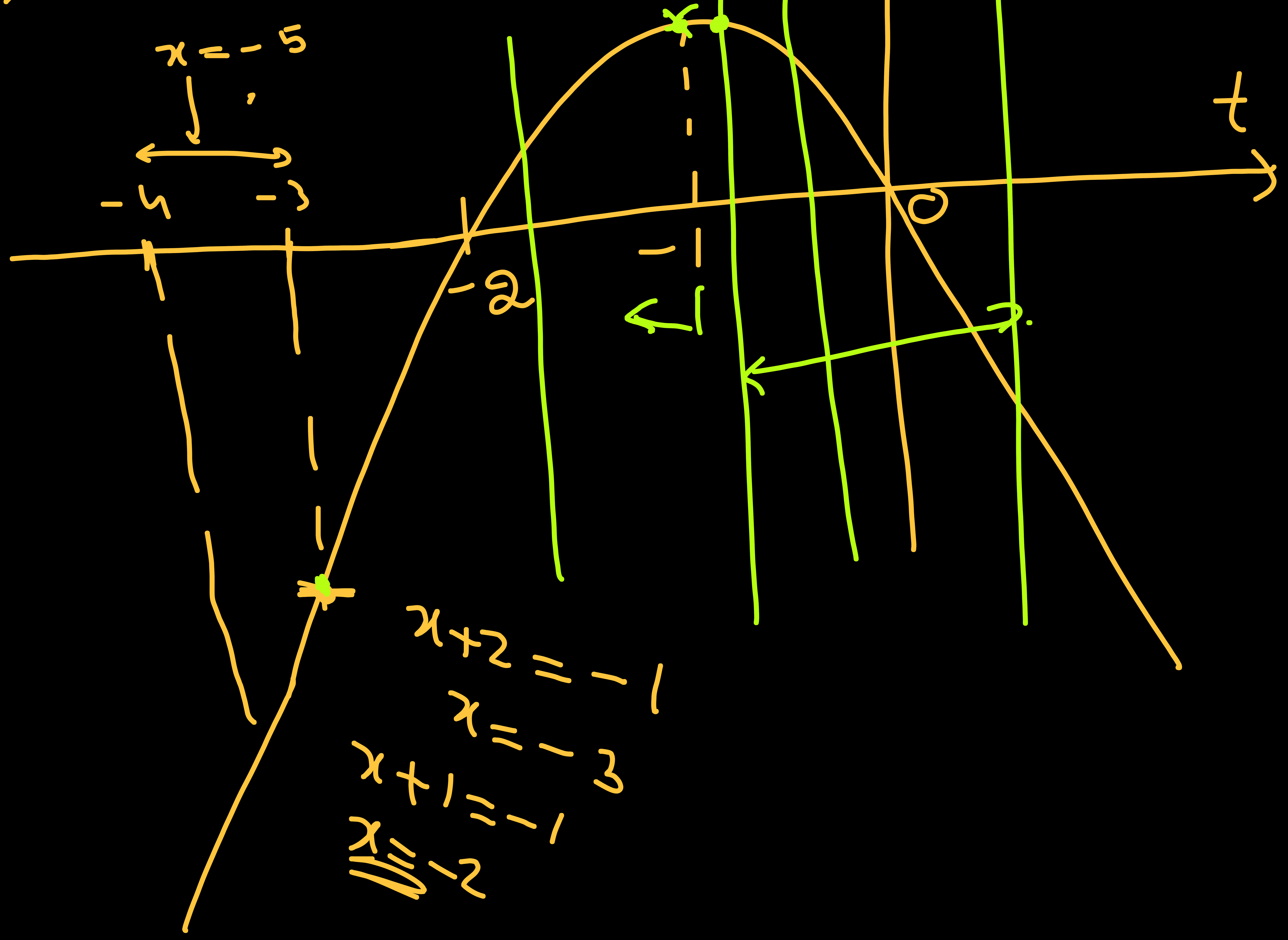
where $f(x) = -x^2 - 2x$

Sol

$$g(x) = f(x+2), \quad x < -3$$
$$= f(-1), \quad -3 \le x \le -2.$$
$$= f(x+1), \quad -2 < x < 0$$
$$= 1 - x, \quad x \geq 0.$$

$y = f(t) = -t(t+2)$

$x+1 \le t \le x+2$



$x+2 = -1$

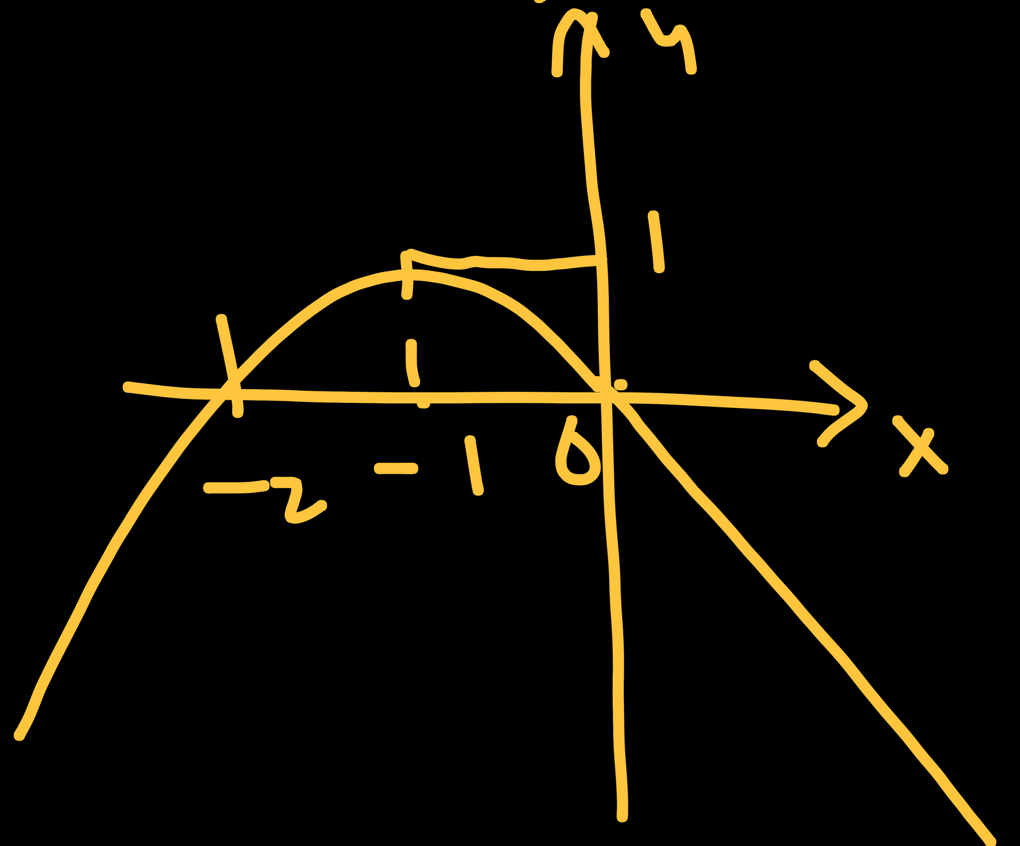
$x = -3$

$x+1 = -1$

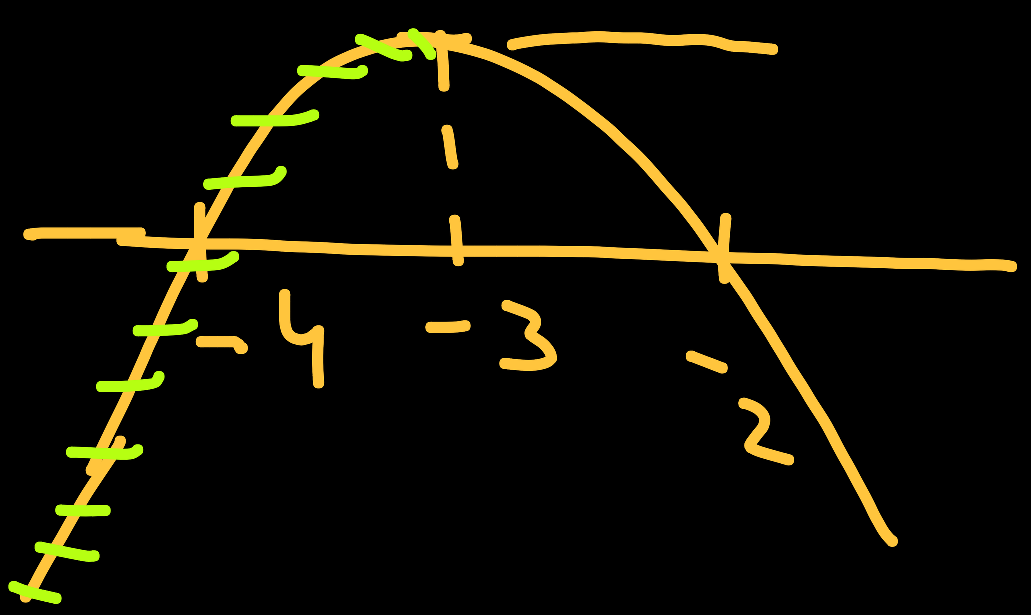
$x = -2$

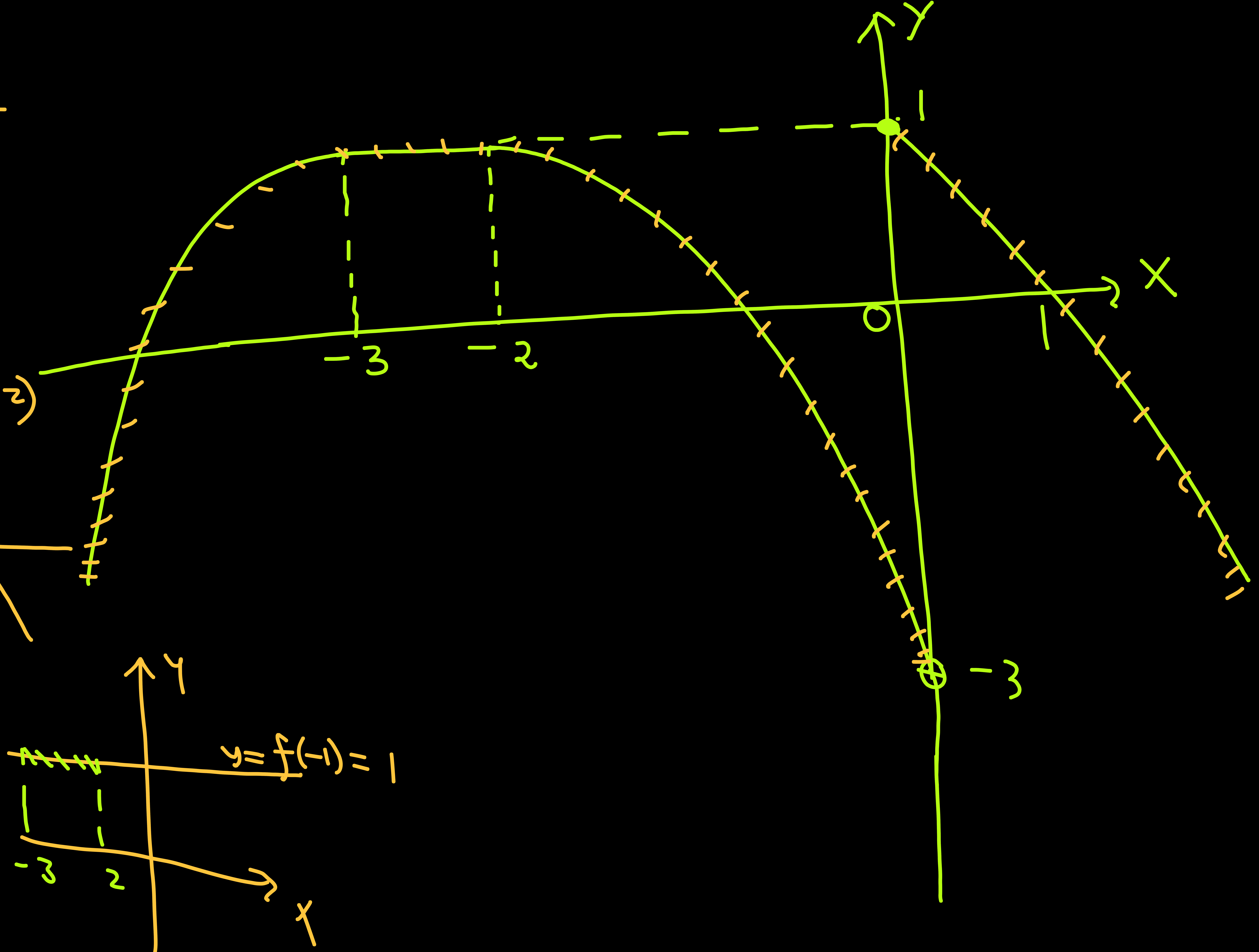
$$g(x) = \begin{cases} f(x+2), & x < -3 \\ f(-1), & -3 \le x \le -2 \\ f(x+1), & -2 < x < 0 \\ 1 - x, & x \ge 0. \end{cases}$$

$y = f(x) = -x(x+2)$

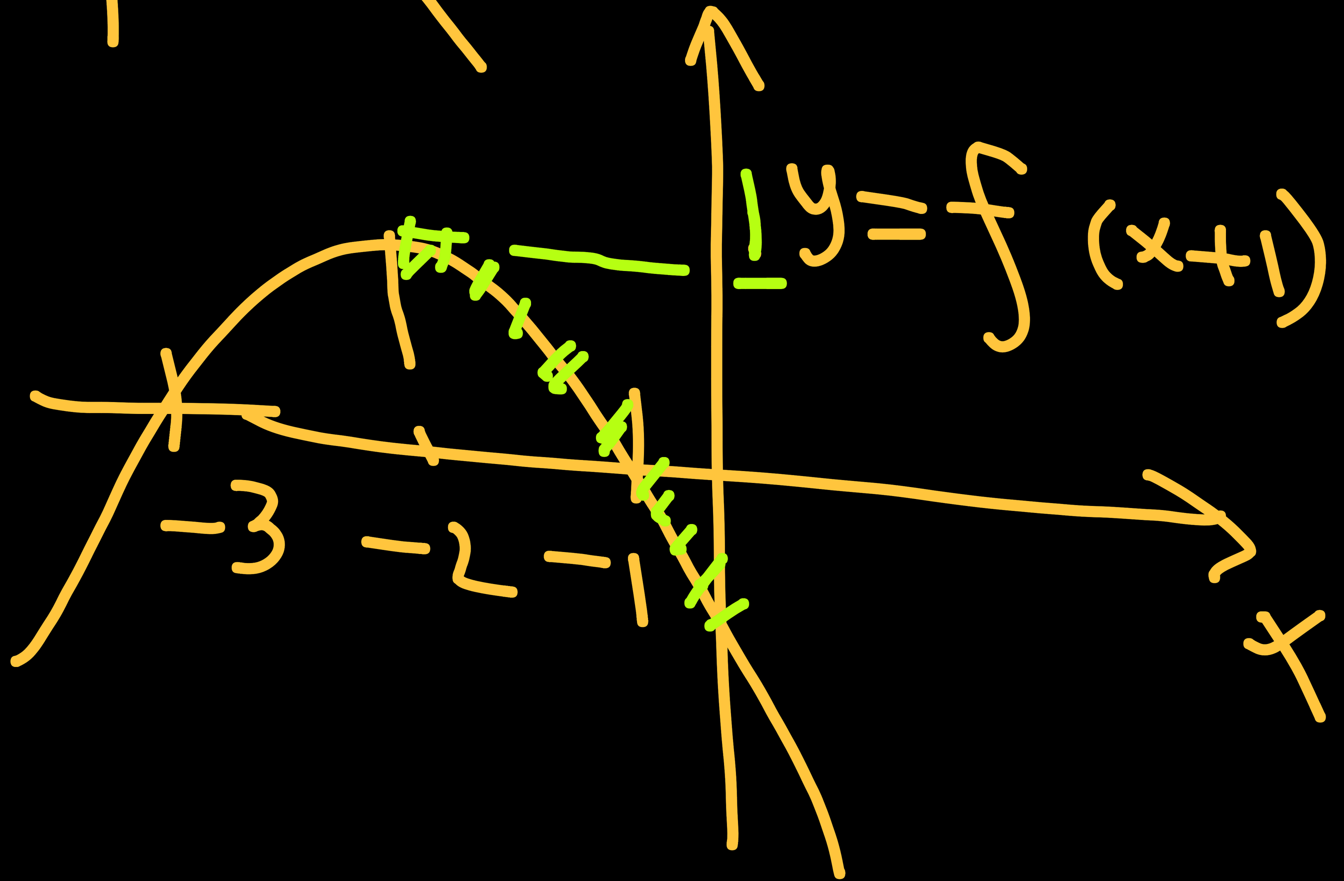


$y = f(x+2)$





$y = f(x+1)$



$y = f(-1) = 1$

Que Find a polynomial function $f(x)$, satisfying

$$f(2x) = f'(x) \cdot f''(x).$$

Soln

$$f(2x) = f'(x) \cdot f''(x)$$

$$n = n-1 + n-2$$

$$n = 3.$$

$$f(x) = ax^3 + bx^2 + cx + d$$

$$\Rightarrow 8ax^3 + 4bx^2 + 2cx + d = (3ax^2 + 2bx + c) \cdot (6ax + 2b)$$

$$8a = 18a^2 \Rightarrow a = \frac{4}{9}.$$

$$4b = 12ab + 6ab = 18ab \Rightarrow 4b = \cancel{18}^{2} \cdot \frac{4}{\cancel{9}} \cdot b$$

$$\boxed{b = 0}$$

$$2c = 6ac + 4b^2 \xrightarrow[a = 4/9]{b = a} \boxed{c = 0}$$

$$d = 2bc \Rightarrow d = 0$$

$$\Rightarrow \boxed{f(x) = \frac{4}{9}x^3}$$ Ans

Que. If $\sqrt{x+y} + \sqrt{y-x} = 5$ then find $\frac{d^2y}{dx^2}$

Sol

$$\sqrt{x+y} + \sqrt{y-x} = 5$$

$$\Rightarrow \sqrt{x+y} = 5 - \sqrt{y-x}$$

$$x + \cancel{y} = 25 + \cancel{y} - x - 10\sqrt{y-x}$$

$$2x - 25 = -10\sqrt{y-x}$$

$$4x^2 + 625 - \cancel{100x} = 100(y - \cancel{x})$$

$$4x^2 + 625 = 100y$$

$( )''$

$$\Rightarrow 8 = 100y'' \Rightarrow y'' = \frac{8}{100} = \boxed{\frac{2}{25}}$$

Ans

Que Let $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\left(\sqrt{x} + \sqrt{x^2+1}\right)$ satisfies

$\lambda y = xy' + \ln(y')$, then find λ.

Sol

$$y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\cdot\ln\left(x+\sqrt{x^2+1}\right).$$

$$y' = x + \frac{\sqrt{x^2+1}}{2} + \frac{1}{2}\frac{x^2}{\sqrt{x^2+1}} + \frac{1}{2}\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= x + \frac{1}{2}\left(\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right) = x + \frac{1}{2}\left(\sqrt{x^2+1}+\sqrt{x^2+1}\right) = x + \sqrt{x^2+1}$$

Consider $xy' + \ln(y') = x\cdot\left(x+\sqrt{x^2+1}\right) + \ln\left(x+\sqrt{x^2+1}\right) = x^2 + x\sqrt{x^2+1} + \ln\left(x+\sqrt{x^2+1}\right)$

$$= 2y \quad \Rightarrow \boxed{\lambda = 2}$$

Que Let $f(x)$ is a continuous function defined on $[-1, 1]$ and satisfies $f(2x^2-1) = 2x \cdot f(x)$, then find $f(x)$.

Soln

$$f(2x^2-1) = 2x \cdot f(x).$$

$$\Big\downarrow \; x = \cos(\theta), \; \theta \in [0, \pi]$$

$$f(2\cos^2(\theta) - 1) = 2\cos(\theta) \cdot f(\cos(\theta))$$

$$\Rightarrow f(\cos(2\theta)) = 2\cos(\theta) \cdot f(\cos(\theta))$$

$$= 2\cos(\theta) \cdot 2 \cdot \cos\left(\frac{\theta}{2}\right) \cdot f\left(\cos\left(\frac{\theta}{2}\right)\right) = 2^2 \cdot \cos(\theta) \cdot \cos\left(\frac{\theta}{2}\right) \cdot f\left(\cos\left(\frac{\theta}{2}\right)\right)$$

$$= 2^3 \cdot \cos(\theta) \cdot \cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot f\left(\cos\left(\frac{\theta}{4}\right)\right)$$

$$\vdots$$

$$= 2^{n+1} \cdot \cos(\theta) \cdot \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{4}\right) \cdots \cos\left(\frac{\theta}{2^n}\right) \cdot f\left(\cos\left(\frac{\theta}{2^n}\right)\right)$$

$$f(\cos(2\theta)) = 2^{n+1} \cdot \underbrace{\cos(\theta)\cos\left(\frac{\theta}{2}\right) \cdots \cos\left(\frac{\theta}{2^n}\right)} \cdot f\left(\cos\left(\frac{\theta}{2^n}\right)\right)$$

$$= \cancel{2^{n+1}} \cdot \frac{\sin(2\theta)}{\cancel{2^{n+1}} \sin\left(\frac{\theta}{2^n}\right)} \cdot f\left(\cos\left(\frac{\theta}{2^n}\right)\right)$$

$$f(\cos(2\theta)) = \frac{\sin(2\theta)}{\sin\left(\frac{\theta}{2^n}\right)} \cdot f\left(\cos\left(\frac{\theta}{2^n}\right)\right).$$

$$\lim_{t\to 1} f(t) = f(1).$$

$n \to \infty$

$$f(\cos(2\theta)) = \sin(2\theta) \cdot \lim_{n\to\infty} \frac{f\left(\cos\left(\frac{\theta}{2^n}\right)\right)}{\sin\left(\frac{\theta}{2^n}\right)} = \sin(2\theta) \cdot \lim_{n\to\infty} \frac{f\left(1-2\sin^2\left(\frac{\theta}{2^{n+1}}\right)\right)}{2\cdot\sin\left(\frac{\theta}{2^{n+1}}\right)\cdot\cos\left(\frac{\theta}{2^{n+1}}\right)}$$

1, 0, $\to 0$, $\to 0$

$$f(2x^2-1) = 2x \cdot f(x)$$

$x=-1 \to f(2x^2-1) = -2x\, f(-x)$

$-$

$$0 = 2x\,(f(x) + f(-x))$$

$\Rightarrow f(x)$ is an odd function — (1)

$x = 1 \to f(1) = 2f(1) \Rightarrow f(1) = 0. \to$ (2)

$$f(2x^2-1) = 2x \cdot f(x)$$

$$\to f(x) = \frac{f(2x^2-1)}{2x}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(2x^2-1)}{2x}.$$

$\lim_{x\to 0}$

$f(0) = 0$ (As $f(x)$ is an odd continuous function)

$$f(\cos(2\theta)) = \sin(2\theta) \cdot \lim_{n\to\infty} \frac{f\left(1-2\sin^2\left(\frac{\theta}{2^{n+1}}\right)\right)}{2\sin\left(\frac{\theta}{2^{n+1}}\right)\cdot \cancel{\cos\left(\frac{\theta}{2^{n+1}}\right)}} \cdot$$

$\cos\left(\frac{\theta}{2^{n+1}}\right) \to 1$

$$= \sin(2\theta) \lim_{n\to\infty} \frac{-f\left(2\sin^2\left(\frac{\theta}{2^{n+1}}\right)-1\right)}{2\,\underbrace{\sin\left(\frac{\theta}{2^{n+1}}\right)}_{t}}$$

$$t \to 0$$

$$\frac{f(2t^2-1)}{2t}$$

$$= \sin(2\theta) \cdot (0)$$

$$f(\cos(2\theta)) = 0.$$

$$\boxed{f(x) = 0}, \ \forall x \in [-1, 1].$$

Que Find $f(x)$ satisfying

$$f(x+2y) = f(x)\cdot e^{2y} + f(2y)\cdot e^{x} + x^2(1-e^{2y}) + 4y^2(1-e^{x}) + 4xy, \forall x,y \in R$$

and $f'(0) = 1$ then find $f(x)$.

Sol

$2y = y$

$$f(x+y) = f(x)\cdot e^{y} + f(y)\cdot e^{x} + x^2 - x^2\cdot e^{y} + y^2 - y^2\cdot e^{x} + 2xy$$

$$f(x+y) = e^{y}(f(x)-x^2) + e^{x}(f(y)-y^2) + (x+y)^2$$

$$f(x+y) - (x+y)^2 = e^{y}(f(x)-x^2) + e^{x}(f(y)-y^2)$$

$e^{-(x+y)}$

$$e^{-(x+y)}\left(f(x+y)-(x+y)^2\right) = \underbrace{e^{-x}(f(x)-x^2)}_{= g(x) \text{ (say)}} + e^{-y}(f(y)-y^2)$$

$$\Rightarrow g(x+y) = g(x) + g(y).$$

$$\Rightarrow g(x) = kx$$

Result for diff fn.

$f(x+y) = f(x) + f(y)$ $\Rightarrow f(x) = kx.$
$f(x+y) = f(x)\cdot f(y)$ $f(x) = e^{kx}$ @ 0.
$f(xy) = f(x) + f(y), x,y \in R^+$ $f(x) = k\ln(x)$
$f(xy) = f(x)\cdot f(y), \forall x,y \in R$ $f(x) = x^k$ @ 0

$$g(x) = kx$$

$$e^{-x}\left(f(x) - x^2\right) = kx$$

$$f(x) = x^2 + kx \cdot e^x$$

given $f'(0) = 1$

$$\Downarrow$$

$$f'(x) = 2x + kxe^x + ke^x$$

$x = 0$

$$k = 1$$

$$\boxed{f(x) = x^2 + x \cdot e^x}$$

Result

$f(x)$ is polynomial & satisfies $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$.

$$\Downarrow$$

$$\boxed{f(x) = 1 \pm x^n}$$

Que Let $-1 \le p \le 1$, Then find number of real roots of the equation $4x^3 - 3x - p = 0$ in the interval $[\frac{1}{2}, 1]$.

Sol $x \in [\frac{1}{2}, 1]$ $p \in [-1, 1]$

$4x^3 - 3x = p$

$x = \cos(\theta)$, $\theta \in [0, \frac{\pi}{3}]$

$4\cos^3(\theta) - 3\cos(\theta) = p$

$\cos(3\theta) = p$

$3\theta \in [0, \pi]$

$\cos(3\theta) \in [-1, 1]$

$\Rightarrow$ as $\cos(3\theta)$ can take all values from $[-1, 1]$ and in $[0, \pi]$ it will be one-one function hence

$\Rightarrow$ for any value of 'p' $\exists$ exactly one θ

$\Downarrow$

$\exists$ exactly one 'x' in $[\frac{1}{2}, 1]$

$\Rightarrow$ No. of solution $= \boxed{1}$.

Alt:

$$4x^3 - 3x = p, \quad x \in \left[\tfrac{1}{2}, 1\right] \qquad p \in [-1, 1]$$

$$\underset{y}{\parallel} \qquad\qquad \underset{y}{\parallel}$$

$$y = 4x^3 - 3x$$

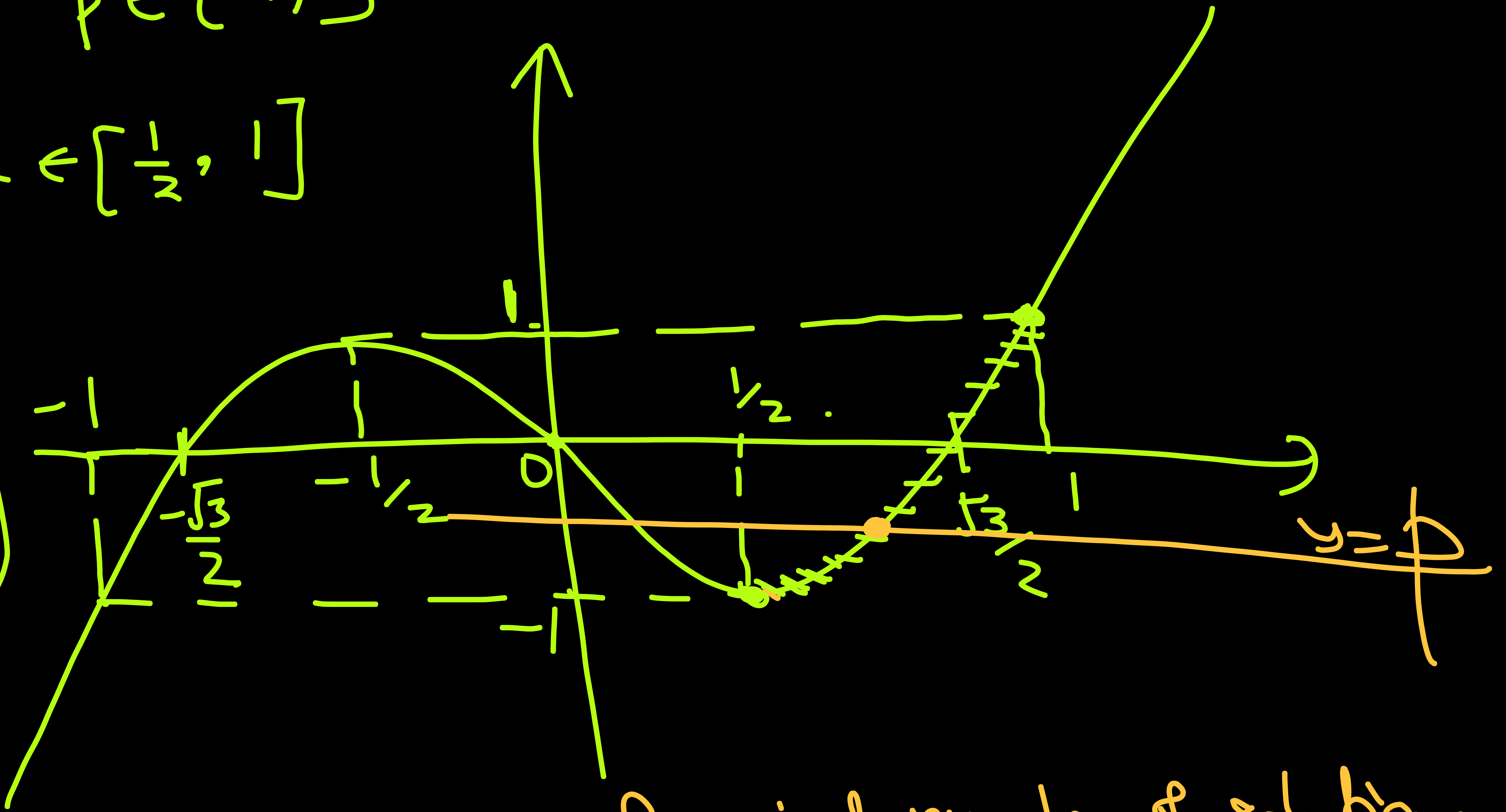
$$= 4x\left(x^2 - \frac{3}{4}\right)$$

$$= 4 \cdot x\left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right)$$

$$y' = 12x^2 - 3 = 12\left(x^2 - \frac{1}{4}\right)$$

$$y\left(\frac{1}{2}\right) = \frac{\cancel{4}}{\cancel{8}2} - \frac{3}{2} = \frac{1}{2} - \frac{3}{2} = -1$$

$$y(1) = 4 - 3 = 1.$$



$\Rightarrow$ Required number of solutions $= \boxed{1}$ Ans

Que Let $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = x \cdot e^{x^2} + e^{-x^2}$ and $h(x) = x^2 \cdot e^{x^2} + e^{-x^2}$ are 3- functions defined on $[0,1]$ then find $(f(x))_{max.} + (g(x))_{max.} + (h(x))_{max.}$

Sol

$f(x) = e^{x^2} + e^{-x^2}$, $x \in [0,1]$

$x \in [0,1] \Rightarrow \underbrace{x^2}_{t} \in [0,1]$

$x \uparrow$, $t \uparrow$

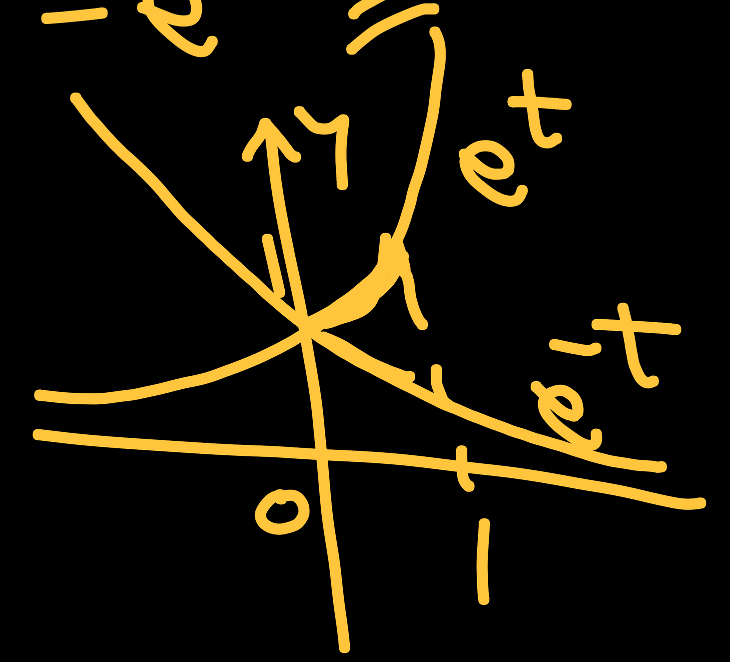
$F(t) = e^t + e^{-t}$

$F'(t) = e^t - e^{-t} \geq 0$

$\Downarrow$

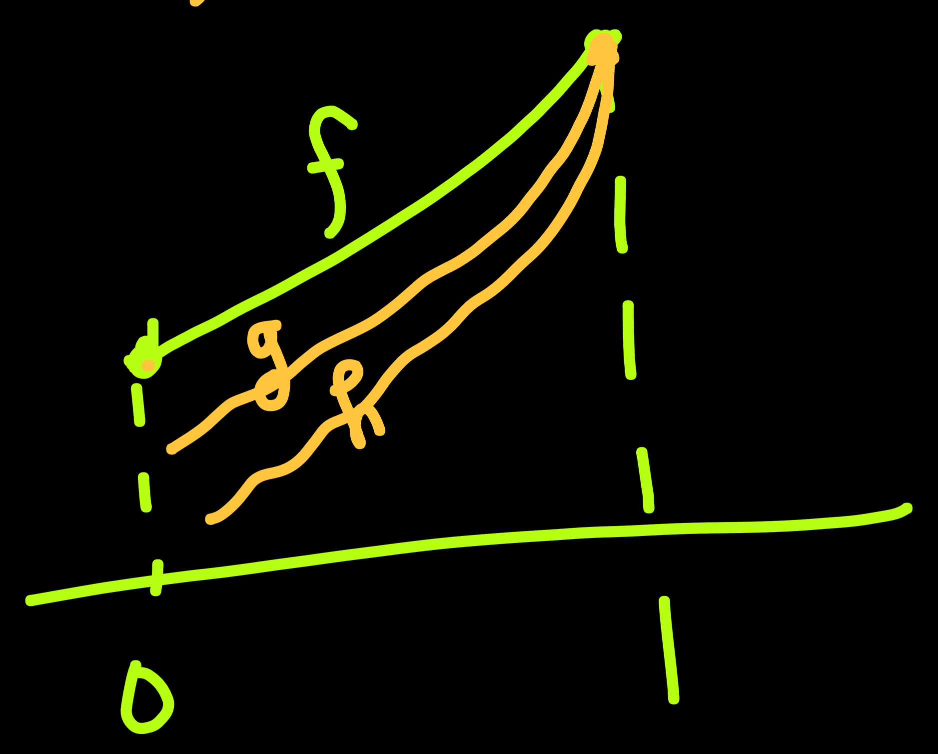
$F(t) \uparrow$

$f(x) \uparrow$.



$(f(x))_{max.} = f(1) = e + \frac{1}{e}$

$$\left.\begin{aligned} f(x) &= e^{x^2} + e^{-x^2} \\ g(x) &= x \cdot e^{x^2} + e^{-x^2} \\ h(x) &= x^2 \cdot e^{x^2} + e^{-x^2} \end{aligned}\right\} \Rightarrow f(x) \geq g(x) \geq h(x)$$



$(g(x))_{max.} = g(1) = f(1)$

$h(x)_{max.} = h(1) = f(1)$

$(f(x))_{max.} + (g(x))_{max.} + (h(x))_{max.}$

$3f(1) = 3\left(e + \frac{1}{e}\right) = \boxed{\frac{3(e^2+1)}{e}}$

Ans

Que Let $f(x) = \dfrac{\ln(\pi + x)}{\ln(e+x)}$, $\forall x \in R^+$ then find the interval in which $f(x)$ increases.

Soln

$$f(x) = \frac{\ln(\pi+x)}{\ln(e+x)}$$

$$f'(x) = \frac{\dfrac{\ln(e+x)}{\pi+x} - \dfrac{\ln(\pi+x)}{e+x}}{(\ln(e+x))^2}$$

$$= \frac{(e+x)\ln(e+x) - (\pi+x)\cdot\ln(\pi+x)}{(\pi+x)(e+x)\cdot(\ln^2(e+x))}$$

$< 0 \Rightarrow$ fun $\downarrow^{ing}$

If $x > 0$ then

$e + x > e$ and $\pi + x > \pi$

$(\pi + x) > (e + x)$

$\ln(\pi+x) > \ln(e+x) > 1$

$(\pi+x)\cdot\ln(\pi+x) > (e+x)\cdot\ln(e+x)$

In R^+, $f(x)$ never increases

~~Ans~~