Que. Discuss continuity and differentiability of the following function

$$f(x) = (x-e)\cdot 2^{-2^{\frac{1}{e-x}}}, \quad x \neq e$$
$$= 0, \quad x = e.$$

D_1

D_2

e

Soln

$$f'(e) = \lim_{x\to e} \frac{f(x)-f(e)}{x-e} = \lim_{x\to e} \frac{\cancel{(x-e)}\cdot 2^{-2^{\frac{1}{e-x}}} - 0}{\cancel{(x-e)}}$$

$$= \lim_{x\to e} 2^{-2^{\left(\frac{1}{e-x}\right)}}$$

$$f'(e^-) = \lim_{x\to e^-} 2^{-2^{\left(\frac{1}{e-x}\right)}}$$

$\frac{1}{e-x} \to \infty$, $2^{\frac{1}{e-x}} \to \infty$, $-2^{\frac{1}{e-x}} \to -\infty$, $2^{-2^{\frac{1}{e-x}}} \to 0$ $= \boxed{0}$

$\frac{1}{e-x} \to -\infty$, $2^{\frac{1}{e-x}} \to 0$, $-2^{\frac{1}{e-x}} \to 0$, $2^{-2^{\frac{1}{e-x}}} \to 1$ $= \boxed{1}$

$$f'(e^+) = \lim_{x\to e^+} 2^{-2^{\left(\frac{1}{e-x}\right)}}$$

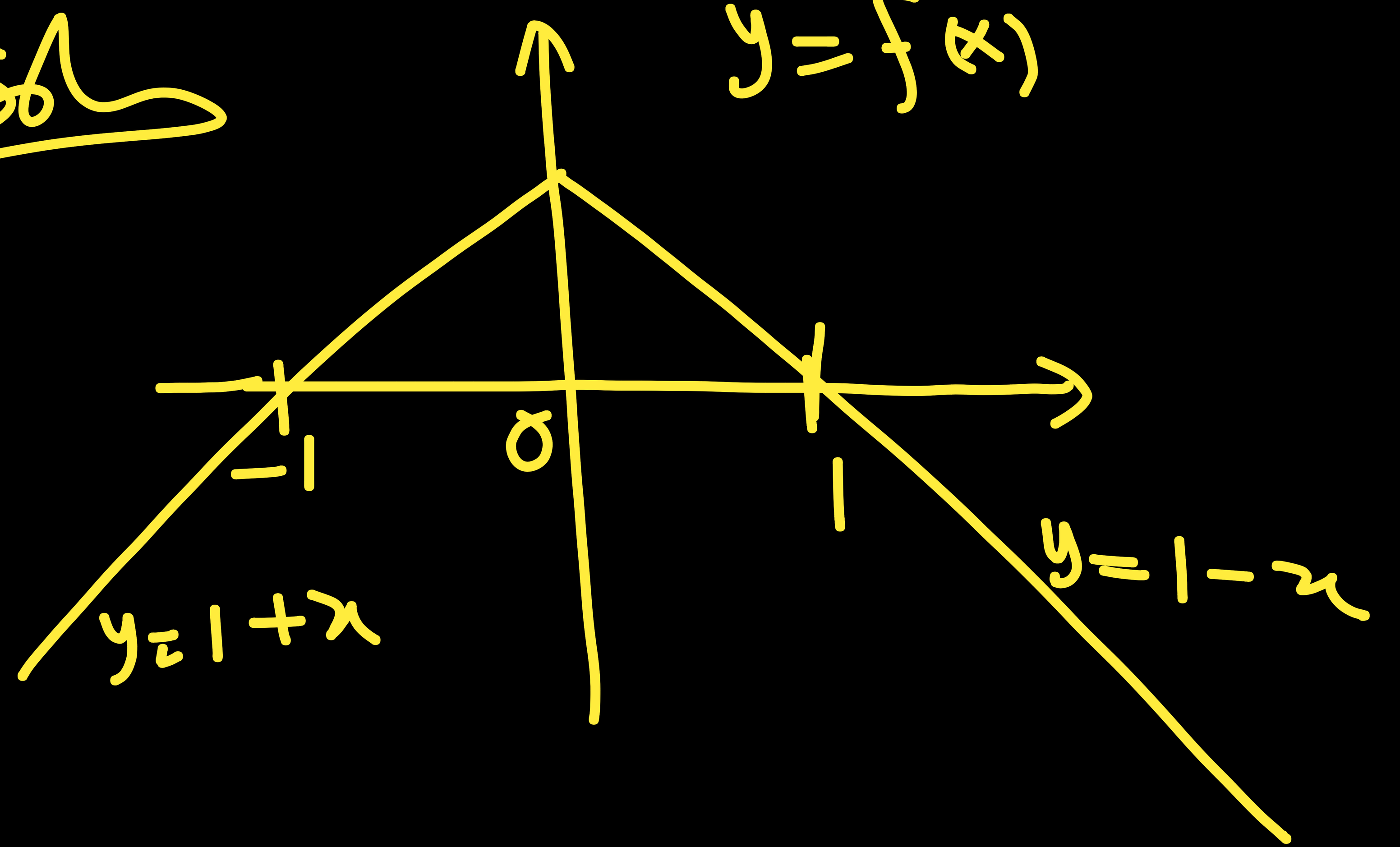
L.H.D. = 1
& R.H.D = 0.

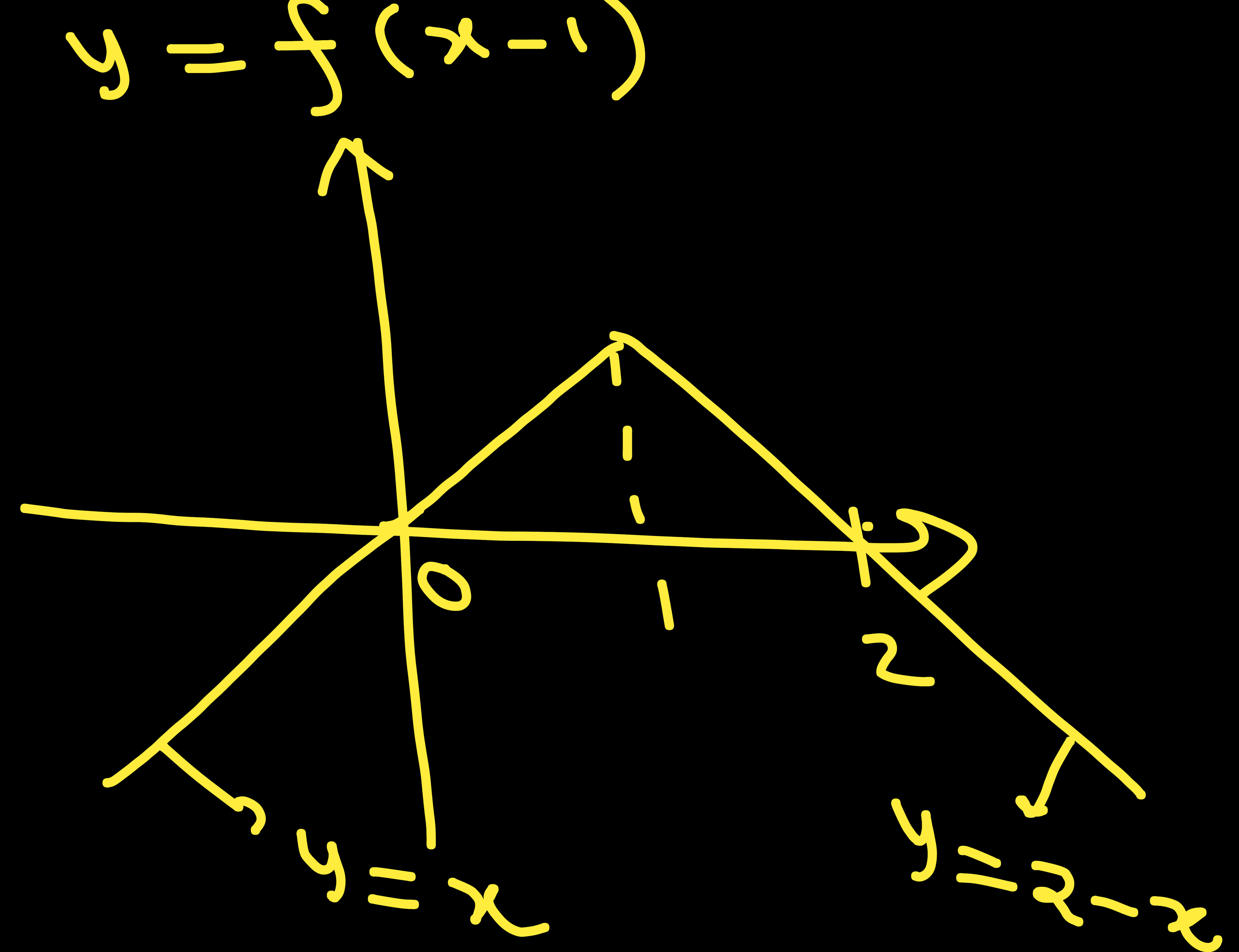
$\Rightarrow$ f(x) is continuous in R
f(x) is differentiable everywhere except at $x = e$.

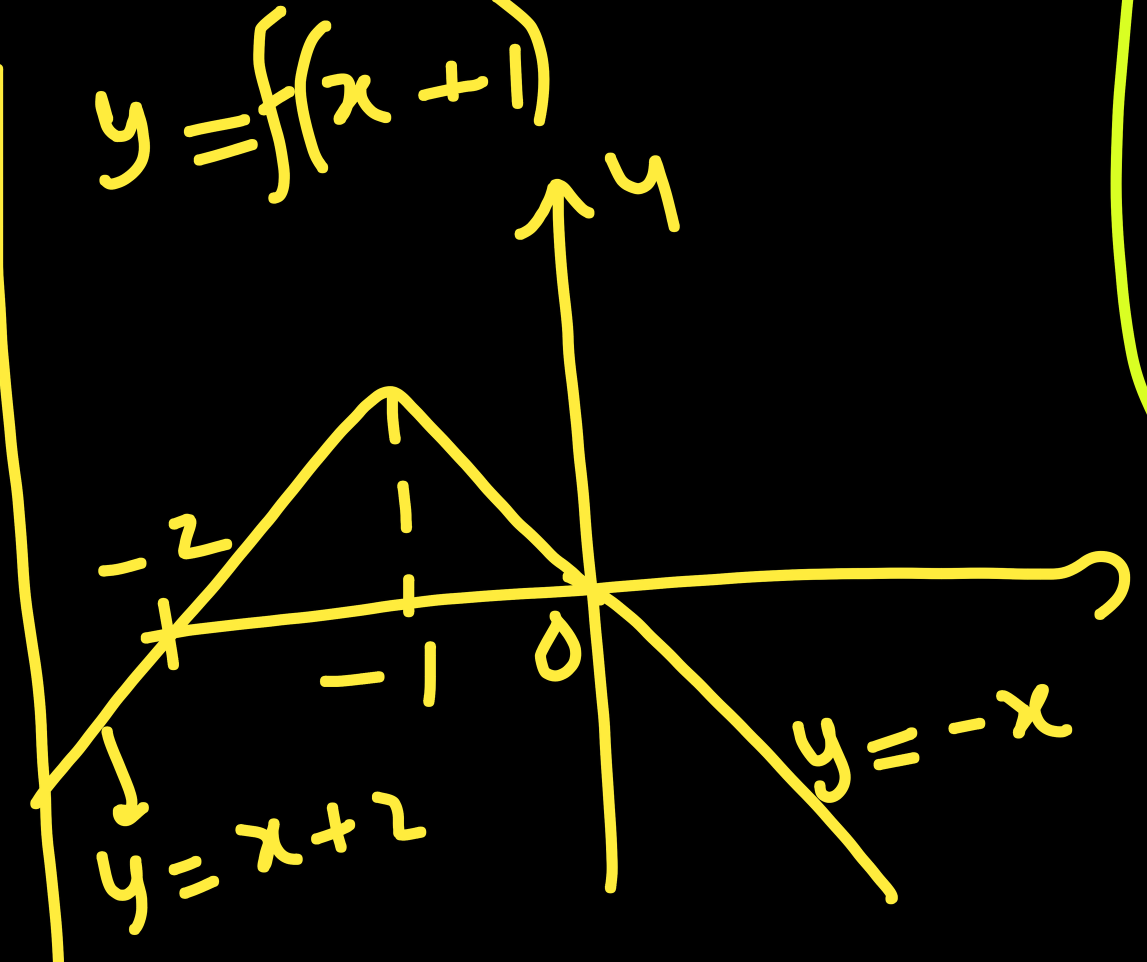
Que Let $f: R \to R$, $f(x) = 1-|x|, \; |x| \le 1$
$\qquad = 0, \; |x|>1$, and let

$g(x) = f(x-1) + f(x+1), \; \forall x \in R$, Discuss continuity and differentiability of $g(x)$.

Sol







$$g(x) = \begin{cases} 2(x+1), & x < -1 \\ 0, & x \in [-1, 1] \\ 2(1-x), & x > 1 \end{cases}$$

$f(x+1)$	$x+2$	$-x$	$-x$
$f(x-1)$	x	x	$2-x$
$g(x)$	$2x+2$	0	$2-2x$
	-1		1

$g(x)$ will be continuous in R (Using $C \pm C \equiv C$).

& $g(x)$ will be Non-differentiable at $x = -1$ & 1 only

(using $N.D + D \equiv N.D.$)

Que Discuss the continuity of the function $f(g(x))$ if

$$f(x) = \begin{cases} x^2+4x+1, & x<0 \\ x^2-4x+2, & x \geq 0, \end{cases} \quad \text{where} \quad g(x) = \begin{cases} x^2+x+1, & x<-1 \\ \frac{1}{16}(x-1)^2(x-2)(x-3), & x \geq -1 \end{cases}$$

Soln

$$f(x) = x^2+4x+1, \quad x<0$$
$$= x^2-4x+2, \quad x \geq 0.$$

$$f(0^-) = 1, \quad f(0^+) = f(0) = 2$$

$\Rightarrow f(x)$ is discontinuous at $x=0$ only

$$g(x) = x^2+x+1, \quad x<-1$$
$$= \frac{1}{16}(x-1)^2 \cdot (x-2)(x-3), \quad x \geq -1$$

$g(-1^-) = 1$ $\Rightarrow g(x)$ is discont. only at $x=-1$

$$g(-1^+) = g(-1) = \frac{1}{16} 4 \cdot (-3)(-4) = 3.$$

For $f(g(x))$

Check continuity at $x=-1$ (b/c of $g(x)$) and at the points where $g(x)=0$ (b/c of $f(x)$)

C-1: $x<-1$

$$x^2+x+1=0$$
$$\downarrow$$
$$x \in \phi$$

C-2: $x \geq -1$

$$\frac{1}{16}(x-1)^2(x-2)(x-3)=0$$
$$\Downarrow$$
$$x = 1, 2, 3.$$

hence we need to verify $x = -1, 1, 2, 3.$

$x = -1, 1, 2, 3$

$$f(x) = \begin{cases} x^2 + 4x + 1, & x < 0 \\ x^2 - 4x + 2, & x \geq 0. \end{cases}$$

$$g(x) = \begin{cases} x^2 + x + 1, & x < -1 \\ \frac{1}{16}(x-1)^2(x-2)(x-3), & x \geq -1. \end{cases}$$

$f(0^-) = 1$

$f(0^+) = f(0) = 2.$

$g(-1^-) = 1$

$g(-1^+) = 3$

$g(-1)$

at $x = -1$

$y = f(g(x))$

$\xrightarrow{x = -1^-} y = f(g(-1^-)) = f(1) = -1.$

$\xrightarrow{x = -1^+,\ x = -1} y = f(g(-1^+ \text{ or } -1))$

$= f(3)$

$= -1.$

$\Rightarrow f(g(x))$ is continuous at $\boxed{x = -1}$

at $x = 1$

$y = f(g(x))$

1^-	1	1^+
$f(g(1^-))$	$f(g(1))$	$f(g(1^+))$
$= f(0^+)$	$= f(0)$	$= f(0^+)$
$= 2$	$= 2$	$= 2$

$\Rightarrow f(g(x))$ is continuous at $x = 1$

at $x = 2$

$f(g(2^-)) = f(0^+) = 2$

$f(g(2^+)) = f(0^-) = 1$

$\Rightarrow$ Discontinuous at $\boxed{x = 2}$

at $x = 3$

$f(g(3^-)) = f(0^-) = 1$

$f(g(3^+)) = f(0^+) = 2$

$\Rightarrow$ discont. at $x = 3.$

Hence

$y = f(g(x))$ will be continuous everywhere except at the points $x = 2$ & $x = 3$.

Que Discuss continuity and differentiability of $y = f(x)$, where
$x = 2t - |t|$ and $y = t^2 + t|t|$, $t \in R$.

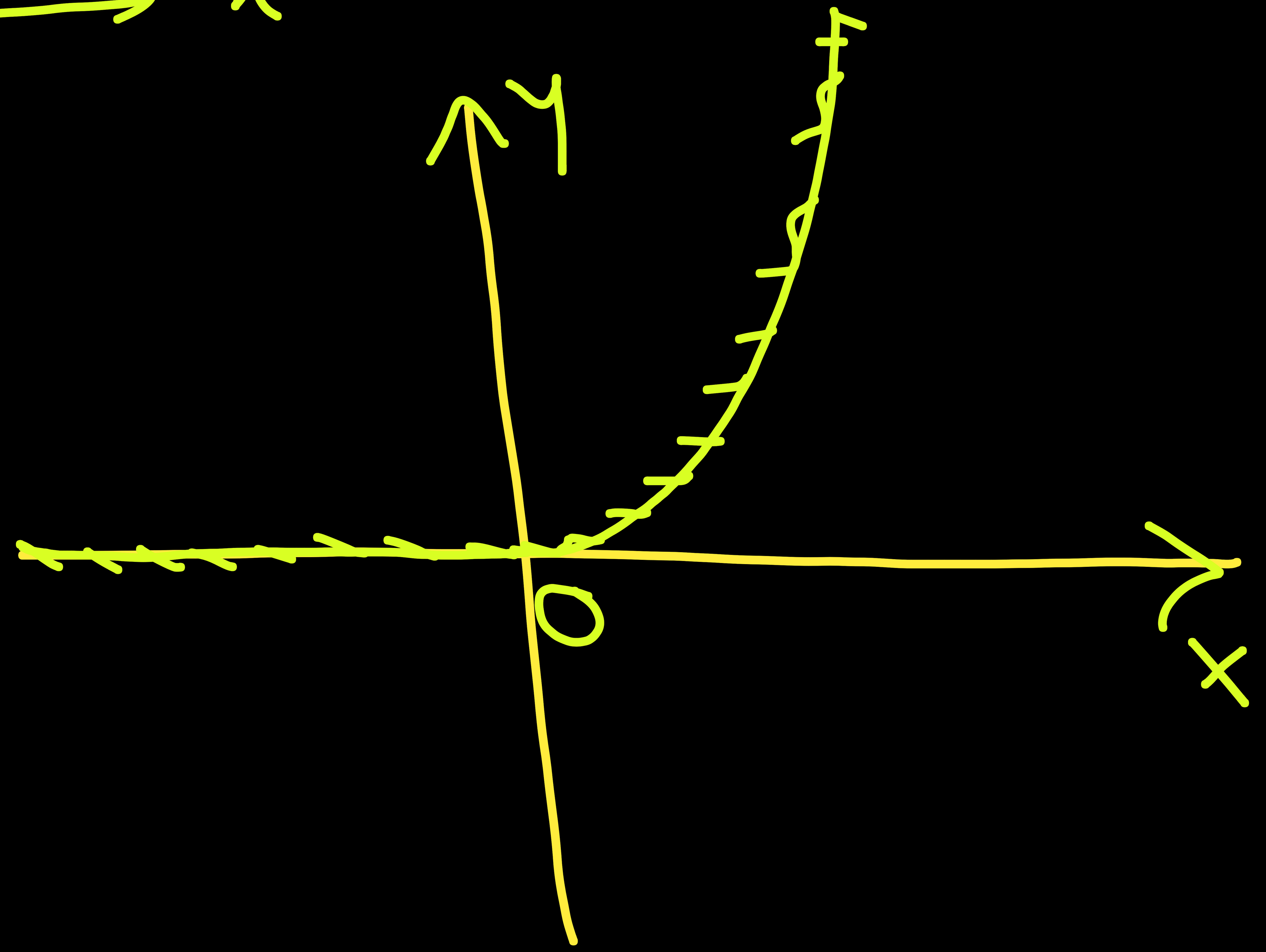
Soln.

$x = 2t - |t|$, $y = t^2 + t|t|$

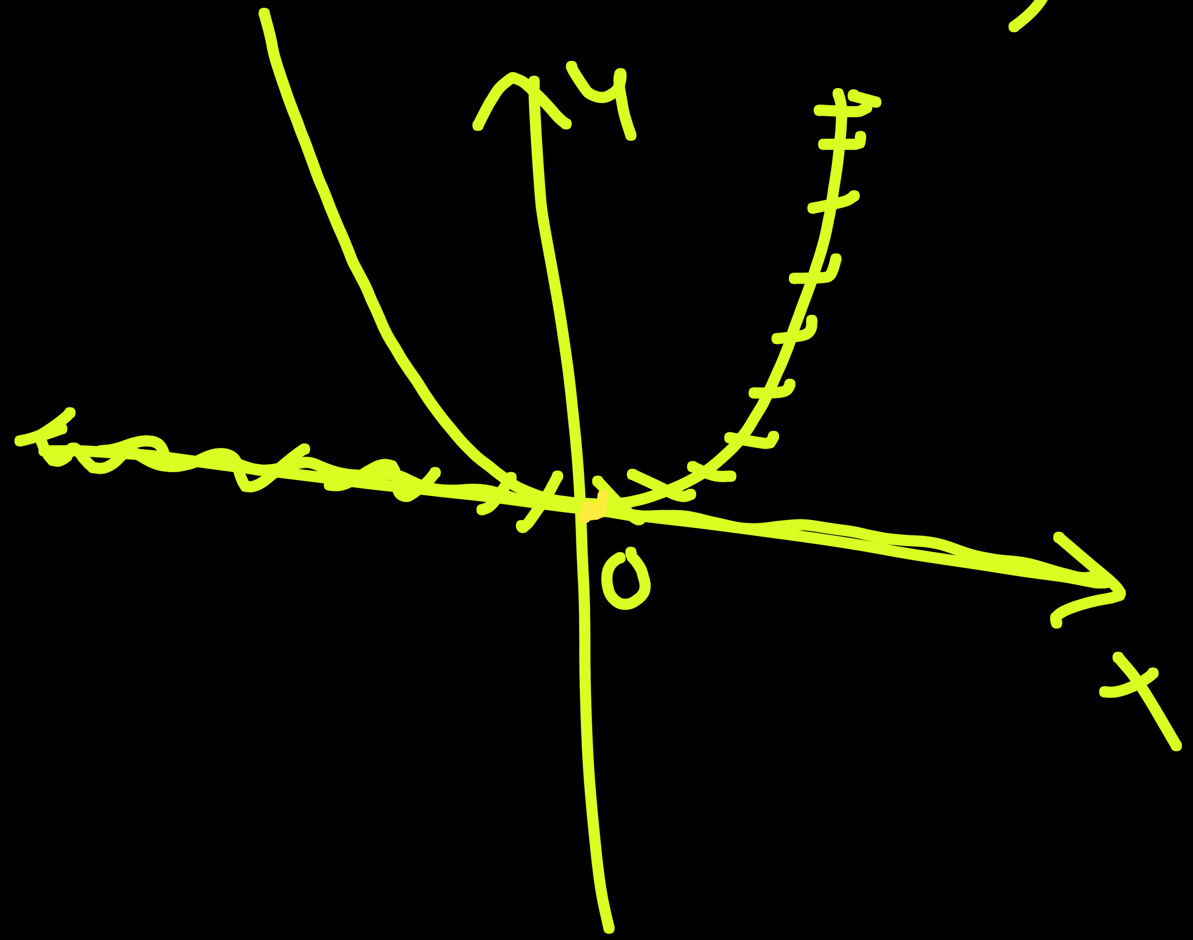
C-1 $t \geq 0 \rightarrow x = 2t - t = t$ & $y = t^2 + t^2 = 2t^2 \Rightarrow y = 2x^2, x \geq 0.$

C-2 $t < 0 \rightarrow x = 2t + t = 3t$ & $y = t^2 - t^2 = 0 \Rightarrow y = 0, x < 0.$

$$y = \begin{cases} 0, & x < 0 \\ 2x^2, & x \geq 0. \end{cases}$$



$\Rightarrow$ function is continuous and differentiable in 'R'.



Que Discuss continuity and differentiability of

$$f(x) = (x^2)^n \cdot \cos\left(\frac{1}{x}\right), \quad x \neq 0 \qquad , \text{ where } n \in R.$$
$$= 0, \quad x = 0.$$

Soln. <u>for continuity at $x=0$</u>

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x^2)^n \cdot \cos\left(\frac{1}{x}\right)$$

$\cos\left(\frac{1}{x}\right) \to [-1,1]$

$n \le 0$: D.N.E.

$n > 0$: 0

$f(0) = 0$

$\Rightarrow$ <u>for $n \le 0$</u> $f(x)$ is discontinuous at $x=0$.

for $n \in R^+$ $f(x)$ is continuous at $x=0$

<u>for differentiability at $x=0$</u>

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(x^2)^n \cdot \cos\left(\frac{1}{x}\right) - 0}{x}$$

$$= \lim_{x\to 0} \frac{(x^2)^n \cdot \cos\left(\frac{1}{x}\right)}{x}$$

$$= \lim_{x\to 0} \frac{(x^2)^n}{x} \cdot \cos\left(\frac{1}{x}\right)$$

$\cos\left(\frac{1}{x}\right) \to [-1, 1]$

$\Rightarrow$ If $n \le \frac{1}{2}$ then function will be non-differentiable

if $n > \frac{1}{2}$ then function will be differentiable.

x	$f(x)$
$x \in (-\infty, 0]$	Discontinuous & Non-differentiable
$x \in (0, \frac{1}{2}]$	Continuous & Non-differentiable
$x \in (\frac{1}{2}, \infty)$	Differentiable.

Que Let $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim_{n\to\infty} y_n(x)$, then discuss continuity and differentiability of (i) $y_n(x)$ (ii) $y(x)$, where $n \in N$.

Soln $y_n(x) = x^2 + \frac{x^2}{1+x^2} + \dots + \frac{x^2}{(1+x^2)^{n-1}} = x^2\left(1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \dots + \frac{1}{(1+x^2)^{n-1}}\right)$

$$= 0, \quad x = 0$$

$$= x^2\left(\frac{1-\left(\frac{1}{1+x^2}\right)^n}{1-\left(\frac{1}{1+x^2}\right)}\right), \quad x \neq 0$$

$\Rightarrow y_n(x) = 0, \quad x = 0.$

$$= (1+x^2)\left(1 - \left(\frac{1}{1+x^2}\right)^n\right), \quad x \neq 0.$$

$y_n(0) = 0 \qquad y_n(0^-) = 0$

$y_n(0^+) = 0$

$\Downarrow$

continuous at $x = 0$.

$$y_n(x) = 0, \quad x = 0$$
$$= \frac{(1+x^2)^n - 1}{(1+x^2)^{n-1}}, \quad x \neq 0$$

for differentiability

$$y_n'(0) = \lim_{x\to 0} \frac{\frac{(1+x^2)^n - 1}{(1+x^2)^{n-1}} - 0}{x - 0}.$$

$$= \lim_{x\to 0} \frac{(1+x^2)^n - 1}{x\,\underbrace{(1+x^2)^{n-1}}_{\to 1}}$$

$$= \lim_{x\to 0} \frac{(\cancel{1}+nx^2) - \cancel{1}}{\cancel{x}}$$

$$= 0 \Rightarrow \text{Diff. at } x = 0$$

hence in 'R'.

$$y(x) = \lim_{n\to\infty} y_n(x)$$

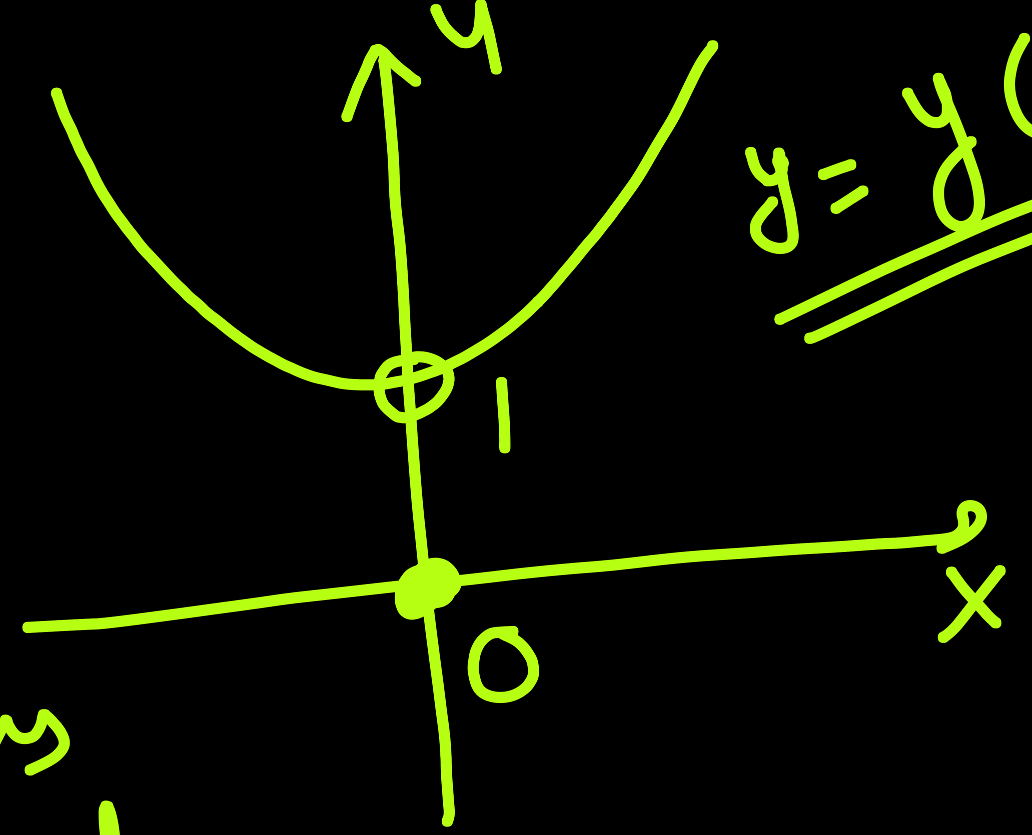
$$= 0, \quad x = 0.$$

$$= \lim_{n\to\infty} \frac{(1+x^2)^n - 1}{(1+x^2)^{n-1}}, \quad x \neq 0$$

$$= \lim_{n\to\infty} \frac{(1+x^2)^n}{(1+x^2)^{n-1}} = \lim_{n\to\infty} (1+x^2)$$

$$= 1+x^2.$$

$$y(x) = 0, \quad x = 0.$$
$$= 1+x^2, \quad x \neq 0$$



$y = y(x)$

$\Rightarrow y(x)$ is discontinuous and non-differentiable at $\underline{\underline{x=0}}$ only.

Que Discuss the continuity of the function

$$f(x) = \lim_{n \to \infty} \frac{(1+\sin(x))^n + \ln(x)}{2 + (1+\sin(x))^n}$$

Soln

$$f(x) = \lim_{n \to \infty} \frac{(1+\sin(x))^n + \ln(x)}{2 + (1+\sin(x))^n}.$$

$= \frac{1+\ln(x)}{2+1}$, $\sin(x) = 0 \Rightarrow x = n\pi,\ n \in N.$

$= \frac{\ln(x)}{2}$, $\sin(x) < 0 \Rightarrow x \in ((2n-1)\pi, 2n\pi)$

$= 1$, $\sin(x) > 0 \Rightarrow x \in (2n\pi, (2n+1)\pi)$

$$f(x) = \frac{1+\ln(x)}{3}, \quad x = n\pi$$
$$= \frac{\ln(x)}{2}, \quad x \in ((2n-1)\pi, 2n\pi)$$
$$= 1, \quad x \in (2n\pi, (2n+1)\pi)$$

Result

$$\lim_{n \to \infty} (a)^n = 1,\ a = 1$$
$$= 0,\ 0 < a < 1$$
$$= \text{D.N.E.}\ (\to \infty),\ a > 1$$

$$n\pi \Rightarrow f(x) = \frac{\ln(ex)}{3}, \quad x = n\pi$$

$$= \frac{\ln(ne\pi)}{3}$$

$\Rightarrow f(x)$ is discontinuous at $x = n\pi$ only when $n \in N$.

⊗ it will be continuous in $x \in R^{+} - n\pi$.