$\boxed{Q-1}$ $\lim_{x\to 0} \dfrac{e-(1+x)^{1/x}}{\tan(x)}$

Soln

$k = e^{\ln(k)}$

$a^b = e^{\ln(a^b)} = e^{b\ln(a)}$

$\lim_{x\to 0} \dfrac{e - e^{\frac{\ln(1+x)}{x}}}{\cancel{\left(\frac{\tan(x)}{x}\right)} x}$ $\to 1$

$= \lim_{x\to 0} \dfrac{-e\left(e^{\frac{\ln(1+x)}{x}-1}-1\right)}{x}$

$e^{A}-1 \to A$, $A\to 0$

$= \lim_{x\to 0} -e\left(\dfrac{\frac{\ln(1+x)}{x}-1}{x}\right) = \lim_{x\to 0} -e\left(\dfrac{\ln(1+x)-\cancel{x}}{x^2}\right) = \lim_{x\to 0} \dfrac{e\cdot \cancel{x^2}}{2\cdot \cancel{x^2}} = \boxed{e/2}$ Ans

$\cancel{x}-\frac{x^2}{2}$

$\boxed{Q-2}$ Find $a, b, c, d \in R$ if

$\lim_{x\to\infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2} - \sqrt{x^4+2x^3-cx^2+3x-d}\right) = \underline{4}$

$L = \lim_{x\to\infty}\left(\dfrac{(a-2)x^3+(3+c)x^2+(b-3)x+2+d}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}\right)$

$= \lim_{x\to\infty}\left(\dfrac{(a-2)x^3+(c+3)x^2+(b-3)x+2+d}{2x^2}\right)$

$\boxed{a=2}$ & $\frac{c+3}{2}=4$ & $b-3\in R$, $2+d\in R$

$a=2$ & $c=5$ & $b\in R$ & $d\in R$

Ans

Short cut

$x \to 0$.

$\sin(x)$, $\tan(x)$, $\tan^{-1}(x)$, $\sin^{-1}(x)$, $\ln(1+x)$, $e^x - 1 \longrightarrow 1 \cdot x$

$$\frac{\ln(1+x)}{\boxed{x}} = 1$$

$$\frac{e^x - 1}{x} = 1$$

$$1 - \cos(A) \longrightarrow \frac{A^2}{2}$$

$$\frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

$\cancel{(\sin^{-1}(x) - x)}$

$\boxed{\text{Que}}$ $\lim_{x\to 0^+} \dfrac{\log_{\sin(x)}(\cos(x))}{\log_{\sin(x/2)}(\cos(x/2))}$

Sol^n $\lim_{x\to 0^+} \dfrac{\left(\dfrac{\ln(\cos(x))}{\ln(\sin(x))}\right)}{\left(\dfrac{\ln(\cos(\frac{x}{2}))}{\ln(\sin(\frac{x}{2}))}\right)}$

$= \lim_{x\to 0^+} \dfrac{\ln(\cos(x))\cdot\ln(\sin(\frac{x}{2}))}{\ln(\sin(x))\cdot\ln(\cos(\frac{x}{2}))}$

$\cos(x)\to 1$, $\ln(\cos(x))\to 0$, $\ln(\sin(\frac{x}{2}))\to -\infty$, $\sin(x)\to 0^+$, $\ln(\sin(x))\to -\infty$, $\ln(\cos(\frac{x}{2}))\to 0$

$= \lim_{x\to 0} \left(\dfrac{\ln(\cos(x))}{\ln(\cos(\frac{x}{2}))}\right)\cdot\left(\dfrac{\ln(\sin(\frac{x}{2}))}{\ln(\sin(x))}\right)$

$\boxed{\text{Que}}$ $\lim_{x\to 0} \dfrac{\sqrt{1-\cos^2(x)}}{\sqrt{x^2+8x^3}}$

$= \lim_{x\to 0} \dfrac{\sqrt{\sin^2(x)}}{\sqrt{x^2}\,\sqrt{1+8x}}$, $\sqrt{1+8x}\to 1$

$= \lim_{x\to 0} \dfrac{|\sin(x)|}{|x|}$

$= \lim_{x\to 0} \left|\dfrac{\sin(x)}{x}\right| = 1.$

Que $\lim_{x\to 0} \dfrac{\sqrt{1-\cos^2(x)}}{x}$

$= \lim_{x\to 0} \dfrac{|\sin(x)|}{x} = \text{D.N.E}$

$\boxed{\text{Que}}$ $\lim_{x\to 0} 9^{\frac{1}{\cos(x)-1}}$

$= \lim_{x\to 0} (9)^{\frac{1}{\cos(x)-1}} = \boxed{0}$

$x\to 0 \Rightarrow \cos(x)\to 1^-$

$\cos(x)-1 \to 0^-$

$\dfrac{1}{\cos(x)-1} \to -\infty$

$(9)^{\frac{1}{\cos(x)-1}} \to 0^+$

1

$$\lim_{x\to 0^+}\left(\frac{\ln(\cos(x))}{\ln\left(\cos\left(\frac{x}{2}\right)\right)}\right)\left(\frac{\ln\left(\sin\left(\frac{x}{2}\right)\right)}{\ln(\sin(x))}\right)$$

$$\sin(x) \to 2\sin\left(\tfrac{x}{2}\right)\cos\left(\tfrac{x}{2}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cos(x)-1}{\cos\left(\frac{x}{2}\right)-1}\right)\cdot\left(\frac{\ln\left(\sin\left(\frac{x}{2}\right)\right)}{\ln\left(\sin\left(\frac{x}{2}\right)\right)+\ln\left(2\cos\left(\frac{x}{2}\right)\right)}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cancel{2}\sin^2\left(\frac{x}{2}\right)}{\cancel{2}\sin^2\left(\frac{x}{4}\right)}\right)\cdot\left(\frac{1}{1+\cancel{\dfrac{\ln\left(2\cos\left(\frac{x}{2}\right)\right)}{\ln\left(\sin\left(\frac{x}{2}\right)\right)}}^{\,0}}\right)$$

$$= \frac{\left(\frac{x}{2}\right)^2}{\left(\frac{x}{4}\right)^2}\cdot 1$$

$$= \boxed{4}\ \text{Ans}$$

$$\ln(1+\overset{0}{x}) \longrightarrow (1+x)-1$$

$$\ln(\underset{\to 1}{A}) \longrightarrow A-1$$

Que Let $S_n = \sin(\theta) + \sin(2\theta) + \sin(3\theta) + \cdots + \sin(n\theta)$, given $\sin\left(\frac{\theta}{2}\right) \neq 0$.

Evaluate $\lim_{n\to\infty} \frac{S_1 + S_2 + S_3 + \cdots + S_n}{n}$

Soln

$$S_n = \sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta)$$

$$= \frac{\sin\left(n\cdot\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left(\frac{\theta + n\theta}{2}\right) = \frac{\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2} + n\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{1}{2\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left(\frac{\theta}{2} + n\theta\right)$$

$$S_1 + S_2 + \cdots + S_n = \sum_{r=1}^{n} S_r = \sum_{r=1}^{n}\left(\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \frac{\cos\left(\frac{\theta}{2} + r\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}\right)$$

$$= \frac{n}{2}\cot\left(\frac{\theta}{2}\right) - \frac{1}{2\sin\left(\frac{\theta}{2}\right)}\left(\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left(\frac{\frac{\theta}{2} + \theta + \frac{\theta}{2} + n\theta}{2}\right)\right)$$

$$\lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n} = \lim_{n\to\infty} \frac{\frac{\cancel{n}}{2}\cot\left(\frac{\theta}{2}\right)}{\cancel{n}} - \lim_{n\to\infty} \frac{\sin\left(\frac{n\theta}{2}\right)\cdot\cos\left(\theta + \frac{n\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)\cdot n}$$

$\to 0$

$$= \frac{1}{2}\cot\left(\frac{\theta}{2}\right)$$

Ans

Que $\lim_{n\to\infty} (3^n + 5^n + 7^n)^{1/n}$

$\infty \uparrow + \infty\uparrow + \infty\uparrow \longrightarrow \infty\uparrow$

$(\infty\uparrow)^{0\uparrow}$

$$= \lim_{n\to\infty} 7\left(\left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n + 1\right)^{\frac{1}{n}}$$

$\to 0 \quad \to 0 \quad \to 1^{0\uparrow}$

$[1]^{0\uparrow} \to 1$

$= 7$ Ans

Rem/K

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$$

Que $\lim_{n\to\infty} n^2\left(\sqrt[n]{a} - \sqrt[n+1]{a}\right)$,

given $n \in N$ & $a \in R^+$

$\infty \cdot 0\uparrow$

$$= \lim_{n\to\infty} n^2\left(a^{\frac{1}{n}} - a^{\frac{1}{n+1}}\right)$$

$$= \lim_{n\to\infty} a^{\frac{1}{n+1}} \cdot n^2 \cdot \left(a^{\frac{1}{n} - \frac{1}{n+1}} - 1\right)$$

$a^{\frac{1}{n+1}} \to 1$

$$= \lim_{n\to\infty} \frac{n^2 \cdot}{n(n+1)} \frac{\left(a^{\frac{1}{n(n+1)}} - 1\right)}{\left(\frac{1}{n(n+1)}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \cdot \ln(a)$$

$\to \ln(a)$

$= \ln(a)$ Ans

$\boxed{\text{Que}}$ $\lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right)$, $p, q \in N$

$= \lim_{x\to 1}\left(\frac{p}{1-x^p}-\frac{q}{1-x^q}\right) \neq \frac{\cancel{p}}{\cancel{\frac{1-x^p}{1-x}}(1-x)} - \ldots$

$\boxed{\frac{1}{1-x}} - \frac{1}{1-x} = 0$

$= \lim_{x\to 1}\left(\frac{p(1-x^q)-q(1-x^p)}{(1-x^p)(1-x^q)}\right)$

$= \lim_{x\to 1}\frac{p-q-px^q+qx^p}{\cancel{\left(\frac{1-x^p}{1-x}\right)}\cancel{\left(\frac{1-x^q}{1-x}\right)}(1-x)^2}$

$\frac{1-x^p}{1-x} \to p$, $\frac{1-x^q}{1-x} \to q$

$x = 1+t$

$x-1 = t$

$= \lim_{t\to 0}\frac{p-q-p(1+t)^q+q(1+t)^p}{p\cdot q\cdot t^2} = \frac{\cancel{p}-\cancel{q}-p\left(\cancel{1}+\cancel{qt}+\frac{q(q-1)}{2}t^2\right)+q\left(\cancel{1}+\cancel{pt}+\frac{p(p-1)}{2!}t^2\right)}{pq\,t^2}$

$= \frac{\left(-\frac{\cancel{pq}(q-1)}{2}+\frac{\cancel{pq}(p-1)}{2}\right)\cancel{t^2}}{\cancel{(pq)}\cancel{t^2}} = \boxed{\frac{p-q}{2}}$ Ans

$\underline{\text{Que}}$ $\lim_{x\to\infty}\left(x-\ln(\cosh(x))\right)$ where

$\cosh(t) = \frac{e^t+e^{-t}}{2}$.

$$\lim_{x\to\infty}\left(x - \ln(\cosh x)\right)$$

$$\cosh(t) = \frac{e^{t}+e^{-t}}{2}$$

$$\ln(t)$$

$$= \lim_{x\to\infty}\left(x - \ln\left(\frac{e^{x}+e^{-x}}{2}\right)\right)$$

$$= \lim_{x\to\infty}\left(\ln(e^{x}) - \ln\left(\frac{e^{x}+e^{-x}}{2}\right)\right)$$

$$= \lim_{x\to\infty}\left(\ln\left(\frac{e^{x}}{\left(\frac{e^{x}+e^{-x}}{2}\right)}\right)\right) = \lim_{x\to\infty}\ln\left(\frac{2}{1+\underbrace{e^{-2x}}_{\to 0}}\right)$$

$$= \boxed{\ln(2)} \text{ Ans}$$

One

Results to Remember

① $\lim_{x \to 0^+} x^m \cdot \ln^n(x) = 0$, $m \in R^+$

$x^m \to 0$, $\ln^n(x) \to -\infty$

② $\lim_{x \to 0^+} x^x = e^{\underline{x\ln(x)}} = e^0 = 1$.

$A \to 0$

$e^A - 1 \longrightarrow A$

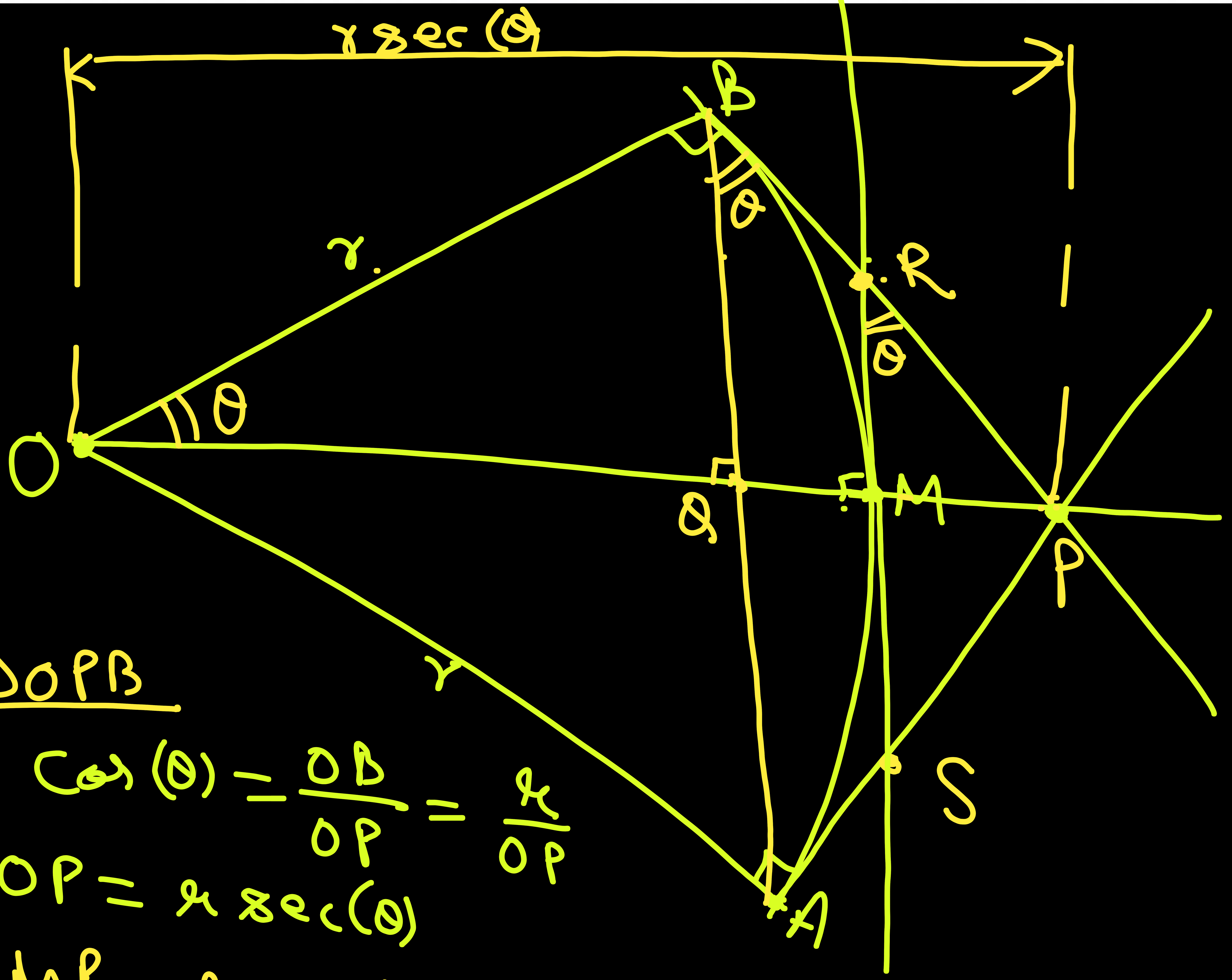
$$\underline{a}^{\underline{m}} = e^{\ln(a^m)}$$

$$= e^{m\ln(a)}$$

$$= \underline{\exp}(\underline{m\ln(a)})$$

One

At the end-points and at the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the two triangles thus formed tends to 4 as the arc AB decreases indefinitely.



In ΔOPB

$$\cos(\theta) = \frac{OB}{OP} = \frac{r}{OP}$$

$$OP = r\sec(\theta)$$

$$MP = r\sec(\theta) - r$$

In ΔMPR $\quad \tan(\theta) = \frac{MP}{MR} \Rightarrow MR = MP\cot(\theta)$

In ΔOBQ $\quad BQ = r\sin(\theta)$

In ΔPQB $\quad \tan(\theta) = \frac{PQ}{BQ} \Rightarrow PQ = BQ\tan(\theta)$

$$\frac{Ar(\Delta PAB)}{Ar(\Delta PRS)} = \frac{\frac{1}{\cancel{2}} \cdot PQ \cdot \cancel{2}BQ}{\frac{1}{\cancel{2}} \cdot PM \cdot \cancel{2}RM} = \frac{BQ\tan(\theta) \cdot BQ}{MP \cdot MP\cot(\theta)}$$

$$= \frac{BQ^2}{MP^2} \cdot \tan^2(\theta)$$

$$= \frac{\cancel{r^2}\sin^2(\theta) \cdot \tan^2(\theta)}{\cancel{r^2}(\sec(\theta)-1)^2} = \frac{\sin^4(\theta)\,\cancel{\cos^2(\theta)}}{\cancel{\cos^2(\theta)} \cdot (1-\cos(\theta))^2}$$

<u>Now</u> $\overset{\frown}{AB} \longrightarrow 0 \Rightarrow \theta \longrightarrow 0.$

$$\lim_{\theta \to 0} \frac{Ar(\Delta PAB)}{Ar(\Delta PRS)} = \lim_{\theta \to 0} \frac{\sin^4(\theta)}{(1-\cos(\theta))^2}$$

$$= \lim_{\theta \to 0} \frac{\theta^4}{\left(\theta^2/2\right)^2} = \lim_{\theta \to 0} 4 \cdot \frac{\cancel{\theta^4}}{\cancel{\theta^4}}$$

$$= \boxed{4} \quad \underline{Ans}$$

$1-\cos(A) \longrightarrow A^2/2$

Que Let $P_n = a^{P_{n-1}} - 1$, $\forall n \geq 2$ and $P_1 = a^x - 1$ then evaluate

$$\lim_{x \to 0} \frac{P_{2021}}{x}.$$

Sol

$$P_1 = a^x - 1$$

$$P_n = a^{P_{n-1}} - 1$$

$n = 2$

$$P_2 = a^{P_1} - 1$$

$n = 3$

$$P_3 = a^{P_2} - 1$$

$\vdots$

Que $\displaystyle \lim_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$

$\displaystyle \lim_{x\to a}(f)^{g}$, $f \to 1$

$\exp\left((f-1)\cdot g\right)$

Sol

$$\lim_{n\to\infty}\left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$$

$$=\exp\left(\lim_{n\to\infty}\left(2\sqrt{n^2+n}-1\right)\left(\frac{\sqrt{n^2+n}-1}{n}-1\right)\right)$$

$$=\exp\left(\lim_{n\to\infty}\left(2\sqrt{n^2+n}-1\right)\left(\frac{\sqrt{n^2+n}-(1+n)}{n}\right)\right)$$

$$=\exp\left(\lim_{n\to\infty}\left(\frac{2\cancel{n}}{\cancel{n}}\right)\left(\frac{n^2+n-(1+n)^2}{\sqrt{n^2+n}+1+n}\right)\right)=\exp\left(\lim_{n\to\infty}\cancel{2}\left(\frac{\cancel{n^2}+\cancel{n}-1-\cancel{n^2}-2n}{\cancel{2}n}\right)\right)$$

$$=\exp(-1)=e^{-1}=\boxed{\frac{1}{e}}$$

Ans

Que

Let $x_0 = 2\cos\frac{\pi}{6}$ and $x_n = \sqrt{2+x_{n-1}}$, $n = 1, 2, 3, \ldots\ldots$ find $\lim\limits_{n\to\infty} 2^{n+1}\sqrt{2-x_n}$

Let $x_0 = 2\cos\left(\frac{\pi}{6}\right) = 2\cos(\theta)$.

& $x_n = \sqrt{2+x_{n-1}}$, $n = 1, 2, 3, - -$

find $\lim\limits_{n\to\infty} 2^{n+1}\sqrt{2-x_n}$.

Soln

$$x_n = \sqrt{2+x_{n-1}} \quad \Big\}\ x_0 = 2\cos(\theta)$$

$$x_1 = \sqrt{2+x_0} = \sqrt{2+2\cos(\theta)} = 2\cos\left(\frac{\theta}{2}\right)$$

$$x_2 = \sqrt{2+x_1} = \sqrt{2+2\cos\left(\frac{\theta}{2}\right)} = 2\cos\left(\frac{\theta}{2^2}\right)$$

$$\vdots$$

$$x_n = 2\cos\left(\frac{\theta}{2^n}\right).$$

Que

$$\lim_{n\to\infty} 2^{n+1}\cdot\sqrt{2-x_n}$$

$$= \lim_{n\to\infty} 2^{n+1}\cdot\sqrt{2-2\cos\left(\frac{\theta}{2^n}\right)}$$

$$= \lim_{n\to\infty} 2^{n+1}\sqrt{4\sin^2\left(\frac{\theta}{2^{n+1}}\right)}$$

$$= \lim_{n\to\infty} 2^{n+1}\cdot 2\cdot \underbrace{\sin\left(\frac{\theta}{2^{n+1}}\right)}_{\to 0}$$

$$= \lim_{n\to\infty} \cancel{2^{n+1}}\cdot 2\cdot\frac{\theta}{\cancel{2^{n+1}}} = 2\theta = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$