**Q** Find the value of $\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)$.

**Soln.** <u>M-1</u> Let

$$x = \sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)$$

$$\Rightarrow x^2 = \sin^2\left(\frac{2\pi}{7}\right)+\sin^2\left(\frac{4\pi}{7}\right)+\sin^2\left(\frac{8\pi}{7}\right)+\underbrace{2\left(s\left(\frac{2\pi}{7}\right)s\left(\frac{4\pi}{7}\right)+s\left(\frac{4\pi}{7}\right)\cdot s\left(\frac{8\pi}{7}\right)+s\left(\frac{8\pi}{7}\right)\cdot\sin\left(\frac{2\pi}{7}\right)\right)}_{=\,y\ (\text{say})}$$

<u>Now</u>

$$y = \left(\cos\left(\frac{2\pi}{7}\right)-\cancel{\cos\left(\frac{6\pi}{7}\right)}\right)+\left(\underset{\pi-\frac{3\pi}{7}}{\cos\left(\frac{4\pi}{7}\right)}-\underset{2\pi-\frac{2\pi}{7}}{\cos\left(\frac{12\pi}{7}\right)}\right)+\left(\cancel{\cos\left(\frac{6\pi}{7}\right)}-\underset{\pi+\frac{3\pi}{7}}{\cos\left(\frac{10\pi}{7}\right)}\right)$$

$$= \cancel{\cos\left(\frac{2\pi}{7}\right)}-\cancel{\cos\left(\frac{3\pi}{7}\right)}-\cancel{\cos\left(\frac{2\pi}{7}\right)}+\cancel{\cos\left(\frac{3\pi}{7}\right)} = 0.$$

<u>Now again</u>

$$x^2 = \sin^2\left(\frac{2\pi}{7}\right)+\sin^2\left(\frac{4\pi}{7}\right)+\sin^2\left(\frac{8\pi}{7}\right)+0$$

$$= \frac{1-\cos\left(\frac{4\pi}{7}\right)}{2}+\frac{1-\cos\left(\frac{8\pi}{7}\right)}{2}+\frac{1-\cos\left(\frac{16\pi}{7}\right)}{2} = \frac{3}{2}-\frac{1}{2}\left(\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{8\pi}{7}\right)+\cos\left(\frac{16\pi}{7}\right)\right)$$

$$x^2 = \frac{3}{2} - \frac{1}{2}\left(\cos\left(\frac{4\pi}{7}\right) + \cos\left(\underset{2\pi - \frac{6\pi}{7}}{\underset{\downarrow}{\frac{8\pi}{7}}}\right) + \cos\left(\underset{2\pi + \frac{2\pi}{7}}{\underset{\downarrow}{\frac{16\pi}{7}}}\right)\right)$$

$$= \frac{3}{2} - \frac{1}{2}\left(\cos\left(\frac{4\pi}{7}\right) + \underbrace{\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)}\right)$$

$$= \frac{3}{2} - \frac{1}{2}\left(\frac{\sin\left(3\cdot\frac{\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)} \cdot \cos\left(\frac{4\pi}{7}\right)\right)$$

$$= \frac{3}{2} - \frac{1}{2}\left(\frac{\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}\right)$$

$$= \frac{3}{2} - \frac{1}{2}\cdot\frac{1}{2}\;\cancel{\frac{\sin\left(\frac{8\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}}\,(-1) = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$$

$$\Rightarrow x = \pm\frac{\sqrt{7}}{2} \Rightarrow x = \frac{\sqrt{7}}{2}$$

Ans

M-2

$$\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)$$

Consider $7\theta = 2n\pi \Rightarrow \theta = \frac{2n\pi}{7}$ → $n=1$ … $n=3$

$\theta = \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$ ; $\frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}$

$\alpha$: $\sin\left(\frac{2\pi}{7}\right) = -\sin\left(\frac{12\pi}{7}\right)$

$\beta$: $\sin\left(\frac{4\pi}{7}\right) = -\sin\left(\frac{10\pi}{7}\right)$

$\sin\left(\frac{6\pi}{7}\right) = -\sin\left(\frac{8\pi}{7}\right)$ ; $\gamma$

$\Rightarrow 4\theta = 2n\pi - 3\theta$

$\sin()$: $\sin(4\theta) = -\sin(3\theta)$

$2\cdot 2\cdot \cancel{s(\theta)}\, c(\theta)\left(1-2\sin^2(\theta)\right) = 4\cancel{\sin^3}(\theta) - 3\cancel{\sin(\theta)}$ $\quad \sin^2(\theta)$

$()^2$: $4c(1-2s^2) = 4s^2-3$

$16(1-s^2)(1+4s^4-4s^2) = 16s^4+9-24s^2$

$\Rightarrow 64s^6+(16-64-64)s^4+s^2(-24+16+64)-7=0.$

$64s^6-112s^4+56s^2-7=0.$ → $\alpha,\beta,\gamma,-\alpha,-\beta,-\gamma$

$s^2=t$ → $64t^3-112t^2+56t-7=0.$ → $\alpha^2,\beta^2,\gamma^2$

$\alpha^2+\beta^2+\gamma^2 = \frac{112}{64} = \frac{7}{4}$

$\Rightarrow \sin^2\left(\frac{2\pi}{7}\right)+\sin^2\left(\frac{4\pi}{7}\right)+\sin^2\left(\frac{8\pi}{7}\right) = \frac{7}{4}.$

Consider

$$\left(\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)\right)^2$$

$$= \sum \sin^2\left(\frac{2\pi}{7}\right) + \underbrace{2\sum \sin\left(\frac{2\pi}{7}\right)\cdot\sin\left(\frac{4\pi}{7}\right)}_{= y = 0}$$

$$= \frac{7}{4}$$

$$\sqrt{\phantom{x}} \quad \sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right) = \frac{\sqrt{7}}{2}.$$

Q) In a $\Delta ABC$, if $\sin^3(\theta) = \sin(A-\theta)\cdot\sin(B-\theta)\cdot\sin(C-\theta)$ then find the value of $\dfrac{\sum\cot(A)}{\cot(\theta)}$

Sol<sup></sup>n M-1 Given

$$\sin^3(\theta) = \prod \sin(A-\theta)$$

$$\Rightarrow 1 = \prod\left(\frac{\sin(A)\cos(\theta) - \cos(A)\sin(\theta)}{\sin(\theta)}\right)$$

$$= \prod\left(\sin(A)\cdot\cot(\theta) - \cos(A)\right)$$

$$1 = \prod \sin(A)\cdot\left(\cot(\theta) - \cot(A)\right)$$

$$1 = \prod \sin(A)\left(\prod\left(\cot(\theta) - \cot(A)\right)\right)$$

$$\prod \operatorname{cosec}(A) = \cot^3(\theta) - \left(\sum\cot(A)\right)\cot^2(\theta) + \left(\sum\cot(A)\cot(B)\right)\cot(\theta) - \prod\cot(A)$$

$$\sum\cot(A)\cot(B) = 1$$

$$\underbrace{\cot^3(\theta) + \cot(\theta)}\ - \cot^2(\theta)\left(\sum\cot(A)\right) - \left(\prod\cot(A) + \prod\operatorname{cosec}(A)\right)$$

$$\prod\cot(A) + \prod\operatorname{cosec}(A) = \frac{\prod c(A) + 1}{\prod s(A)} = \frac{\sum s(A)\,s(B)\,c(C)}{\prod s(A)} = \sum\cot(A)$$

$$\Rightarrow \cot(\theta)\left(\cancel{\cot^2(\theta) + 1}\right) - \left(\sum\cot(A)\right)\left(\cancel{\cot^2(\theta) + 1}\right) = 0$$

$$\Rightarrow \cot(\theta) = \sum\cot(A)$$

$$\Rightarrow \frac{\sum\cot(A)}{\cot(\theta)} = \boxed{1}\ \text{Ans}$$

formula used

In a $\Delta ABC$

R-1 $\quad \sum \cot(A)\cot(B) = 1$

R-2 $\quad \cos(A+B+C) = \Pi \cos(A) - \sum s(A)s(B)c(C)$

$A+B+C=\pi$

$$-1 = \Pi \cos(A) - \sum s(A)s(B)c(C).$$

$$1 + \Pi c(A) = \sum s(A)s(B)c(C)$$

R-3

$$\sum \sin(2A) = 4\,\Pi \sin(A)$$

$$\sum \cos(2A) = -1 - 4\,\Pi \cos(A)$$

M-2

$$\sin^3(\theta) = \prod \sin(A-\theta)$$

$$4\sin^3(\theta) = 2\sin(A-\theta)\cdot\left(2\sin(B-\theta)\sin(C-\theta)\right)$$

$$= 2\sin(A-\theta)\left(\cos(B-C) - \cos(B+C-2\theta)\right)$$

$$= \left(\left(\sin(\underbrace{A+B}_{\pi-C}-C-\theta) + \sin(\underline{A+C}-B-\theta)\right) - \left(\sin(\underbrace{A+B+C}_{=\pi}-3\theta) + \sin(A+\theta-B-C)\right)\right)$$

$$\sin(\theta+2C) + \sin(\theta+2B) + \sin(\theta+2A)$$

$$4\sin^3(\theta) + \sin(3\theta) = \sin(\theta+2C) + \sin(\theta+2B) + \sin(\theta+2A).$$

$$3\sin(\theta) = \sin(\theta+2C) + \sin(\theta+2B) + \sin(\theta+2A)$$

$$3\sin(\theta) = \sin(\theta)\left(\sum\cos(2A)\right) + \cos(\theta)\left(\sum\sin(2A)\right)$$

$$\sin(\theta)\left(3 - \sum\cos(2A)\right) = \cos(\theta)\left(\sum\sin(2A)\right) \Rightarrow \sin(\theta)\left(4+4\prod\cos(A)\right) = \cos(\theta)\left(4\prod\sin(A)\right)$$

$$\cot(\theta) = \frac{1+\prod\cos(A)}{\prod\sin(A)}$$

$$\cot(\theta) = \frac{\sum s.s.c}{s.s.s} = \sum \cot(A)$$

$$\Rightarrow \frac{\sum \cot(A)}{\cot(\theta)} = \boxed{1}$$

Q. If $2\tan^2(\alpha)\cdot\tan^2(\beta)\cdot\tan^2(\gamma) + \sum \tan^2(\alpha)\cdot\tan^2(\beta) = 1$, then find the value of $\sum \sin^2(\alpha)$.

Sln M-1 $\gamma = 0$

$0 + t^2(\alpha)\,t^2(\beta) + 0 + 0 = 1$

$\underline{t^2(\alpha)\cdot t^2(\beta) = 1}$

$s^2(\alpha) + s^2(\beta) + \cancel{s^2(\gamma)} = $ 0

$\frac{t^2(\alpha)}{1+t^2(\alpha)} + \frac{t^2(\beta)}{1+t^2(\beta)}$

$= \frac{t^2(\alpha) + t^2(\beta) + 2\cancel{t^2(\alpha)t^2(\beta)}}{1 + t^2(\alpha) + t^2(\beta) + \cancel{t^2(\alpha)t^2(\beta)}}$ 1

$= \boxed{1}$ Ans

M-2 $t^2(\alpha) = t_1$, $t^2(\beta) = t_2$, $t^2(\gamma) = t_3$

Give $2t_1t_2t_3 + t_1t_2 + t_2t_3 + t_3t_1 = 1$

$2\prod t_1 + \sum t_1t_2 = 1$

$\tan^2(\alpha) = t_1 = \frac{s^2(\alpha)}{c^2(\alpha)} = \frac{s^2(\alpha)}{1-s^2(\alpha)}$

$t_1 = \frac{s_1}{1-s_1}$

$\Rightarrow 2\prod \frac{s_1}{1-s_1} + \sum \frac{s_1}{1-s_1}\frac{s_2}{1-s_2} = 1$

$\Rightarrow 2\prod s_1 + \sum s_1 s_2 (1 - s_3) = \prod(1-s_1)$.

$\Rightarrow \cancel{2\prod s_1} + \cancel{\sum s_1 s_2} - \cancel{3\prod s_1} = 1 - \sum s_1 + \cancel{\sum s_1 s_2} - \cancel{\prod s_1}$

$\Rightarrow \sum s_1 = 1 \Rightarrow \sum \sin^2(\alpha) = \boxed{1}$ Ans

Q. Find number of polynomials with integral coefficients which satisfy $f(a)=b$, $f(b)=c$ & $f(c)=a$, where $a, b, c$ are distinct integers.

Soln

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

$$f(a)-f(b) = a_n(a^n - b^n) + a_{n-1}(a^{n-1} - b^{n-1}) + \dots + a_1(a-b)$$

$$= (a-b)(I_1).$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})$$

$$\Rightarrow f(a) - f(b) = (a-b)\cdot I_1$$

$f(a)=b$ & $f(b)=c$ given

Similarly $\Rightarrow$

$$(a-b)\cdot I_1 = b-c$$
$$(b-c)\cdot I_2 = c-a$$
$$(c-a)\, I_3 = a-b$$

— ①

$$\cancel{\Pi(a-b)}\cdot I_1 I_2 I_3 = \cancel{\Pi(b-c)}$$

$I_1 I_2 I_3 = 1$. from ①

$I_1 I_2 I_3 \Rightarrow$

| $I_1$ | $I_2$ | $I_3$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | -1 | -1 |

$a-b = b-c$
$b-c = c-a$
$b - c = a+b$
$\Rightarrow 2b = a+c$
$b = 2c$

C-2

$a-b = b-c$

$\cancel{c}-b = \cancel{c}-a \Rightarrow a=b$ X

$a-c = a-b$

$\Rightarrow$ There is no such polynomial. Ans 0

Q. If $P(x)$ is the remainder when $x^{81}+x^{49}+x^{25}+x^{9}+x$ is divided by $x^3-x$, then find $P(2)$.

Sol

$$x^{81}+x^{49}+x^{25}+x^{9}+x = (x^3-x)\cdot Q(x) + \underbrace{ax^2+bx+c}_{P(x)}.$$

$x=0 \rightarrow c=0$

$x=1 \rightarrow 5 = a+b+\cancel{c}$ $\quad 0$

$x=-1 \rightarrow -5 = a-b$

$a=0 \Rightarrow b=5 \quad \Rightarrow \boxed{P(x) = 5x}$

$\Downarrow$

$\boxed{P(2)=10}$ Ans

Q Let $f(x)$ is a polynomial satisfying
$x f(x-1) = (x+1)\cdot f(x)$, $\forall x$ then find $f\left(\frac{2020}{2021}\right)$.

Sol

$x \cdot f(x-1) = (x+1)\cdot f(x) \Rightarrow f(x) = \left(\frac{x+1}{x}\right)\cdot f(x-1)$

$x=0$

$0 = 1\cdot f(0)$

$\Downarrow$

$x=0$ is a root

$x=1$

$1\cdot \cancel{f(0)} = 2\cdot f(1)$
$0$

$\Rightarrow f(1)=0$

$\Leftarrow$ 1 is also root

$f(\alpha) = \left(\frac{\alpha+1}{\alpha}\right)\cdot f(\alpha-1)$.

If $\alpha-1$ is root then R.H.S. $=0$

$\Downarrow$

$f(\alpha)=0$

$\Downarrow$ $\alpha$ is also a root.

$x=0$ is root
$\Downarrow$
$x=1$ is root
$\Downarrow$
$x=2$ is root
$\Downarrow$
⋮

$\Rightarrow$ There are $\infty$-roots

$\Rightarrow f(x)=0$.

$\Rightarrow \boxed{f\left(\frac{2020}{2021}\right) = 0}$

Ans

Q Let $(1+x+x^2+\cdots+x^{2021})^2 - x^{2021} = P_{2020}(x)\cdot P_{2022}(x)$, where $P_{2020}(x)$ and $P_{2022}(x)$ are polynomials of degree 2020 and 2022 respectively then find $P_{2020}(1)$.

Sol

$$\text{L.H.S.} = (\underbrace{1+x+x^2+\cdots+x^{2020}}_{\alpha}+x^{2021})^2 - x^{2021}$$

$$= (\alpha+x^{2021})^2 - x^{2021}$$

$$= (x^{2021})^2 + 2\alpha\cdot x^{2021} + \alpha^2 - x^{2021}$$

$$= x^{2021}(x^{2021}-1) + 2\alpha\cdot x^{2021} + \alpha^2.$$

$$= x^{2021}\cdot(x-1)\cdot\alpha + 2\alpha\cdot x^{2021} + \alpha^2$$

$$= \alpha(x^{2022} - \cancel{x^{2021}} + \cancel{2}\cdot x^{2021} + \alpha)$$

$$= (1+x+x^2+\cdots+x^{2020})\cdot(1+x+x^2+\cdots+x^{2020}+x^{2021}+x^{2022})$$

Let

$$\alpha = 1+x+x^2+\cdots+x^{2020}$$

$$= \frac{1-x^{2021}}{1-x} = \frac{x^{2021}-1}{x-1}$$

$$x^{2021}-1 = (x-1)\cdot\alpha.$$

$$P_{2020}(x)\cdot P_{2022}(x)$$

$$\Rightarrow P_{2020}(1) = 2021$$

Q Find elements of the set $A = \left\{ x \in Z \,\middle|\, \frac{x^3 - 3x + 2}{2x+1} \in Z \right\}$, where $Z$ is set of integers.

Sol <u>according to Question</u> $\frac{x^3 - 3x + 2}{2x+1} = \text{Integer}$

$\Rightarrow x^3 - 3x + 2$ should be div. by $2x+1$.

$\Rightarrow 8(x^3 - 3x + 2)$ " " " "

$\Rightarrow (2x)^3 + 1 - 12(2x+1) + 27$ " " " "

$\Rightarrow 27$ must be div by $2x+1$

$2x+1 = \pm 1, \pm 3, \pm 9, \pm 27$

$\Rightarrow x = 0, -1, 1, -2, 4, -5, 13, -14$ Ans