VECTORS

1. INTRODUCTION :

Vectors constitute one of the several Mathematical systems which can be usefully employed to provide mathematical handling for certain types of problems in Geometry, Mechanics and other branches of Applied Mathematics.

Vectors facilitate mathematical study of such physical quantities as possess Direction in addition to Magnitude. Velocity of a particle, for example, is one such quantity.

2. Physical quantities are broadly divided in two categories viz **(a) Vector Quantities** & **(b) Scalar quantities.**

(a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

Note :

Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

For example, the rotations of a rigid body through finite angles have both magnitude & direction but do not satisfy the law of vector addition therefore not a vector.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

3. MATHEMATICAL DESCRIPTION OF VECTOR & SCALAR :

To understand vectors mathematically we will first understand directed line segment.

Directed line segment :

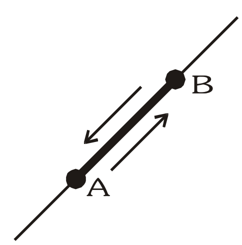
Any given portion of a given straight line where the two end points are distinguished as **Initial** and **Terminal** is called a **Directed Line Segment**.

The directed line segment with initial point A and terminal point B is denoted by the symbol $\overrightarrow{AB}$.

The two end points of a directed line segment are not interchangeable and the directed line segments. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ must be thought of as different.

(a) Vector :

A directed line segment is called vector. Every directed line segment have three essential characteristics.



(i) **Length** : The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$

Clearly, we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

(ii) **Support** : The line of unlimited length of which a directed line segment is a part is called its line of support or simply the Support.

(iii) **Sense :** The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ from B to A so that the sense of a directed line segment is from its initial to the terminal point.

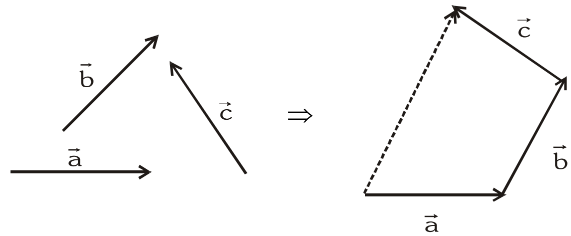
(b) **Scalar :**

Any real number is a scalar.

4. CLASSIFICATION OF VECTORS :

There is a very important classification of vector in which vectors are divided into two categories.

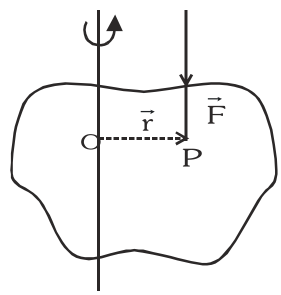
(a) **Free vectors :** If a vector can be translated anywhere in space without changing its magnitude & direction, then such a vector is called **free vector**. In other words, the initial point of free vector can be taken anywhere in space keeping its magnitude & direction same.



(b) **Localised vectors :** For a vector of given magnitude and direction, if its initial point is fixed in space, then such a vector is called **localised vector**.

Examples : Torque, Moment of Inertia etc.

Unless & until stated, vectors are treated as free vectors.



$\vec{\tau} = \vec{r} \times \vec{F}$

5. EQUALITY OF TWO VECTORS :

Two vectors are said to be equal if they have

(a) the same length,

(b) the same or parallel supports and

(c) the same sense.

Note : Components of two equal vectors taken in any arbitrary direction are equal. i.e If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, where $\hat{i}, \hat{j}$ & $\hat{k}$ are the unit vectors taken along co-ordinate axes, then $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2, a_3 = b_3$

Illustration 1 : Let $\vec{r} = 3\hat{i} + 2\hat{j} - 5\hat{k}, \vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{c} = -2\hat{i} + \hat{j} - 3\hat{k}$. $\vec{r} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$, then find $\lambda + \mu + \nu$.

Solution :

$$3\hat{i} + 2\hat{j} - 5\hat{k} = \lambda(2\hat{i} - \hat{j} + \hat{k}) + \mu(\hat{i} + 3\hat{j} - 2\hat{k}) + \nu(-2\hat{i} + \hat{j} - 3\hat{k})$$

$$= (2\lambda + \mu - 2\nu)\hat{i} + (-\lambda + 3\mu + \nu)\hat{j} + (\lambda - 2\mu - 3\nu)\hat{k}$$

Equating components of equal vectors

$2\lambda + \mu - 2\nu = 3$ ....(i)

$-\lambda + 3\mu + \nu = 2$ ....(ii)

$\lambda - 2\mu - 3\nu = -5$ ....(iii)

on solving (i), (ii) & (iii)

we get $\lambda = 3, \mu = 1, \nu = 2$

So $\lambda + \mu + \nu = 6$ **Ans.**

Do yourself - 1 :

(i) If $\vec{a} = 2\tilde{i} + \mu\tilde{j} - 7\tilde{k}$ and $\vec{b} = \lambda\tilde{i} + \sqrt{3}\,\tilde{j} - 7\tilde{k}$ are two equal vectors, then find $\lambda^2 + \mu^2$.

(ii) If $\vec{a}$, $\vec{b}$ are two vectors then which of the following statements is/are correct -

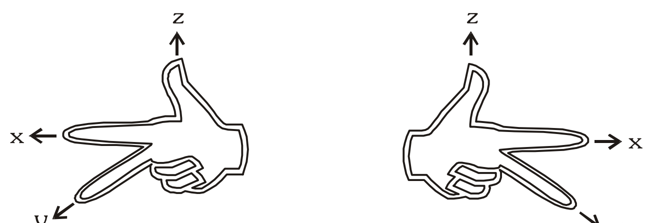
(A) $\vec{a} = -\vec{b} \Rightarrow |\vec{a}| = |\vec{b}|$

(B) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm\vec{b}$

(C) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$

(D) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm 2\vec{b}$

6. LEFT AND RIGHT - HANDED ORIENTATION (CONFIGURATIONS) :



Right - Handed Triplet   Left - Handed Triplet

For each hand take the directions Ox, Oy and Oz as shown in the figure. Thus we get two rectangular coordinate systems. Can they be made congruent ? They cannot be, because the two hands have different orientations. Therefore these two systems are different.

A rectangular coordinate system which can be made congruent with the system formed with the help of right hand (or left hand) is called a right handed (or left handed) rectangular coordinates system.

Thus we have the following condition to identify these two systems using sense of rotation :

(a) If the rotation from Ox to Oy is in the anticlockwise direction and Oz is directed upwards (see right hand), then the system is right handed.

(b) If the rotation from Ox to Oy is clockwise and Oz is directed upward (see left hand) then the system is left handed.

Here after we shall use the right-handed rectangular Cartesian coordinate system (or Ortho-normal system).

7. ALGEBRA OF VECTORS :

It is possible to develop an Algebra of Vectors which proves useful in the study of Geometry, Mechanics and other branches of Applied Mathematics.

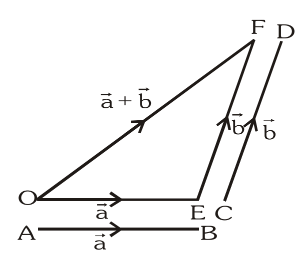
(a) Addition of two vectors :

The vectors have magnitude as well as direction, therefore their addition is different than addition of real numbers.

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, which are represented by $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Their addition can be performed in the following two ways :

(i) Triangle law of addition of vectors : If two vectors can be represented in magnitude and direction by the two sides of a triangle, taken in order, then their sum will be represented by the third side in reverse order.

Let O be the fixed point in the plane of vectors. Draw a line segment $\overrightarrow{OE}$ from O, equal and parallel to $\overrightarrow{AB}$, which represents the vector $\vec{a}$. Now from E, draw a line segment $\overrightarrow{EF}$, equal and parallel to $\overrightarrow{CD}$, which represents the vector $\vec{b}$. Line segment $\overrightarrow{OF}$ obtained by joining O and F represents the sum of vectors $\vec{a}$ and $\vec{b}$.



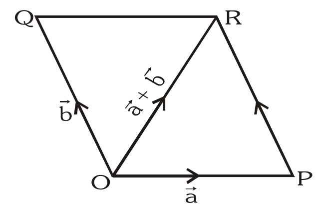
i.e. $\overrightarrow{OE}+\overrightarrow{EF}=\overrightarrow{OF}$

or $\vec{a}+\vec{b}=\overrightarrow{OF}$

This method of addition of two vectors is called **Triangle law of addition of vectors.**

(ii) **Parallelogram law of addition of vectors** : If two vectors be represented in magnitude and direction by the two adjacent sides of a parallelogram then their sum will be represented by the diagonal through the co-initial point.

Let $\vec{a}$ and $\vec{b}$ be vectors drawn from point O denoted by line segments $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Now complete the parallelogram OPRQ. Then the vector represented by the diagonal OR will represent the sum of the vectors $\vec{a}$ and $\vec{b}$.



i.e. $\overrightarrow{OP}+\overrightarrow{OQ}=\overrightarrow{OR}$

or $\vec{a}+\vec{b}=\overrightarrow{OR}$

This method of addition of two vectors is called **Parallelogram law of addition of vectors.**

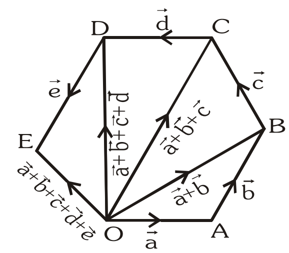
(iii) Properties of vector addition :

(1) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative) (2) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)

(3) $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ (additive identity) (4) $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$ (additive inverse)

(b) **Polygon law of vector Addition (Addition of more than two vectors):**

Addition of more than two vectors is found to be by repetition of triangle law.



Suppose we have to find the sum of five vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and $\vec{e}$. If these vectors be represented by line segment $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{DE}$ respectively, then their sum will be denoted by $\overrightarrow{OE}$. This is the vector represented by rest (last) side of the polygon OABCDE in reverse order. We can also make it clear this way :

By triangle's law

$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$ or $\vec{a}+\vec{b}=\overrightarrow{OB}$

$\overrightarrow{OB}+\vec{c}=\overrightarrow{OC}$ or $(\vec{a}+\vec{b})+\vec{c}=\overrightarrow{OC}$

$\overrightarrow{OC}+\vec{d}=\overrightarrow{OD}$ or $(\vec{a}+\vec{b}+\vec{c})+\vec{d}=\overrightarrow{OD}$

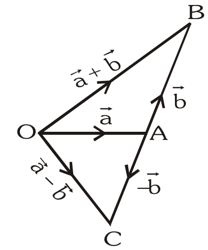
$\overrightarrow{OD}+\vec{e}=\overrightarrow{OE}$ or $(\vec{a}+\vec{b}+\vec{c}+\vec{d})+\vec{e}=\overrightarrow{OE}$

Here, we see that $\overrightarrow{OE}$ is represented by the line segment joining the initial point O of the first vector $\vec{a}$ and the final point of the last vector $\vec{e}$.

In order to find the sum of more that two vectors by this method, a polygon is formed. Therefore this method is known as the **polygon law of addition.**

Note : If the initial point of the first vector and the final point of the last vector are the same, then the sum of the vectors will be a null vector.

(c) Subtraction of Vectors :



Earlier we have described the vector $-\vec{b}$ whose length is equal to vector $\vec{b}$ but direction is opposite. Subtraction of vector $\vec{a}$ and $\vec{b}$ is defined as addition of $\vec{a}$ and $(-\vec{b})$. It is written as follows :

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Geometrical representation :

In the given diagram, $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{AB}$. We extend the line AB in opposite direction upto C, where AB = AC. The line segment $\overrightarrow{AC}$ will represent the vector $-\vec{b}$. By joining the points O and C, the vector represented by $\overrightarrow{OC}$ is $\vec{a}+(-\vec{b})$. i.e. denotes the vector $\vec{a}-\vec{b}$.

Note :

(i) $\vec{a}-\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

(ii) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

Hence subtraction of vectors does not obey the commutative law.

(iii) $\vec{a}-(\vec{b}-\vec{c})\neq(\vec{a}-\vec{b})-\vec{c}$

i.e. subtraction of vectors does not obey the associative law.

(d) Multiplication of vector by scalars :

If $\vec{a}$ is a vector & m is a scalar, then m($\vec{a}$) is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called SCALAR MULTIPLICATION. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

(i) $m(\vec{a})=(\vec{a})m=m\vec{a}$ (ii) $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

(iii) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$ (iv) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

Illustration 2 : ABCD is a parallelogram whose diagonals meet at P. If O is a fixed point, then $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$ equals :-

(A) $\overrightarrow{OP}$ (B) $2\overrightarrow{OP}$ (C) $3\overrightarrow{OP}$ (D) $4\overrightarrow{OP}$

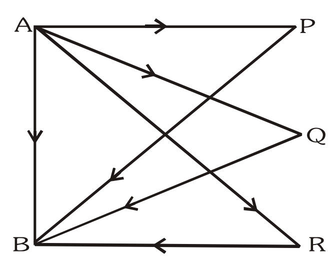
Solution : Since, P bisects both the diagonal AC and BD, so

$\therefore\ \overrightarrow{OA}+\overrightarrow{OC}=2\overrightarrow{OP}$ and $\overrightarrow{OB}+\overrightarrow{OD}=2\overrightarrow{OP} \Rightarrow \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$ **Ans. [D]**

Illustration 3 : A, B, P, Q, R are five points in any plane. If forces $\overrightarrow{AP}$, $\overrightarrow{AQ}$, $\overrightarrow{AR}$ acts on point A and force $\overrightarrow{PB}$, $\overrightarrow{QB}$, $\overrightarrow{RB}$ acts on point B then resultant is :-

(A) $3\overrightarrow{AB}$ (B) $3\overrightarrow{BA}$ (C) $3\overrightarrow{PQ}$ (D) $3\overrightarrow{PR}$

Solution : From figure



$\overrightarrow{AP}+\overrightarrow{PB}=\overrightarrow{AB}$

$\overrightarrow{AQ}+\overrightarrow{QB}=\overrightarrow{AB}$

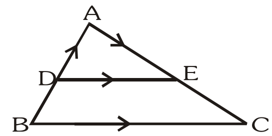
$\overrightarrow{AR}+\overrightarrow{RB}=\overrightarrow{AB}$

So $(\overrightarrow{AP}+\overrightarrow{AQ}+\overrightarrow{AR})+(\overrightarrow{PB}+\overrightarrow{QB}+\overrightarrow{RB})=3\overrightarrow{AB}$

so required resultant $=3\overrightarrow{AB}$. **Ans. [A]**

Illustration 4 : Prove that the line joining the middle points of two sides of a triangle is parallel to the third side and is of half its length.

Solution : Let the middle points of side AB and AC of a ΔABC be D and E respectively.



$\overrightarrow{BA} = 2\overrightarrow{DA}$ and $\overrightarrow{AC} = 2\overrightarrow{AE}$

Now in ΔABC, by triangle law of addition

$\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$

$2\overrightarrow{DA} + 2\overrightarrow{AE} = \overrightarrow{BC}$ $\Rightarrow$ $\overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$

Hence, line DE is parallel to third side BC of triangle and half of it.

Do yourself - 2 :

(i) If $\vec{a}, \vec{b}, \vec{c}$ be the vectors represented by the sides of a triangle taken in order, then prove that $\vec{a} + \vec{b} + \vec{c} = 0$

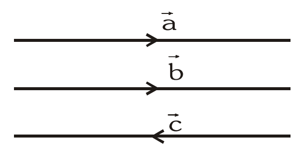
(ii) If $\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR}$, then prove that the points P, Q and R are collinear.

(iii) For any two vectors $\vec{a}$ and $\vec{b}$ prove that

(a) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (b) $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$ (c) $|\vec{a} + \vec{b}| \ge |\vec{a}| - |\vec{b}|$

Note : In general for any non-zero vectors $\vec{a}, \vec{b}$ & $\vec{c}$ one may note that although $\vec{a} + \vec{b} + \vec{c} = 0$ but it will not always represent the three sides of a triangle.

8. COLLINEAR VECTORS :



Two vectors are said to be collinear if their supports are parallel disregards to their direction. Collinear vectors are also called **Parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors.**

Note :

(i) Symbolically two non zero vectors $\vec{a}$ & $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two collinear vectors then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

(iii) If $\vec{a}$ & $\vec{b}$ are two non-zero, non-collinear vectors such that $x\vec{a} + y\vec{b} = 0 \Rightarrow x = y = 0$

9. CO-INITIAL VECTORS :

Vectors having same initial point are called Co-initial Vectors.

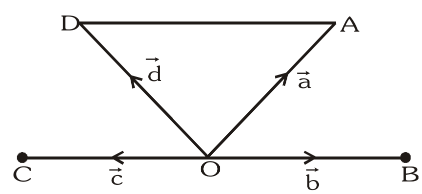


Illustration 5 : If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then find the value of x for which vectors :

$\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3+2x)\vec{a} - 2\vec{b}$ are collinear.

Solution : Since the vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear.

$\therefore$ there exist scalar λ such that $\vec{\alpha} = \lambda\vec{\beta}$

$\Rightarrow$ $(x - 2)\vec{a} + \vec{b} = \lambda\{(3 + 2x)\vec{a} - 2\vec{b}\}$ $\Rightarrow$ $(x - 2 - \lambda(3 + 2x))\vec{a} + (1 + 2\lambda)\vec{b} = 0$

$\Rightarrow$ $x - 2 - \lambda(3 + 2x) = 0$ and $1 + 2\lambda = 0$

$x - 2 - \lambda(3 + 2x) = 0$ and $\lambda = -\frac{1}{2}$

$\Rightarrow$ $x - 2 + \frac{1}{2}(3 + 2x) = 0$ $\Rightarrow$ $4x - 1 = 0$ $\Rightarrow$ $x = \frac{1}{4}$ **Ans.**

Illustration 6 : If A ≡ $(2\hat{i} + 3\hat{j})$, B ≡ $(p\hat{i} + 9\hat{j})$ and C ≡ $(\hat{i} - \hat{j})$ are collinear, then the value of p is :-

(A) 1/2 (B) 3/2 (C) 7/2 (D) 5/2

Solution : $\overrightarrow{AB} = (p-2)\hat{i} + 6\hat{j}$, $\overrightarrow{AC} = -\hat{i} - 4\hat{j}$

Now A, B, C are collinear ⇔ $\overrightarrow{AB} \,||\, \overrightarrow{AC}$ ⇔ $\dfrac{p-2}{-1} = \dfrac{6}{-4}$ ⇔ p = 7/2 **Ans. [C]**

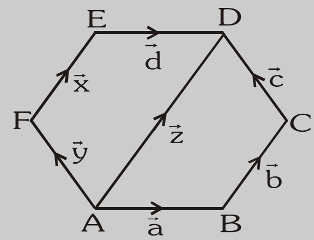
Illustration 7 : The value of λ when $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = 8\hat{i} + \lambda\hat{j} + 4\hat{k}$ are parallel is :-

(A) 4 (B) – 6 (C) – 12 (D) 1

Solution : Since $\vec{a}$ & $\vec{b}$ are parallel ⇒ $\dfrac{2}{8} = -\dfrac{3}{\lambda} = \dfrac{1}{4}$ ⇒ λ = –12 **Ans. [C]**

Do yourself - 3 :

(i) In the given figure, which vectors are (provided vertices A, B, C, D, E, F are all fixed) :



(a) Parallel (b) Equal

(c) Coinitial (d) Parallel but not equal.

(ii) If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 8\hat{i} - 12\hat{j} + 16\hat{k}$ such that $\vec{a} = \lambda\vec{b}$, then λ equals to

(iii) If $3\vec{a} + 2\vec{b} = 5\vec{c}$ and $8\vec{a} - 7\vec{b} = 4\vec{c}$, then which statement is/are true :

(A) $|\vec{a}| > |\vec{b}|$ (B) $|\vec{c}| > |\vec{b}|$

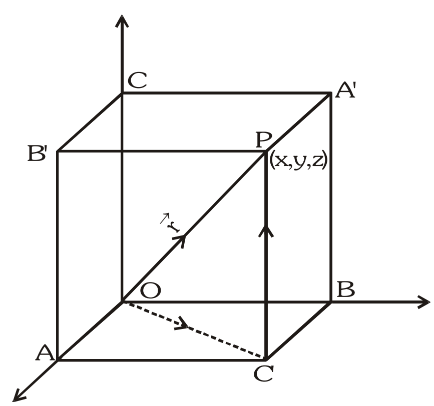
(C) $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear vectors. (D) $|\vec{a}| = |\vec{b}|$

10. COPLANAR VECTORS :

A given number of vectors are called coplanar if their supports are all parallel to the same plane. Note that "TWO VECTORS ARE ALWAYS COPLANAR ".

Note : Coplanar vectors may have any directions or magnitude.

11. REPRESENTATION OF A VECTOR IN SPACE IN TERMS OF 3 ORTHONORMAL TRIAD OF UNIT VECTORS :



Let P(x, y, z) be a point in space with reference to OX, OY and OZ as the coordinate axes, then OA = x, OB = y and OC = z

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively, then

$\overrightarrow{OA} = x\hat{i}$, $\overrightarrow{OB} = y\hat{j}$, $\overrightarrow{OC} = z\hat{k}$

$\overrightarrow{OP} = \overrightarrow{OC'} + \overrightarrow{C'P} = \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC}$ [∴ $\overrightarrow{C'P} = \overrightarrow{OC}$]

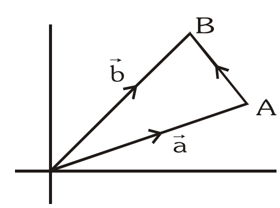
$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = x\hat{i} + y\hat{j} + z\hat{k}$

If $\overrightarrow{OP} = \vec{r}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = OP = \sqrt{x^2 + y^2 + z^2}$

12. POSITION VECTOR :



Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ are position vectors of two point A and B, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B – pv of A.

13. ZERO VECTOR OR NULL VECTOR :

A vector of zero magnitude i.e. which has the same initial & terminal point is called a zero vector. It is denoted by $\vec{O}$. It can have any arbitrary direction and any line as its line of support.

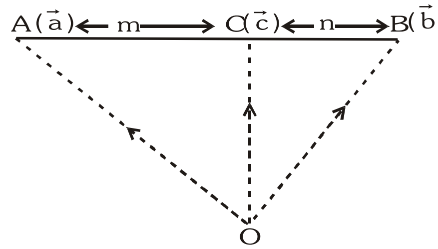
14. UNIT VECTOR :

A vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ (provided $|\vec{a}| \neq 0$)

15. SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point $C(\vec{r})$ which divides AB in the ratio m : n is given by :

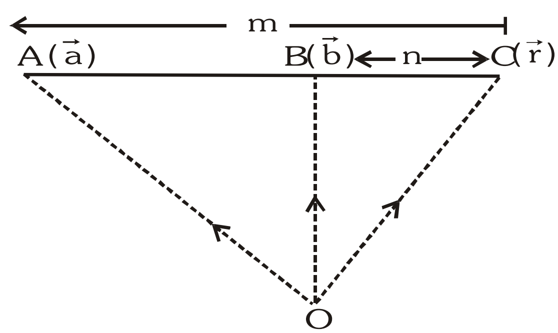
(a) Internal Division :



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

Note : Position vector of mid point of AB $= \dfrac{\vec{a} + \vec{b}}{2}$

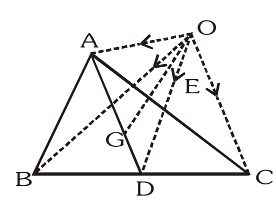
(b) External division :



$$\overrightarrow{OC} = \vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

Illustration 8 : Prove that the medians of a triangle are concurrent.

Solution : Let ABC be a triangle and position vectors of three vertices A, B and C with respect to the origin O be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.



$\therefore \quad \overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$

Again, let D be the middle point of the side BC,

so the position vector of point D is $\overrightarrow{OD} = \dfrac{\vec{b} + \vec{c}}{2}$

[$\because$ Position vector of the middle point of any line =1/2 (Sum of position vectors of end point of line)]

Now take a point G, which divides the median AD in the ratio 2 : 1.

Position vector of point G is $\overrightarrow{OG} = \dfrac{1.\overrightarrow{OA} + 2.\overrightarrow{OD}}{1+2} = \dfrac{1.\vec{a} + 2.\frac{1}{2}\left(\vec{b} + \vec{c}\right)}{1+2} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

Similarly, the position vector of the middle points of the other two medians, which divide the medians in the ratio 2 : 1 will comes out to the same $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$, which is the position vector of G.

Hence, the medians of the triangles meet in G i.e. are concurrent.

Illustration 9 : If the middle points of sides BC, CA & AB of triangle ABC are respectively D, E, F then position vector of centroid of triangle DEF, when position vector of A,B,C are respectively i + j, j + k, k + i is -

(A) $\frac{1}{3}(i + j + k)$ (B) $(i + j + k)$ (C) $2(i + j + k)$ (D) $\frac{2}{3}(i + j + k)$

Solution : The position vector of points D, E, F are respectively $\frac{i+j}{2} + k$, $i + \frac{k+j}{2}$ and $\frac{i+k}{2} + j$

So, position vector of centroid of $\Delta DEF = \frac{1}{3}\left[\frac{i+j}{2}+k+i+\frac{k+j}{2}+\frac{i+k}{2}+j\right]=\frac{2}{3}[i + j + k]$. **Ans. [D]**

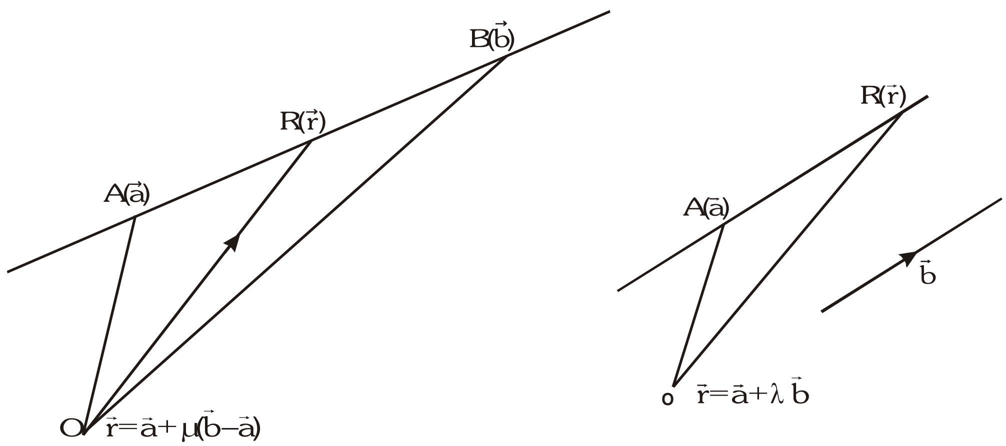
Do yourself - 4 :

(i) Find the position vectors of the points which divide the join of the points $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ internally and externally in the ratio 2 : 3,

(ii) ABCD is a parallelogram and P is the point of intersection of its diagonals. If O is the origin of reference, show that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=4\overrightarrow{OP}$

(iii) Find the unit vector in the direction of $3\hat{i} - 6\hat{j} + 2\hat{k}$.

16. VECTOR EQUATION OF A LINE :



Parametric vector equation of a line passing through two points $A(\vec{a})$ & $B(\vec{b})$ is given by, $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is $\vec{r} = \vec{a} + t\vec{b}$.

Note :

(i) Equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ & $\vec{r} = \vec{a} + \mu\vec{c}$ is, $\vec{r} = \vec{a} + t(\hat{b} + \hat{c})$ & $\vec{r} = \vec{a} + p(\hat{c} - \hat{b})$.

(ii) In a plane, two lines are either intersecting or parallel.

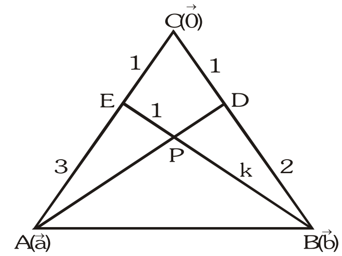
(iii) Two non parallel nor intersecting lines are called **skew lines**.

Illustration 10 : In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods. **(JEE-1993)**

Solution : Let the position vectors of A and B be a and b respectively. Equations of AD and BE are

$\vec{r} = \vec{a} + t(\vec{b}/3 - \vec{a})$ .... (i)

$\vec{r} = \vec{b} + s(\vec{a}/4 - \vec{b})$ .... (ii)



If they intersect at P we must have identical values of r.

Comparing the coefficients of a and b in (i) and (ii), we get

$1 - t = \frac{s}{4},\ \frac{t}{3} = 1 - s$

solving we get $t = \frac{9}{11},\ s = \frac{8}{11}$.

Putting for t or s in (i) or (ii), we get the point P as $\frac{2\vec{a} + 3\vec{b}}{11}$.

Let P divide BE in the ratio k : 1, then P is $\frac{k.\frac{\vec{a}}{4} + \vec{b}}{k+1} = \frac{2\vec{a} + 3\vec{b}}{11}$.

Comparing $\vec{a}$ and $\vec{b}$, we get $11k = 8(k + 1)$ and $11 = 3(k + 1)$ $\therefore$ $k = \frac{8}{3}$

and this satisfies the 2nd relation also. Hence the required ratio is 8 : 3. **Ans.**

Illustration 11 : Find whether the given lines are coplanar or not

$$\vec{r} = \hat{i} - \hat{j} - 10\hat{k} + \lambda(2\hat{i} - 3\hat{j} + 8\hat{k})$$

$$\vec{r} = 4\hat{i} - 3\hat{j} - \hat{k} + \mu(\hat{i} - 4\hat{j} + 7\hat{k})$$

Solution : $L_1 : \vec{r} = (2\lambda + 1)\hat{i} - (1 + 3\lambda)\hat{j} + (8\lambda - 10)\hat{k}$

$L_2 : \vec{r} = (4 + \mu)\hat{i} - (4\mu + 3)\hat{j} + (7\mu - 1)\hat{k}$

The given lines are not parallel. For coplanarity, the lines must intersect.

$\therefore$ $(2\lambda + 1)\hat{i} - (1 + 3\lambda)\hat{j} + (8\lambda - 10)\hat{k} = (4 + \mu)\hat{i} - (4\mu + 3)\hat{j} + (7\mu - 1)\hat{k}$

$2\lambda + 1 = 4 + \mu$ ....(i)

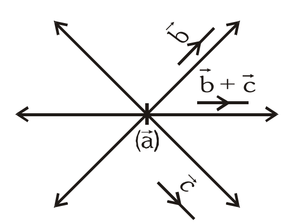
$1 + 3\lambda = 4\mu + 3$ ....(ii)

$8\lambda - 10 = 7\mu - 1$ ....(iii)

Solving (i) & (ii), $\lambda = 2,\ \mu = 1$ and $\lambda = 2,\ \mu = 1$ satisfies equation (iii)

Given lines are intersecting & hence coplanar. **Ans.**

17. TEST OF COLLINEARITY OF THREE POINTS :



(a) 3 points A B C will be collinear if AB = λ BC,

(b) Three points A, B, C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that ; $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$

(c) Collinearly can also be checked by first finding the equation of line through two points and satisfying the third point.

Illustration 12 : Prove that the points with position vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ & $-7\hat{j} + 10\hat{k}$ are collinear.

Solution : If we find, three scalars ℓ, m & n such that $\ell\vec{a}+m\vec{b}+n\vec{c}=0$ where $\ell+m+n=0$ then points are collinear.

$\ell(\hat{i}-2\hat{j}+3\hat{k})+m(2\hat{i}+3\hat{j}-4\hat{k})+n(-7\hat{j}+10\hat{k})=0$

$\Rightarrow \quad (\ell+2m)\hat{i}+(-2\ell+3m-7n)\hat{j}+(3\ell-4m+10n)\hat{k}=0$

$\Rightarrow \quad \ell + 2m = 0,\ -2\ell + 3m - 7n = 0,\ 3\ell - 4m + 10n = 0$

Solving, we get $\ell = 2,\ m = -1,\ n = -1$

since $\ell + m + n = 0$

Hence, the points are collinear.

Aliter :

$\overrightarrow{AB}=\vec{b}-\vec{a}=\left(2\hat{i}+3\hat{j}-4\hat{k}\right)-\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\hat{i}+5\hat{j}-7\hat{k}$

$\overrightarrow{BC}=\vec{c}-\vec{b}=\left(-7\hat{j}+10\hat{k}\right)-\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=-2\hat{i}-10\hat{j}+14\hat{k}=-2\left(\hat{i}+5\hat{j}-7\hat{k}\right)$

$\therefore \quad \overrightarrow{AB}=-2\overrightarrow{BC}$

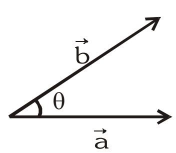
Hence $\vec{a},\vec{b}$ & $\vec{c}$ are collinear.

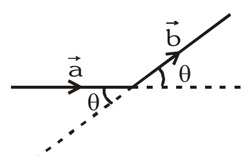
Do yourself - 5 :

(i) The position vectors of the points P, Q, R are $\hat{i}+2\hat{j}+3\hat{k}$, $-2\hat{i}+3\hat{j}+5\hat{k}$ and $7\hat{i}-\hat{k}$ respectively. Prove that P, Q and R are collinear.

18. SCALAR PRODUCT OF TWO VECTORS (DOT PRODUCT) :

Definition : Let $\vec{a}$ and $\vec{b}$ be two non zero vectors inclined at an angle θ. Then the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a}\,.\,\vec{b}$ and is defined as $\vec{a}\,.\,\vec{b} = |\vec{a}\,||\,\vec{b}\,| \cos\theta \quad ; \quad 0 \le \theta \le \pi$.





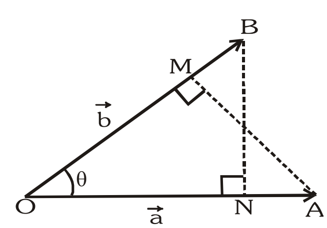
Geometrical Interpretation of Scalar product :

$|\overrightarrow{OA}| = |\vec{a}|,\ |\overrightarrow{OB}| = |\vec{b}|$

Now $\quad \vec{a}\,.\,\vec{b} = |\vec{a}\,||\,\vec{b}\,| \cos\theta$

$= |\vec{a}|\ (OB \cos\theta)$

$=$ (magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$)



Again, $\quad \vec{a}\,.\,\vec{b} = |\vec{a}\,||\,\vec{b}\,| \cos\theta$

$= |\vec{b}|\ (|\vec{a}| \cos\theta)$

$=$ (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$)

(a) $\quad \vec{a}\,.\,\vec{b} = |\vec{a}\,||\,\vec{b}\,| \cos\theta (0 \le \theta \le \pi)$

Note that if θ is acute then $\vec{a}.\vec{b}>0$ & if θ is obtuse then $\vec{a}.\vec{b}<0$

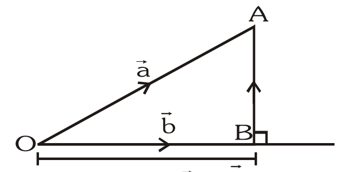
(b) (i) $\vec{a}\,.\,\vec{a}=|\vec{a}|^2=\vec{a}^2$ $\qquad$ (ii) $\vec{a}\,.\,\vec{b}=\vec{b}\,.\,\vec{a}$ (commutative)

(c) $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$ (distributive)

(d) $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$; $(\vec{a}, \vec{b} \neq 0)$

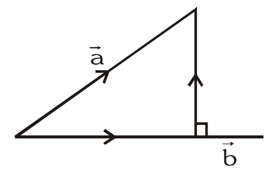
(e) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(f) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. (Provided $|\vec{b}| \neq 0$)



Note :

(i) The vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ [by triangle law of vector Addition]



(ii) The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $0 \le \phi \le \pi$

(iii) If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1+a_2b_2+a_3b_3$

$|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$, $|\vec{b}| = \sqrt{b_1^2+b_2^2+b_3^2}$

(iv) Maximum value of $\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|$

(v) Minimum values of $\vec{a}.\vec{b} = -|\vec{a}|\,|\vec{b}|$

(vi) Any vector $\vec{a}$ can be written as, $\vec{a} = (\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k}$

(g) **Vector equation of angle bisector :**

A vector in the direction of the bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}$. Hence bisector of the angle between the two vectors $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}+\hat{b})$, where $\lambda \in R^+$. Bisector of the exterior angle between $\vec{a}$ & $\vec{b}$ is $\lambda(\hat{a}-\hat{b})$, $\lambda \in R^+$

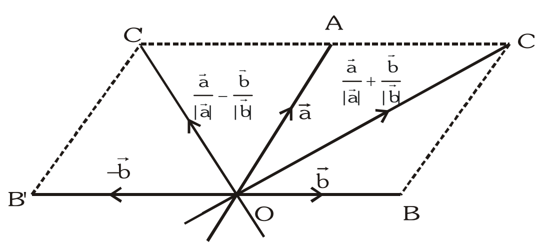


Illustration 13 : $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of four coplanar points A, B, C and D respectively. If $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0 = (\vec{b}-\vec{d}).(\vec{c}-\vec{a})$, then for the ΔABC, D is :-

(A) incentre (B) orthocentre (C) circumcentre (D) centroid

Solution : $(\vec{a}-\vec{d}).(\vec{b}-\vec{c}) = 0 \Rightarrow (\vec{a}-\vec{d}) \perp (\vec{b}-\vec{c}) \Rightarrow \overrightarrow{AD} \perp \overrightarrow{BC}$

Similarly $(\vec{b}-\vec{d}).(\vec{c}-\vec{a}) = 0 \Rightarrow \overrightarrow{BD} \perp \overrightarrow{AC}$

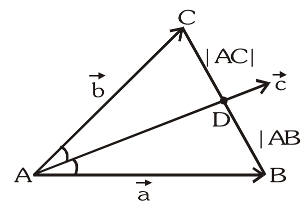
$\therefore$ D is the orthocentre of ΔABC. **Ans. [B]**

Illustration 14 : The vector $\vec{c}$, directed along the internal bisector of the angle between the vector $7\tilde{i} - 4\tilde{j} - 4\tilde{k}$ and $-2\tilde{i} - \tilde{j} + 2\tilde{k}$ with $|\vec{c}| = 5\sqrt{6}$ is -

(A) $\frac{5}{3}(\tilde{i} - 7\tilde{j} + 2\tilde{k})$ (B) $\frac{5}{3}(5\tilde{i} + 5\tilde{j} + 2\tilde{k})$ (C) $\frac{5}{3}(\tilde{i} + 7\tilde{j} + 2\tilde{k})$ (D) none of these

Solution : Let $\vec{a} = 7\tilde{i} - 4\tilde{j} - 4\tilde{k}$

and $\vec{b} = -2\tilde{i} - \tilde{j} + 2\tilde{k}$



internal bisector divides the BC in the ratio of $|\overline{AB}| : |\overline{AC}|$,

$|\overline{AB}| = 9$, $|\overline{AC}| = 3$

$$|\overline{AD}| = \left(\frac{9(-2\tilde{i} - \tilde{j} + 2\tilde{k}) + 3(7\tilde{i} - 4\tilde{j} - 4\tilde{k})}{9+3}\right) = \frac{\tilde{i} - 7\tilde{j} + 2\tilde{k}}{4}$$

$$\vec{c} = \pm\left(\frac{\overrightarrow{AD}}{|\overrightarrow{AD}|}\right)5\sqrt{6} = \pm\frac{5}{3}(\tilde{i} - 7\tilde{j} + 2k)$$ **Ans.[A]**

Illustration 15 : If modulii of vectors $\vec{a}, \vec{b}, \vec{c}$ are 3, 4 and 5 respectively and $\vec{a}$ and $\vec{b} + \vec{c}$, $\vec{b}$ and $\vec{c} + \vec{a}$, $\vec{c}$ and $\vec{a} + \vec{b}$ are perpendicular to each other, then modulus of $\vec{a} + \vec{b} + \vec{c}$ is -

(A) $5\sqrt{2}$ (B) $2\sqrt{5}$ (C) 50 (D) 20

Solution : $\because$ $\vec{a} \perp (\vec{b} + \vec{c}) \Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} = 0$

Similarly $\vec{b} \perp (\vec{c} + \vec{a}) \Rightarrow \vec{b}.\vec{c} + \vec{b}.\vec{a} = 0$

and $\vec{c} \perp (\vec{a} + \vec{b}) \Rightarrow \vec{c}.\vec{a} + \vec{c}.\vec{b} = 0$

$\therefore$ $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = 0$

Now $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 9 + 16 + 25 = 50$

$\therefore$ $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$ **Ans. [A]**

Illustration 16 : If p^{th}, q^{th}, r^{th} terms of a G.P. are the positive numbers a, b, c then angle between the vectors $\log a^2\tilde{i} + \log b^2\tilde{j} + \log c^2\tilde{k}$ and $(q-r)\tilde{i} + (r-p)\tilde{j} + (p-q)\tilde{k}$ is :-

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\sin^{-1}\left(\frac{1}{\sqrt{a^2+b^2+c^2}}\right)$ (D) none of these

Solution : Let x_0 be first term and x the common ratio of the G.P.

$\therefore$ $a = x_0 x^{p-1}$, $b = x_0 x^{q-1}$, $c = x_0 x^{r-1}$ $\Rightarrow$ $\log a = \log x_0 + (p-1)\log x$;

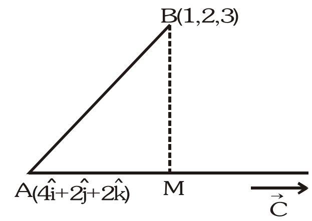
$\log b = \log x_0 + (q-1)\log x$; $\log c = \log x_0 + (r-1)\log x$

If $\vec{a} = \log a^2\hat{i} + \log b^2\hat{j} + \log c^2\hat{k}$ and $\vec{b} = (q-r)\hat{i} + (r-p)\hat{j} + (p-q)\hat{k}$

$$\therefore \quad \vec{a}.\vec{b} = \Sigma 2(\log a)(q-r) = 2\sum(\log x_0 + (p-1)\log x)(q-r) = 0 \Rightarrow \angle\vec{a},\vec{b} = \frac{\pi}{2}.$$ **Ans.**

Illustration 17 : Find the distance of the point $B(\tilde{i} + 2\tilde{j} + 3\tilde{k})$ from the line which is passing through $A(4\tilde{i} + 2\tilde{j} + 2\tilde{k})$ and which is parallel to the vector $\vec{C} = 2\tilde{i} + 3\tilde{j} + 6\tilde{k}$. **(Roorkee 1993)**

Solution : $AB = \sqrt{3^2+1^2} = \sqrt{10}$



$AM = \overrightarrow{AB}.\hat{i} = (-3\hat{i}+\hat{k}).\dfrac{(2\hat{i}+3\hat{j}+6\hat{k})}{7}$

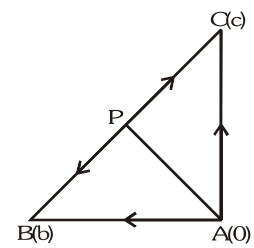
$= -6+6=0$

$BM^2 = AB^2 - AM^2$

So, $BM = AB = \sqrt{10}$ **Ans.**

Illustration 18 : Prove that the medians to the base of an isosceles triangle is perpendicular to the base.

Solution : The triangle being isosceles, we have



$AB = AC$ .... (i)

Now $\overrightarrow{AP} = \dfrac{\vec{b}+\vec{c}}{2}$ where P is mid-point of BC.

Also $\overrightarrow{BC} = \vec{c}-\vec{b}$

$\therefore \quad \overrightarrow{AP}.\overrightarrow{BC} = \dfrac{\vec{b}+\vec{c}}{2}.(\vec{c}-\vec{b}) = \dfrac{1}{2}(c^2-b^2)$

$= \dfrac{1}{2}(AC^2 - AB^2) = 0 \qquad \{\text{by (i)}\}$

$\therefore$ Median AP is perpendicular to base BC.

Do yourself - 6 :

(i) Find the angle between two vectors $\vec{a}$ & $\vec{b}$ with magnitude 2 and 1 respectively and such that $\vec{a}.\vec{b} = \sqrt{3}$.

(ii) Find the value of $(\vec{a}+3\vec{b}).(2\vec{a}-\vec{b})$ if $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}+2\hat{j}-\hat{k}$.

(iii) The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1, find λ.

(iv) Find the projection of the vector $\vec{a} = 4\hat{i}-2\hat{j}+\hat{k}$ on the vector $\vec{b} = 3\hat{i}+6\hat{j}+2\hat{k}$. Also find component of $\vec{a}$ along $\vec{b}$ and perpendicular to $\vec{b}$.

(v) Find the unit vectors along the angle bisectors between the vectors $\vec{a} = \hat{i}+2\hat{j}-2\hat{k}$ and $\vec{b} = -3\hat{i}+6\hat{j}+2\hat{k}$.

19. LINEAR COMBINATIONS :

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}$,.......... then the vector $\vec{r} = x\vec{a}+y\vec{b}+z\vec{c}+$........... is called a **linear combination** of $\vec{a}, \vec{b}, \vec{c}$,........ for any x, y, z........ $\in R$.

FUNDAMENTAL THEOREM IN PLANE :

Let $\vec{a}, \vec{b}$ be non zero, non collinear vectors. then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ i.e. there exist some unique $x, y \in R$ such that $x\vec{a}+y\vec{b} = \vec{r}$.

Illustration 19 : Find a vector $\vec{c}$ in the plane of $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}+\hat{k}$ such that $\vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}.(-2\hat{i}+3\hat{j}-\hat{k}) = -1$

Solution : Any vector in the plane of $\vec{a}$ & $\vec{b}$ can be written as $x\vec{a}+y\vec{b}$

let $\vec{c} = x\vec{a}+y\vec{b}$ [by fundamental theorem in plane]

Now, given that

$\vec{c}.\vec{b}=0 \Rightarrow (x\vec{a}+y\vec{b}).\vec{b}=0$

$x\vec{a}.\vec{b}+y\vec{b}^2=0$

$\Rightarrow \quad x(-2+1-1)+y(3)=0$

$\quad -2x+3y=0 \quad$ ....(i)

Also $\quad (x\vec{a}+y\vec{b}).(-2\tilde{i}+3\tilde{j}-\tilde{k})=-1$

$\Rightarrow \quad x\vec{a}.(-2\tilde{i}+3\tilde{j}-\tilde{k})+y\vec{b}.(-2\tilde{i}+3\tilde{j}-\tilde{k})=-1$

$\Rightarrow \quad x(-4+3+1)+y(2+3-1)=-1$

$y=-\dfrac{1}{4}$

$x=\dfrac{3y}{2}=-\dfrac{3}{8}$

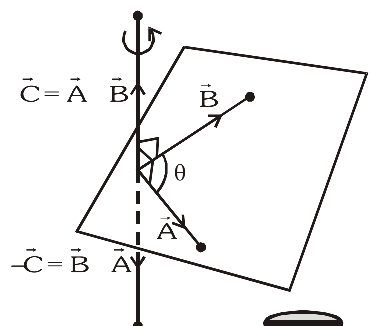
Hence the required vector $\vec{c}=-\dfrac{3}{8}(2\tilde{i}+\tilde{j}-\tilde{k})-\dfrac{1}{4}(-\tilde{i}+\tilde{j}+\tilde{k})$

$=\dfrac{1}{8}[-6\tilde{i}-3\tilde{j}+3\tilde{k}+2\tilde{i}-2\tilde{j}-2\tilde{k}]=\dfrac{1}{8}[-4\tilde{i}-5\tilde{j}+\tilde{k}]$ **Ans.**

Do yourself - 7 :

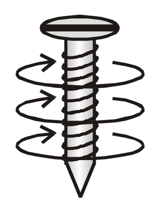
(i) Find a vector $\vec{r}$ in the plane of $\vec{p}=-\tilde{i}+\tilde{j}$ and $\vec{q}=-\tilde{j}+\tilde{k}$ such that $\vec{r}$ is perpendicular to $\vec{p}$ and $\vec{r}.\vec{q}=-2$

20. VECTOR PRODUCT OF TWO VECTORS (CROSS PRODUCT) :



(a) If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them, then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\tilde{n}$, where $\tilde{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\tilde{n}$ forms a right handed screw system.

Sign convention :



Right handed screw system : $\vec{a}$, $\vec{b}$ and $\tilde{n}$ form a right handed system it means that if we rotate vector $\vec{a}$ towards the direction of $\vec{b}$. through the angle θ, then $\tilde{n}$ advances in the same direction as a right handed screw would, if turned in the same way.

(b) **Lagranges Identity** : For any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}.\vec{b})^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b}\\ \vec{a}.\vec{b} & \vec{b}.\vec{b}\end{vmatrix}$

(c) Formulation of vector product in terms of scalar product : The vector product $\vec{a}\times\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}|=\sqrt{\vec{a}^2\vec{b}^2-(\vec{a}.\vec{b})^2}$ (ii) $\vec{c}.\vec{a}=0$; $\vec{c}.\vec{b}=0$ and (iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

(d) (i) $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0,\ \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar

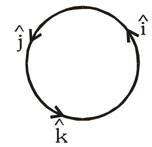
(ii) $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ **(not commutative)**

(iii) $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ where m is a scalar.

(iv) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (distributive over addition)

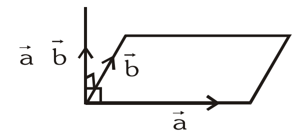
(v) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(vi) $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$



(e) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(f) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.



(g) (i) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(ii) A vector of magnitude 'r' & perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm \dfrac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(iii) If θ is the angle between $\vec{a}$ & $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

(h) Vector area :

(i) If $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC $= \dfrac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$.

(ii) The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = 0$

(iii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$.

Illustration 20 : Find the vectors of magnitude 5 which are perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution : Unit vectors perpendicular to $\vec{a}$ & $\vec{b} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$\because\quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix} = -5\hat{i} - 5\hat{j} - 5\hat{k}$

$\therefore$ Unit vectors $= \pm \dfrac{(-5\hat{i} - 5\hat{j} - 5\hat{k})}{5\sqrt{3}}$

Hence the required vectors are $\pm \dfrac{5\sqrt{3}}{3}(\hat{i} + \hat{j} + \hat{k})$ **Ans.**

Illustration 21 : If $\vec{a}, \vec{b}, \vec{c}$ a re three non zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$, prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$.

Solution : $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{a}=\vec{b}\times\vec{c}$

$\Rightarrow$ $\vec{c}\perp\vec{a},\ \vec{c}\perp\vec{b}$ and $\vec{a}\perp\vec{b},\ \vec{a}\perp\vec{c}$

$\Rightarrow$ $\vec{a}\perp\vec{b},\ \vec{b}\perp\vec{c}$ and $\vec{c}\perp\vec{a}$

$\Rightarrow$ $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors.

Again, $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$

$\Rightarrow$ $|\vec{a}\times\vec{b}|=|\vec{c}|$ and $|\vec{b}\times\vec{c}|=|\vec{a}|$

$\Rightarrow$ $|\vec{a}|\,|\vec{b}|\sin\frac{\pi}{2}=|\vec{c}|$ and $|\vec{b}|\,|\vec{c}|\sin\frac{\pi}{2}=|\vec{a}|$ $\left(\because \vec{a}\perp\vec{b} \text{ and } \vec{b}\perp\vec{c}\right)$

$\Rightarrow$ $|\vec{a}|\,|\vec{b}|=|\vec{c}|$ and $|\vec{b}|\,|\vec{c}|=|\vec{a}|$

$\Rightarrow$ $|\vec{b}|^2\,|\vec{c}|=|\vec{c}|$

$\Rightarrow$ $|\vec{b}|^2=1$

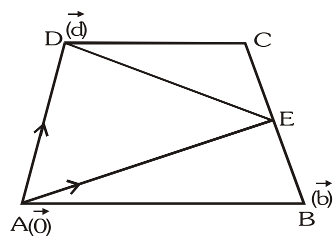
$\Rightarrow$ $|\vec{b}|=1$

putting in $|\vec{a}|\,|\vec{b}|=|\vec{c}|$

$\Rightarrow$ $|\vec{a}|=|\vec{c}|$

Illustration 22 : Show that the area of the triangle formed by joining the extremities of an oblique side of a trapezium to the midpoint of opposite side is half that of the trapezium.

Solution : Let ABCD be the trapezium and E be the midpoint of BC. Let A be the initial point and let $\vec{b}$ be the position vector of B and $\vec{d}$ that of D. Since DC is parallel to AB, $t\vec{b}$ is a vector along DC, so that the position vector of c is $\vec{d}+t\vec{b}$.



$\Rightarrow$ the position vector of E is $\dfrac{\vec{b}+\vec{d}+t\vec{b}}{2}=\dfrac{\vec{d}+(1+t)\vec{b}}{2}$

Area of $\Delta AED=\dfrac{1}{2}\left|\dfrac{\vec{d}+(1+t)\vec{b}}{2}\times\vec{d}\right| = \dfrac{1}{4}(1+t)\,|\vec{b}\times\vec{d}|$

Area of the trapezium = Area (ΔACD) + Area (ΔABC).

$$=\frac{1}{2}|\vec{b}\times(\vec{d}+t\vec{b})|+\frac{1}{2}|(\vec{d}+t\vec{b})\times\vec{d}|$$

$$=\frac{1}{2}|\vec{b}\times\vec{d}|+\frac{t}{2}|\vec{b}\times\vec{d}|=\frac{1}{2}(1+t)|\vec{b}\times\vec{d}| = 2\Delta AED$$

Illustration 23 : Let $\vec{a}$ & $\vec{b}$ be two non-collinear unit vectors. If $\vec{u}=\vec{a}-\left(\vec{a}.\vec{b}\right)\vec{b}$ & $\vec{v}=\left(\vec{a}\times\vec{b}\right)$, then $|\vec{v}|$ is - **[JEE 99]**

(A) $|\vec{u}|$ (B) $|\vec{u}|+|\vec{u}.\vec{a}|$ (C) $|\vec{u}|+|\vec{u}.\vec{b}|$ (D) $\vec{u}+\vec{u}.\left(\vec{a}+\vec{b}\right)$

Solution : $\vec{u}.\vec{a}=\vec{a}.\vec{a}-(\vec{a}.\vec{b})(\vec{a}.\vec{b})$

$=1-|\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$=1-\cos^2\theta=\sin^2\theta$

$|\vec{v}|=|\vec{a}\times\vec{b}|=\sin\theta$

$|\vec{u}|=\sqrt{\vec{u}.\vec{u}}$

$=\sqrt{\vec{a}.\vec{a}-2(\vec{a}.\vec{b})^2+(\vec{a}.\vec{b})^2|\vec{b}|^2} = \sqrt{1-(\vec{a}.\vec{b})^2} = \sin\theta$

$\therefore$ $|\vec{v}|=|\vec{u}|$ also $\vec{u}.\vec{b}=0$

Hence, $|\vec{v}|=|\vec{u}|=|\vec{u}|+|\vec{u}.\vec{b}|$ **Ans. (A, C)**

Do yourself - 8 :

(i) If $\vec{a} \quad \vec{b} = \vec{c} \quad \vec{d}$ and $\vec{a} \quad \vec{c} = \vec{b} \quad \vec{d}$, then show that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$ when $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

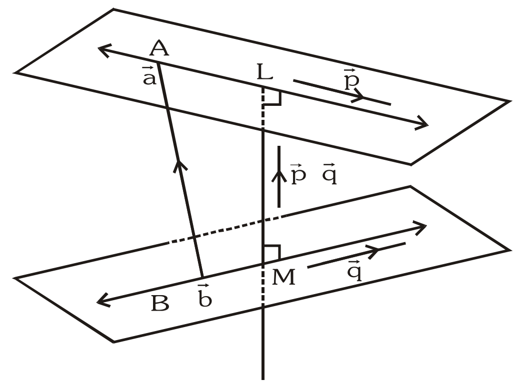
(ii) Find $\vec{a} \quad \vec{b}$, if $\vec{a} = 2\tilde{i} + \tilde{k}$ and $\vec{b} = \tilde{i} + \tilde{j} + \tilde{k}$.

(iii) For any two vectors $\vec{u}$ & $\vec{v}$, prove that [JEE 98]

(a) $(\vec{u}.\vec{v})^2 + |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2$ (b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2) = (1-\vec{u}.\vec{v})^2 + |\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

21. SHORTEST DISTANCE BETWEEN TWO LINES :

If two lines in space intersect at a point, then obviously the shortest distance between them is zero. Lines which do not intersect & are also not parallel are called **skew lines**. **In other words the lines which are not coplanar are skew lines**. For Skew lines the direction of the shortest distance vector would be perpendicular to both the lines. The magnitude of the shortest distance vector would be equal to that of the projection of $\overrightarrow{AB}$ along the direction of the line of shortest distance, $\overrightarrow{LM}$ is parallel to $\vec{p} \times \vec{q}$



i.e. $\overrightarrow{LM}$ = | Projection of $\overrightarrow{AB}$ on $\overrightarrow{LM}$ | = | Projection of $\overrightarrow{AB}$ on $\vec{p} \times \vec{q}$ | $= \left|\dfrac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \left|\dfrac{(\vec{b}-\vec{a}).(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$

(a) **The two lines directed along $\vec{p}$ & $\vec{q}$ will intersect only if shortest distance = 0 i.e.**

$(\vec{b}-\vec{a}).(\vec{p}\times\vec{q}) = 0$ i.e. $(\vec{b}-\vec{a})$ lies in the plane containing $\vec{p}$ & $\vec{q} \Rightarrow \left[(\vec{b}-\vec{a}) \;\; \vec{p} \;\; \vec{q}\right] = 0$

(b) If two lines are given by $\vec{r}_1 = \vec{a}_1 + K_1\vec{b}$ & $\vec{r}_2 = \vec{a}_2 + K_2\vec{b}$ i.e. they are parallel then, $d = \left|\dfrac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$

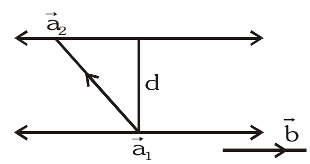


Illustration 24 : Find the shortest distance between the lines

$\vec{r} = (4\tilde{i} - \tilde{j}) + \lambda(\tilde{i} + 2\tilde{j} - 3\tilde{k})$ and $\vec{r} = (\tilde{i} - \tilde{j} + 2k) + \mu(2\tilde{i} + 4\tilde{j} - 5\tilde{k})$

Solution : We known, the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

Comparing the given equation with the equations $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $r = \vec{a}_2 + \lambda\vec{b}_2$ respectively,

we have $\vec{a}_1 = 4\tilde{i} - \tilde{j}, \vec{a}_2 = \tilde{i} - \tilde{j} + 2\tilde{k}$, $\vec{b}_1 = \tilde{i} + 2\tilde{j} - 3\tilde{k}$ and $\vec{b}_2 = 2\tilde{i} + 4\tilde{j} - 5\tilde{k}$

Now, $\vec{a}_2 - \vec{a}_1 = -3\tilde{i} + 0\tilde{j} + 2\tilde{k}$ and $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\tilde{i} - \tilde{j} + 0\tilde{k}$

$\therefore$ $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2) = (-3\tilde{i}+0\tilde{j}+2\tilde{k}).(2\tilde{i}-\tilde{j}+0\tilde{k}) = -6$ and $|\vec{b}_1\times\vec{b}_2| = \sqrt{4+1+0} = \sqrt{5}$

$\therefore$ Shortest distance $d = \left|\dfrac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right| = \left|\dfrac{-6}{\sqrt{5}}\right| = \dfrac{6}{\sqrt{5}}$. **Ans.**

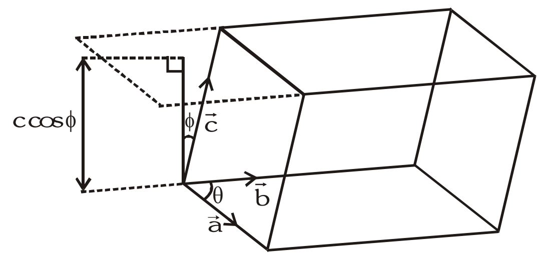
Do yourself - 9 :

(i) Find the shortest distance between the lines :

$\vec{r}_1 = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ & $\vec{r}_2 = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$.

22. SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

(i) The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as : $(\vec{a} \times \vec{b}).\vec{c} = |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a} \times \vec{b}$ & $\vec{c}$. It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product.



(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\,\vec{b}\,\vec{c}]$

(iii) In a scalar triple product the position of dot & cross can be interchanged i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a}.(\vec{b} \times \vec{c}) = -a.(\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general , if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$; & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\,\vec{m}\,\vec{n}]$; where $\vec{l}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors.

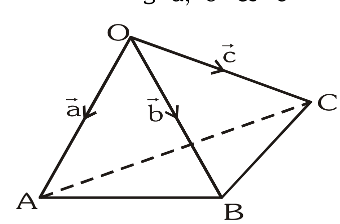
(v) If $\vec{a}, \vec{b}, \vec{c}$ **are coplanar** $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ **are linearly dependent.**

(vi) Scalar product of three vectors, two of which are equal or parallel is 0 i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Note : If $\vec{a}, \vec{b}, \vec{c}$ are non- coplanar then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system & $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(vii) $[i\,j\,k] = 1$ (viii) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$ (ix) $[(\vec{a} + \vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(viii) The Volume of the tetrahedron OABC with O as origin & the pv's of A, B and C being $\vec{a}, \vec{b}$ & $\vec{c}$ are given by $V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$



The position vector of the centroid of a tetrahedron if the pv's of its angular vertices are $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$

Note that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

(ix) $[\vec{a} - \vec{b}\ \ \vec{b} - \vec{c}\ \ \vec{c} - \vec{a}] = 0$ & $[\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

Illustration 25 : For any three vectors $\vec a, \vec b, \vec c$ prove that $[\vec a+\vec b\ \ \vec b+\vec c\ \ \vec c+\vec a] = 2[\vec a\,\vec b\,\vec c]$

Solution : We have $[\vec a+\vec b\ \ \vec b+\vec c\ \ \vec c+\vec a]$

$= \{(\vec a+\vec b)\times(\vec b+\vec c)\}.(\vec c+\vec a) = \{\vec a\times\vec b+\vec a\times\vec c+\vec b\times\vec b+\vec b\times\vec c\}.(\vec c+\vec a)$ $\{\because \vec b\times\vec b = 0\}$

$= \{\vec a\times\vec b+\vec a\times\vec c+\vec b\times\vec c\}.(\vec c+\vec a) = (\vec a\times\vec b).\vec c+(\vec a\times\vec c).\vec c+(\vec b\times\vec c).\vec c+(\vec a\times\vec b).\vec a+(\vec a\times\vec c).\vec a+(\vec b\times\vec c).\vec a$

$= [\vec a\,\vec b\,\vec c] + 0 + 0 + 0 + 0 + [\vec b\,\vec c\,\vec a]$ $\{\because [\vec a\,\vec c\,\vec c]=0, [\vec b\,\vec c\,\vec c]=0, [\vec a\,\vec b\,\vec a]=0, [\vec a\,\vec c\,\vec a]=0\}$

$= [\vec a\,\vec b\,\vec c]+[\vec a\,\vec b\,\vec c] = 2[\vec a\,\vec b\,\vec c]$. **Ans.**

Illustration 26 : If $\vec a, \vec b$ are non-zero and non-collinear vectors then show

$\vec a\times\vec b = [\vec a\,\vec b\,\hat i]\hat i + [\vec a\,\vec b\,\hat j]\hat j + [\vec a\,\vec b\,\hat k]\hat k$

Solution : Let $\vec a\times\vec b = x\hat i + y\hat j + z\hat k$

$(\vec a\times\vec b).\hat i = (x\hat i+y\hat j+z\hat k).\hat i$

$(\vec a\times\vec b).\hat i = x$

also $(\vec a\times\vec b).\hat j = y$ & $(\vec a\times\vec b).\hat k = z$

$\therefore$ $\vec a\times\vec b = [\vec a\,\vec b\,\hat i]\hat i + [\vec a\,\vec b\,\hat j]\hat j + [\vec a\,\vec b\,\hat k]\hat k$ **Ans.**

Do yourself - 10 :

(i) If $\vec a$, $\vec b$, $\vec c$ are three non coplanar mutually perpendicular unit vectors then find $[\vec a, \vec b, \vec c]$.

(ii) If $\vec r$ be a vector perpendicular to $\vec a+\vec b+\vec c$, where $[\vec a\ \ \vec b\ \ \vec c] = z$ and $\vec r = \ell(\vec b\ \ \vec c) + m(\vec c\ \ \vec a) + n(\vec a\ \ \vec b)$, then find $l + m + n$.

(iii) Find the volume of the parallelepiped whose coterminous edges are represented by $\vec a = 2\hat i - 3\hat j + 4\hat k$, $\vec b = \hat i + 2\hat j - \hat k$ and $\vec c = 3\hat i - \hat j + 2\hat k$

(iv) Examine whether the vectors $\vec a = 2\hat i + 3\hat j + 2\hat k$, $\vec b = \hat i - \hat j + 2\hat k$ and $\vec c = 3\hat i + 2\hat j - 4\hat k$ form a left handed or right handed system.

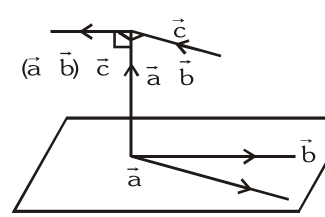
(v) For three vectors $\vec u, \vec v, \vec w$ which of the following expressions is not equal to any of the remaining three ? [JEE 98]

(A) $\vec u.(\vec v\times\vec w)$ (B) $(\vec v\times\vec w).\vec u$ (C) $\vec v.(\vec u\times\vec w)$ (D) $(\vec u\times\vec v).\vec w$

23. VECTOR TRIPLE PRODUCT :

Let $\vec a$, $\vec b$ & $\vec c$ be any three vectors, then the expression $\vec a\times(\vec b\times\vec c)$ is a vector & is called a **vector triple product**.

Geometrical interpretation of $\vec a$ ($\vec b$ $\vec c$)



Consider the expression $\vec a\times(\vec b\times\vec c)$ which itself is a vector, since it is a cross product of two vectors $\vec a$ & $(\vec b\times\vec c)$. Now $\vec a\times(\vec b\times\vec c)$ is vector perpendicular to the plane containing $\vec a$ & $(\vec b\times\vec c)$ but $(\vec b\times\vec c)$ is a vector perpendicular to the plane $\vec b$ & $\vec c$, therefore $\vec a\times(\vec b\times\vec c)$ is vector lies in the plane of $\vec b$ & $\vec c$ and perpendicular to $\vec a$. Hence we can express $\vec a\times(\vec b\times\vec c)$ in terms of $\vec b$ & $\vec c$ i.e. $\vec a\times(\vec b\times\vec c) = x\vec b + y\vec c$ where x & y are scalars.

(a) $\vec a\times(\vec b\times\vec c) = (\vec a.\vec c)\vec b - (\vec a.\vec b)\vec c$ (b) $(\vec a\times\vec b)\times\vec c = (\vec a.\vec c)\vec b - (\vec b.\vec c)\vec a$

(c) $(\vec a\times\vec b)\times\vec c \neq \vec a\times(\vec b\times\vec c)$

Illustration 27 : Prove that $\left[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\right]=\left[\vec{a}\vec{b}\vec{c}\right]^2$

Solution : We have, $\left[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\right]=\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}.\left(\vec{c}\times\vec{a}\right)$

$=\left\{\vec{d}\times\left(\vec{b}\times\vec{c}\right)\right\}.\left(\vec{c}\times\vec{a}\right)$ (where, $\vec{d}=(\vec{a}\times\vec{b})$)

$=\left\{\left(\vec{d}.\vec{c}\right)\vec{b}-\left(\vec{d}.\vec{b}\right)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)=\left\{\left(\left(\vec{a}\times\vec{b}\right).\vec{c}\right)\vec{b}-\left(\left(\vec{a}\times\vec{b}\right).\vec{b}\right)\vec{c}\right\}.\left(\vec{c}\times\vec{a}\right)$

$=\left\{\left[\vec{a}\vec{b}\vec{c}\right]\vec{b}-0\right\}.\left(\vec{c}\times\vec{a}\right)$ ($\because$ $[\vec{a}\vec{b}\vec{b}]=0$)

$=\left[\vec{a}\vec{b}\vec{c}\right]\left\{\vec{b}.\left(\vec{c}\times\vec{a}\right)\right\}=\left[\vec{a}\vec{b}\vec{c}\right]\left[\vec{b}\vec{c}\vec{a}\right]=\left[\vec{a}\vec{b}\vec{c}\right]\left[\vec{a}\vec{b}\vec{c}\right]=\left[\vec{a}\vec{b}\vec{c}\right]^2$

Illustration 28 : Show that $(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=0$

Solution : Let $\vec{b}\times\vec{c}=\vec{u},\vec{c}\times\vec{a}=\vec{v},\vec{c}\times\vec{d}=\vec{w}$

$\therefore$ L.H.S $=\vec{u}.(\vec{a}\times\vec{d})+\vec{v}.(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b}).\vec{w}=(\vec{u}\times\vec{a}).\vec{d}+(\vec{v}\times\vec{b}).\vec{d}+\vec{a}.(\vec{b}\times\vec{w})$

$=[(\vec{b}\times\vec{c})\times\vec{a}].\vec{d}+[(\vec{c}\times\vec{a})\times\vec{b}].\vec{d}+\vec{a}.[\vec{b}\times(\vec{c}\times\vec{d})]$

$=[(\vec{b}.\vec{a})\vec{c}-(\vec{c}.\vec{a})\vec{b}].\vec{d}+[(\vec{c}.\vec{b})\vec{a}-(\vec{a}.\vec{b})\vec{c}].\vec{d}+\vec{a}.[(\vec{b}.\vec{d})\vec{c}-(\vec{b}.\vec{c})\vec{d}]$

$=\{(\vec{a}.\vec{b})(\vec{c}.\vec{d})\}-\{(\vec{c}.\vec{a})(\vec{b}.\vec{d})\}+\{(\vec{c}.\vec{b})(\vec{a}.\vec{d})\}-\{(\vec{a}.\vec{b})(\vec{c}.\vec{d})\}+\{(\vec{a}.\vec{c})(\vec{b}.\vec{d})\}-\{(\vec{a}.\vec{d})(\vec{b}.\vec{c})\}=0=$ R.H.S.

Do yourself - 11 :

(i) If $\vec{a}=2\hat{i}-4\hat{j}+7\hat{k}$, $\vec{b}=3\hat{i}+5\hat{j}-9\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ then find $[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]$ and also $\vec{a}\ \ (\vec{b}\ \ \ \vec{c}\,)$.

24. LINEAR INDEPENDENCE AND DEPENDENCE OF VECTORS :

(a) If $\vec{x}_1,\ \vec{x}_2,\dots\dots\vec{x}_n$ are n non zero vectors, & $k_1,\ k_2,\dots\dots k_n$ are n scalars & if the linear combination $k_1\vec{x}_1+k_2\vec{x}_2+\dots k_n\vec{x}_n=0\Rightarrow k_1=0,\ k_2=0\dots k_n=0$ then we say that vectors $\vec{x}_1,\ \vec{x}_2,\dots\vec{x}_n$ are **linearly independent vectors.**

(b) If $\vec{x}_1,\ \vec{x}_2,\dots\dots\vec{x}_n$ are not **linearly independent** then they are said to be **linearly dependent vectors**. i.e. if $k_1\vec{x}_1+k_2\vec{x}_2+\dots\dots k_n\vec{x}_n=0$ & if there exists at least one $k_r\neq 0$ then $\vec{x}_1,\ \vec{x}_2,\dots\vec{x}_n$ are said to be **linearly dependent.**

FUNDAMENTAL THEOREM IN SPACE :

Let $\vec{a},\ \vec{b},\ \vec{c}$ be non-zero, non-coplanar vectors in space. Then any vector $\vec{r}$, can be uniquely expressed as a linear combination of $\vec{a},\ \vec{b},\ \vec{c}$ i.e. There exist some unique $x,\ y,\ z\in R$ such that $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$.

Note :

(i) If $\vec{a}=3\hat{i}+2\hat{j}+5\hat{k}$ then $\vec{a}$ is expressed as a linear combination of vectors $\hat{i},\ \hat{j},\ \hat{k}$. Also, $\vec{a},\hat{i},\ \hat{j},\ \hat{k}$ form a linearly dependent set of vectors. In general, every set of four vectors is a linearly dependent system.

(ii) If $\vec{a},\vec{b},\vec{c}$ are three non-zero, non-coplanar vectors then $x\vec{a}+y\vec{b}+z\vec{c}=0\ \Rightarrow\ x=y=z=0$

(iii) $\hat{i},\ \hat{j},\ \hat{k}$ are linearly independent set of vectors. For $K_1\hat{i}+K_2\hat{j}+K_3\hat{k}=0\Rightarrow K_1=0=K_2=K_3$

(iv) Two vectors $\vec{a}$ & $\vec{b}$ are **linearly dependent** $\Rightarrow\vec{a}$ is a parallel to $\vec{b}$ i.e. $\vec{a}\times\vec{b}=0\Rightarrow$ linear dependence of $\vec{a}$ & $\vec{b}$. Conversely if $\vec{a}\times\vec{b}\neq 0$ then $\vec{a}$ & $\vec{b}$ are **linearly independent.**

(v) If three vectors $\vec{a},\ \vec{b},\ \vec{c}$ are **linearly dependent** , then they are **coplanar** i.e. $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0$ conversely, if $[\vec{a}\ \ \vec{b}\ \ \vec{c}]\neq 0$, then the vectors are linearly independent.

Illustration 29 : Show that points with position vectors $\vec{a}-2\vec{b}+3\vec{c}, -2\vec{a}+3\vec{b}-\vec{c}, 4\vec{a}-7\vec{b}+7\vec{c}$ are collinear. It is given that vectors $\vec{a},\vec{b},\vec{c}$ are non-coplanar.

Solution : The three points are collinear, if we can find λ_1, λ_2 and λ_3, such that

$\lambda_1(\vec{a}-2\vec{b}+3\vec{c})+\lambda_2(-2\vec{a}+3\vec{b}-\vec{c})+\lambda_3(4\vec{a}-7\vec{b}+7\vec{c})=0$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$

equating the coefficients $\vec{a},\vec{b}$ and $\vec{c}$ separately to zero, we get

$\lambda_1 - 2\lambda_2 + 4\lambda_3 = 0$, $-2\lambda_1 + 3\lambda_2 - 7\lambda_3 = 0$ and $3\lambda_1 - \lambda_2 + 7\lambda_3 = 0$

on solving we get, $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 1$

So that $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Hence the given vectors are collinear.

Illustration 30 : If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$ then -

(A) $\alpha = 1$, $\beta = -1$ (B) $\alpha = 1$, $\beta = \pm 1$ (C) $\alpha = -1$, $\beta = \pm 1$ (D) $\alpha = \pm 1$, $\beta = 1$

Solution : If $\vec{a},\vec{b},\vec{c}$ are linearly dependent vectors, then $\vec{c}$ should be a linear combination of $\vec{a}$ and $\vec{b}$.

Let $\vec{c}=p\vec{a}+q\vec{b}$

i.e. $\hat{i}+\alpha\hat{j}+\beta\hat{k}=p(\hat{i}+\hat{j}+\hat{k})+q(4\hat{i}+3\hat{j}+4\hat{k})$

Equating coefficients of $\hat{i},\hat{j},\hat{k}$ we get $1 = p + 4q$, $\alpha = p + 3q$, $\beta = p + 4q$

from first and third, $\beta = 1$.

Now $|\vec{c}| = \sqrt{3}$

$\therefore$ $1 + \alpha^2 + \beta^2 = 3$

$\Rightarrow$ $1 + \alpha^2 + 1 = 3$ {Using $\beta = 1$}

$\Rightarrow$ $\alpha = \pm 1$

Hence, $\alpha = \pm 1$, $\beta = 1$. **Ans. (D)**

25. COPLANARITY OF FOUR POINTS :

Four points A, B, C, D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}+w\vec{d}=0$

where, $x + y + z + w = 0$

26. RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}\cdot\vec{a}'=\vec{b}\cdot\vec{b}'=\vec{c}\cdot\vec{c}'=1$, then the two systems are called Reciprocal System of vectors.

Note : $a' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $b' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$; $c' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

27. PROPERTIES OF RECIPROCAL SYSTEM OF VECTORS :

(a) $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=\vec{b}\cdot\vec{c}'=\vec{c}\cdot\vec{a}'=\vec{c}\cdot\vec{b}'=0$

(b) The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ formed by three non-coplanar vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is the reciprocal of the scalar triple product formed from reciprocal system.

Illustration 31 : Find a set of vectors reciprocal to the vectors $\vec{a},\vec{b}$ and $\vec{a}\times\vec{b}$.

Solution : Let the given vectors be denoted by $\vec{a},\vec{b}$ and $\vec{c}$ where $\vec{c}=\vec{a}\times\vec{b}$

$\therefore$ $[\vec{a}\,\vec{b}\,\vec{c}]=(\vec{a}\times\vec{b}).\vec{c}=(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})=(\vec{a}\times\vec{b})^2$

and let the reciprocal system of vectors be $\vec{a}'\vec{b}'$and$\vec{c}'$

$\therefore$ $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\dfrac{\vec{b}\times(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|^2}$; $\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\dfrac{(\vec{a}\times\vec{b})\times\vec{a}}{|\vec{a}\times\vec{b}|^2}$; $\vec{c}'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$

$\therefore$ $\vec{a}',\vec{b}',\vec{c}'$ are required reciprocal system of vectors for $\vec{a},\vec{b}$ and $\vec{a}\times\vec{b}$. **Ans.**

Illustration 32 : If $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\vec{c}'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ then shown that ; $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=0$

where $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors.

Solution : Here $\vec{a}\times\vec{a}'=\dfrac{\vec{a}\times(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\vec{a}\times\vec{a}'=\dfrac{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Similarly $\vec{b}\times\vec{b}'=\dfrac{(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ & $\vec{c}\times\vec{c}'=\dfrac{(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=\dfrac{(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}+(\vec{b}.\vec{a})\vec{c}-(\vec{b}.\vec{c})\vec{a}+(\vec{c}.\vec{b})\vec{a}-(\vec{c}.\vec{a})\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ $[\because \vec{a}.\vec{b}=\vec{b}.\vec{a}$ etc.]

$= 0.$ **Ans.**

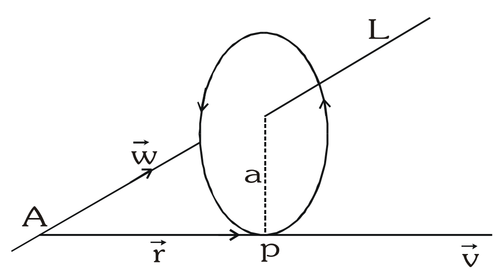
Do yourself - 12 :

(i) If $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{b}\ \vec{c}\ \vec{a}]}$, $\vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]}$, $\vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, then find the value of $(\vec{a}+\vec{b}).\vec{p}+(\vec{b}+\vec{c}).\vec{q}+(\vec{c}+\vec{a}).\vec{r}$.

(ii) Find the set of vectors reciprocal to the set $2\hat{i}+3\hat{j}-\hat{k},\ \hat{i}-\hat{j}-2\hat{k}$ and $-\hat{i}+2\hat{j}+2\hat{k}$. Hence prove that $\vec{a}.\vec{a}'+\vec{b}.\vec{b}'+\vec{c}.\vec{c}'=3$.

(iii) If $\vec{a},\vec{b}$ and $\vec{c}$ are non zero, non coplanar vectors determine whether the vectors $\vec{r}_1=2\vec{a}-3\vec{b}+\vec{c}$, $\vec{r}_2=3\vec{a}-5\vec{b}+2\vec{c}$ and $\vec{r}_3=4\vec{a}-5\vec{b}+\vec{c}$ are linearly independent or dependent.

28. APPLICATION OF VECTORS :

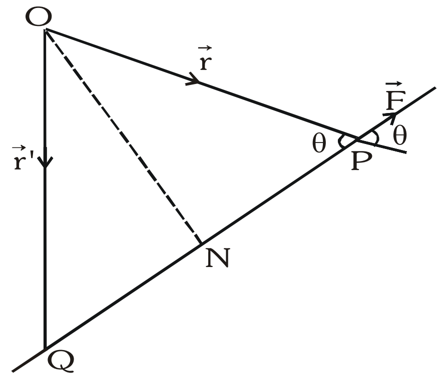


(a) Work done against a constant force $\vec{F}$ over a displacement $\vec{s}$ is defined as $W=\vec{F}.\vec{s}$

(b) The tangential velocity $\vec{V}$ of a body moving in a circle is given by $\vec{V}=\vec{w}\times\vec{r}$ where $\vec{r}$ is the pv of the point P.

(c) The moment of $\vec{F}$ about 'O' is defined as $\vec{M} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the pv of P wrt 'O'. The direction of $\vec{M}$ is along to the normal to the plane OPN such that $\vec{r}$, $\vec{F}$ & $\vec{M}$ form a right handed system.

(d) Moment of couple = $(\vec{r}_1 - \vec{r}_2) \times \vec{F}$ where $\vec{r}_1$ & $\vec{r}_2$ are pv's of the point of the application of the forces $\vec{F}$ & $-\vec{F}$.



Do yourself - 13 :

(i) Force $\vec{F} = 4\hat{i} + \hat{j} - 3\hat{k}$ acting on a particle displaces it from the point $\hat{i} + 2\hat{j} - 3\hat{k}$ to the point $5\hat{i} + 3\hat{j} - 4\hat{k}$. Find the work done by the force $\vec{F}$.

Miscellaneous Illustrations :

Illustration 33 : Forces of magnitudes 5, 4, 3 units act on a particle in the directions $2\hat{i} - 2\hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + 2\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$ respectively, and the particle gets displaced from the point A whose position vector is $6\hat{i} + 2\hat{j} + 3\hat{k}$, to the point B whose position vector is $9\hat{i} + 7\hat{j} + 5\hat{k}$. Find the work done.

Solution : If the forces are $\vec{F}_1, \vec{F}_2, \vec{F}_3$ then $\vec{F}_1 = \frac{5}{3}(2\hat{i} - 2\hat{j} + \hat{k})$; $\vec{F}_2 = \frac{4}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{F}_3 = \frac{3}{3}(-2\hat{i} + \hat{j} - 2\hat{k})$

and hence the sum force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{1}{3}(8\hat{i} + \hat{j} + 7\hat{k})$

Displacement vector $\overline{AB} = \overline{OB} - \overline{OA} = 9\hat{i} + 7\hat{j} + 5\hat{k} - (6\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 5\hat{j} + 2\hat{k}$

Work done = $\frac{1}{3}(8\hat{i} + \hat{j} + 7\hat{k}) . (3\hat{i} + 5\hat{j} + 2\hat{k}) = \frac{1}{3}(24 + 5 + 14) = \frac{43}{3}$ units. **Ans.**

Illustration 34 : Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \quad \vec{u}) = \vec{v}$, then prove that $|(\vec{u} \quad \vec{v}) . \vec{w}| \le \frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.

Solution :

$\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$ ......(i)

$\Rightarrow$ $\vec{w} \times \vec{u} = \vec{v} - \vec{w}$ $\Rightarrow$ $(\vec{w} \times \vec{u})^2 = v^2 + w^2 - 2\vec{v}.\vec{w}$

$\Rightarrow$ $2\vec{v}.\vec{w} = 1 + w^2 - (\vec{u} \times \vec{w})^2$ ......(ii)

also taking dot product of (i) with $\vec{v}$, we get

$\vec{w}.\vec{v} + (\vec{w} \times \vec{u}).\vec{v} = \vec{v}.\vec{v}$

$\Rightarrow$ $\vec{v}.(\vec{w} \times \vec{u}) = 1 - \vec{w}.\vec{v}$ ....(iii) $\{\because \vec{v}.\vec{v} = |\vec{v}|^2 = 1\}$

Now ; $\vec{v}.(\vec{w} \times \vec{u}) = 1 - \frac{1}{2}(1 + w^2 - (\vec{u} \times \vec{w})^2)$ (using (ii) and (iii))

$= \frac{1}{2} - \frac{w^2}{2} + \frac{(\vec{u} \times \vec{w})^2}{2}$ $(\because 0 \le \cos^2\theta \le 1)$

$= \frac{1}{2}(1 - w^2 + w^2 \sin^2 \theta)$ ......(iv) as we know ; $0 \le w^2 \cos^2\theta \le w^2$

$\therefore \quad \frac{1}{2} \ge \frac{1-w^2\cos^2\theta}{2} \ge \frac{1-w^2}{2}$

$\Rightarrow \quad \frac{1-w^2\cos^2\theta}{2} \le \frac{1}{2}$ ......(v)

from (iv) and (v)

$|\vec{v}.(\vec{w}\times\vec{u})| \le \frac{1}{2}$

Equality holds only when $\cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

i.e., $\vec{u} \perp \vec{w} \Rightarrow \vec{u}.\vec{w} = 0 \Rightarrow \vec{w} + (\vec{w}\times\vec{u}) = \vec{v}$

$\Rightarrow \quad \vec{u}\ \vec{w} + \vec{u}.(\vec{w}\times\vec{u}) = \vec{u}.\vec{v}$ (taking dot with $\vec{u}$)

$\Rightarrow \quad 0 + 0 = \vec{u}\ \vec{v} \Rightarrow \vec{u}\ \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$

Illustration 35 : A point $A(x_1, y_1)$ with abscissa $x_1 = 1$ and a point $B(x_2, y_2)$ with ordinate $y_2 = 11$ are given in a rectangular cartesian system of co-ordinates OXY on the part of the curve $y = x^2 - 2x + 3$ which lies in the first quadrant. Find the scalar product of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Solution : Since (x_1, y_1) and (x_2, y_2) lies on $y = x^2 - 2x + 3$.

$\therefore \quad y_1 = x_1^2 - 2x_1 + 3$

$y_1 = 1^2 - 2(1) + 3$ (as ; $x_1 = 1$)

$y_1 = 2$

so the co-ordinates of A(1, 2)

Also, $y_2 = x_2^2 - 2x_2 + 3$

$11 = x_2^2 - 2x_2 + 3 \Rightarrow x_2 = 4,\ x_2 \ne -2$ (as B lie in 1st quadrant)

$\therefore$ co-ordinates of B (4, 11).

Hence, $\overrightarrow{OA} = \tilde{i} + 2\tilde{j}$ and $\overrightarrow{OB} = 4\tilde{i} + 11\tilde{j}$

$\Rightarrow \quad \overrightarrow{OA}\cdot\overrightarrow{OB} = 4 + 22 = 26.$

Illustration 36 : If 'a' is real constant and A, B, C are variable angles and $\sqrt{a^2-4}\tan A + a\tan B + \sqrt{a^2+4}\tan C = 6a$ then find the least value of $\tan^2 A + \tan^2 B + \tan^2 C$

Solution : The given relation can be re-written as ;

$(\sqrt{a^2-4}\,\tilde{i} + a\tilde{j} + \sqrt{a^2+4}\,\tilde{k})\cdot(\tan A\tilde{i} + \tan B\tilde{j} + \tan C\tilde{k}) = 6a$

$\Rightarrow \quad \sqrt{(a^2-4)+a^2+(a^2+4)}.\ \sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\ .\cos\theta = 6a$

(as, a.b = |a| |b| cos θ)

$\Rightarrow \quad \sqrt{3}\,a\ .\ \sqrt{\tan^2 A + \tan^2 B + \tan^2 C}\cos\theta = 6a$

$\Rightarrow \quad \tan^2 A + \tan^2 B + \tan^2 C = 12\sec^2\theta$ ......(i)

also, $12\sec^2\theta \ge 12$ (as, $\sec^2\theta \ge 1$) ......(ii)

from (i) and (ii),

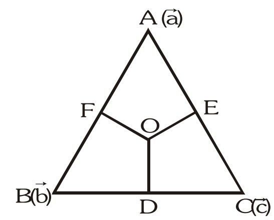
$\tan^2 A + \tan^2 B + \tan^2 C \ge 12$

$\therefore$ least value of $\tan^2 A + \tan^2 B + \tan^2 C = 12.$

Illustration 37 : Prove that the right bisectors of the sides of a triangle are concurrent.

Solution : Let the right bisectors of sides BC and CA meet at O and taking O as origin, let the position vectors of A, B and C be taken as $\vec{a}, \vec{b}, \vec{c}$ respectively. Hence the mid-points D, E, F are

$\frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{a}}{2}, \frac{\vec{a}+\vec{b}}{2}$



$\because \quad \overline{OD} \perp \overline{BC}, \qquad \therefore \quad \frac{\vec{b}+\vec{c}}{2}.(\vec{c}-\vec{b}) = 0$

i.e. $b^2 = c^2$

Again since $\overline{OE} \perp \overline{CA}, \qquad \therefore \quad \frac{\vec{c}+\vec{a}}{2}.(\vec{a}-\vec{c}) = 0$

or $\quad a^2 = c^2 \qquad \therefore \quad a^2 = b^2 = c^2 \qquad$ .... (i)

Now we have to prove that $\overline{OF}$ is also $\perp$ to $\overline{AB}$ which will be true if $\frac{\vec{a}+\vec{b}}{2}.(\vec{b}-\vec{a}) = 0$

i.e. $b^2 = a^2$

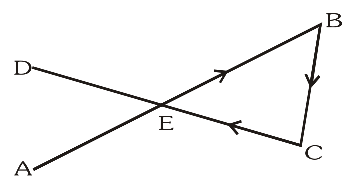
which is true by (i)

Illustration 38 : A, B, C and D are four points such that $\overrightarrow{AB} = m(2\tilde{i}-6\tilde{j}+2\tilde{k}), \overrightarrow{BC} = (\tilde{i}-2\tilde{j})$ and $\overrightarrow{CD} = n(-6\tilde{i}+15\tilde{j}-3\tilde{k})$.

Find the conditions on the scalars m and n so that $\overrightarrow{CD}$ intersects $\overrightarrow{AB}$ at some point E. Also find the area of the triangle BCE. **(Roorkee 1995)**

Solution : $\overrightarrow{AB} = m(2\tilde{i}-6\tilde{j}+2\tilde{k}), \overrightarrow{BC} = (\tilde{i}-2\tilde{j})$

$\overrightarrow{CD} = n(-6\tilde{i}+15\tilde{j}-3\tilde{k})$



If AB and CD intersect at E, then $\overrightarrow{EB} = p\overrightarrow{AB}, \overrightarrow{CE} = q\overrightarrow{CD}$

where both p and q are positive quantities less than 1

Now we know that $\overrightarrow{EB}+\overrightarrow{BC}+\overrightarrow{CE} = \overrightarrow{EE} = 0$

$\therefore \quad p\overrightarrow{AB}+\overrightarrow{BC}+q\overrightarrow{CD} = 0 \qquad \{\text{by (i)}\}$

or $\quad pm(2\tilde{i} - 6\tilde{j} + 2\tilde{k}) + (\tilde{i} - 2\tilde{j}) + q.n(-6\tilde{i} + 15\tilde{j} - 3\tilde{k}) = 0$

Since $\tilde{i}, \tilde{j}, \tilde{k}$ are non-coplanar, the above relation implies that if $x\tilde{i} + y\tilde{j} + z\tilde{k} = 0$, then x = 0, y = 0 and z = 0

$\therefore \quad 2mp + 1 - 6qn = 0, \ -6pm - 2 + 15qn = 0$

$2pm - 3qn = 0$

Solving these for pm and qn, we get

$pm = \frac{1}{2}, \ qn = \frac{1}{3} \qquad \therefore \quad p = \frac{1}{2m}, \ q = \frac{1}{3n}$

$\therefore \quad 0 < \frac{1}{2m} \le 1, \ 0 \le \frac{1}{3n} \le 1 \quad$ or $\quad m \ge \frac{1}{2}, n \ge \frac{1}{3}$

Again area of $\Delta BCE = \frac{1}{2}\left|\overline{EC}\times\overline{EB}\right| = \frac{1}{2}\left|-q\overline{CD}\times p\overline{AB}\right| = \frac{1}{2}pqnm\begin{vmatrix}\tilde{i} & \tilde{j} & \tilde{k}\\ -6 & 15 & -3\\ 2 & -6 & 2\end{vmatrix}$

Put $\quad pm = \frac{1}{2}, qn = \frac{1}{3}$

$= \frac{1}{2}.\frac{1}{2}.\frac{1}{3}|12\tilde{i}+6\tilde{j}-6\tilde{k}| = \frac{1}{12}.6\sqrt{6} = \frac{1}{2}\sqrt{6}$

Illustration 39 : $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar unit vectors such that angle between any two is α. If $\vec{a}\quad\vec{b} + \vec{b}\quad\vec{c} = \ell\vec{a} + m\vec{b} + n\vec{c}$, then determine ℓ, m, n in terms of α. **(JEE-1997)**

Solution : $a^2 = b^2 = c^2 = 1$, $[abc] \neq 0$

$\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{c} = \vec{c}\,.\vec{a} = \cos\alpha$ .... (i)

Multiply both sides of given relation scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$0 + [\vec{a}\,\vec{b}\,\vec{c}\,] = \ell.1 + (m + n)\cos\alpha$ .... (ii)

$0 = m + (n + \ell)\cos\alpha$ .... (iii)

$[\vec{a}\,\vec{b}\,\vec{c}\,] + 0 = (\ell + m)\cos\alpha + n$ .... (iv)

Adding, we get

$2[\vec{a}\,\vec{b}\,\vec{c}\,] = (\ell + m + n) + 2(\ell + m + n)\cos\alpha$

or $\quad 2[\vec{a}\,\vec{b}\,\vec{c}\,] = (\ell + m + n)(1 + 2\cos\alpha)$ .... (v)

From (ii), $(m + n) = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}] - \ell}{\cos\alpha}$

Putting in (v), we get $\quad 2[\vec{a}\,\vec{b}\,\vec{c}] = \left\{\ell + \dfrac{[\vec{a}\,\vec{b}\,\vec{c}] - \ell}{\cos\alpha}\right\}(1 + 2\cos\alpha)$

or $\quad [\vec{a}\,\vec{b}\,\vec{c}]\left\{2 - \dfrac{1 + 2\cos\alpha}{\cos\alpha}\right\} = \ell\left(1 - \dfrac{1}{\cos\alpha}\right)(1 + 2\cos\alpha)$

$\therefore \quad \ell = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{(1 + 2\cos\alpha)(1 - \cos\alpha)} = n \qquad$ {as above}

and $\quad m = -(n + \ell)\cos\alpha = \dfrac{-2[\vec{a}\,\vec{b}\,\vec{c}]\cos\alpha}{(1 + 2\cos\alpha)(1 - \cos\alpha)}$

Thus the values of ℓ, m, n depend on $[\vec{a}\,\vec{b}\,\vec{c}\,]$

Hence we now find the value of scalar $[\vec{a}\,\vec{b}\,\vec{c}\,]$ in terms of α.

Now $[\vec{a}\,\vec{b}\,\vec{c}\,]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\alpha & \cos\alpha \\ \cos\alpha & 1 & \cos\alpha \\ \cos\alpha & \cos\alpha & 1 \end{vmatrix}$ (Apply $C_1 + C_2 + C_3$)

$= (1 + 2\cos\alpha)\begin{vmatrix} 1 & \cos\alpha & \cos\alpha \\ 1 & 1 & \cos\alpha \\ 1 & \cos\alpha & 1 \end{vmatrix}$ (Apply $R_2 - R_1$ and $R_3 - R_1$)

$\therefore \quad [\vec{a}\,\vec{b}\,\vec{c}\,]^2 = (1 + 2\cos\alpha)(1 - \cos\alpha)^2$

$\therefore \quad \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{1 - \cos\alpha} = \sqrt{1 + 2\cos\alpha}$

Putting in the value of ℓ, m, n we have $\quad \ell = \dfrac{1}{\sqrt{(1 + 2\cos\alpha)}} = n, m = \dfrac{-2\cos\alpha}{\sqrt{(1 + 2\cos\alpha)}}$ **Ans.**

Illustration 40 : Vectors $\vec{x}, \vec{y}$ and $\vec{z}$ each of magnitude $\sqrt{2}$, make angles of 60 with each other. If $\vec{x} \times (\vec{y} \times \vec{z}) = \vec{a}$, $\vec{y} \times (\vec{z} \times \vec{x}) = \vec{b}$ and $\vec{x} \times \vec{y} = \vec{c}$, then find $\vec{x}, \vec{y}$ and $\vec{z}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$. **(Roorkee 1997)**

Solution : $\vec{x}.\vec{y} = \sqrt{2}\sqrt{2}\cos 60° = 1 = \vec{y}.\vec{z} = \vec{z}.\vec{x}$ ...... (i)

Also $x^2 = y^2 = z^2 = 2$

Again $\vec{a} = (\vec{x}.\vec{z})\vec{y} - (\vec{x}.\vec{y})\vec{z} = \vec{y} - \vec{z}$ {by (i)}

$\therefore \quad \vec{a} = \vec{y} - \vec{z}, \vec{b} = \vec{z} - \vec{x}$ ....(ii)

Now $\vec{a} \times \vec{c} = (\vec{y} - \vec{z}) \times (\vec{x} \times \vec{y}) = \vec{y} \times (\vec{x} \times \vec{y}) - \vec{z} \times (\vec{x} \times \vec{y})$

$= [(\vec{y}\ .\ \vec{y})\vec{x} - (\vec{y}\ .\ \vec{x})\vec{y}] - [(\vec{z}\ .\ \vec{y})\vec{x} - (\vec{z}\ .\ \vec{x})\vec{y}] = (2\vec{x} - \vec{y}) - (\vec{x} - \vec{y})$ {by (i)}

or $\vec{a} \quad \vec{c} = \vec{x}$

Similarly, $\vec{b} \quad \vec{c} = \vec{y}$

Now $\vec{z} = \vec{y} - \vec{a}$ or $\vec{z} = \vec{b} + \vec{x}$ {by (ii)}

$\therefore \quad \vec{z} = (\vec{b} \quad \vec{c} - \vec{a})$ or $\vec{b} + (\vec{a} \quad \vec{c})$ **Ans.**

Illustration 41 : If $\vec{x} \quad \vec{y} = \vec{a}$, $\vec{y} \quad \vec{z} = \vec{b}$, $\vec{x}\ .\ \vec{b} = \gamma$, $\vec{x}\ .\ \vec{y} = 1$ and $\vec{y}\ .\ \vec{z} = 1$, then find $\vec{x}, \vec{y}$ and $\vec{z}$ in terms of $\vec{a}, \vec{b}$ and γ. **(Roorkee 1998)**

Solution : $\vec{x} \quad \vec{y} = \vec{a}$ .... (i)

$\vec{y} \quad \vec{z} = \vec{b}$ .... (ii)

Also $\vec{x}\ .\ \vec{b} = \gamma,\ \vec{x}\ .\ \vec{y} = 1,\ \vec{y}\ .\ \vec{z} = 1$ .... (iii)

We have to make use of the relations given above.

From (i)

$\vec{x}\ .\ (\vec{x} \quad \vec{y}) = \vec{x}\ .\ \vec{a}$

$\therefore \quad \vec{x}\ .\ \vec{a} = 0 \qquad \because \quad [\vec{x}\,\vec{x}\,\vec{y}] = 0$

Similarly $\vec{y}.\ \vec{a} = 0,\ \vec{y}\ .\ \vec{b} = 0,\ \vec{z}\ .\ \vec{b} = 0$ .... (iv)

Multiplying (i) vectorially by $\vec{b}$,

$\vec{b} \quad (\vec{x} \quad \vec{y}) = \vec{b} \quad \vec{a}$ or $(\vec{b}\ .\ \vec{y})\vec{x} - (\vec{b}\ .\ \vec{x})\vec{y} = \vec{b} \quad \vec{a}$

or $0 - \gamma\vec{y} = -(\vec{a} \quad \vec{b})$ $\therefore \quad \vec{y} = \dfrac{(\vec{a} \times \vec{b})}{\gamma}$ .... (v)

by using relations is (iii) and (iv).

Again multiplying (i) vectorially by $\vec{y}$,

$(\vec{x} \quad \vec{y}) \quad \vec{y} = \vec{a} \quad \vec{y}$ or $(\vec{x}\ .\ \vec{y})\vec{y} - (\vec{y}\ .\ \vec{y})\vec{x} = \vec{a} \quad \vec{y}$

$\vec{y} - \vec{a} \quad \vec{y} = |\vec{y}|^2\vec{x}$ {by (iii)}

$\therefore \quad \vec{x} = \dfrac{1}{|\vec{y}|^2}[\vec{y} - \vec{a} \times \vec{y}]$

where $\vec{y} = \dfrac{\vec{a} \times \vec{b}}{\gamma}$ {by (v)}

Hence x is known in terms of $\vec{a}, \vec{b}$ and γ.

Again multiplying (ii) vectorially by $\vec{y}$, we get

$(\vec{y} \quad \vec{z}) \quad \vec{y} = \vec{b} \quad \vec{y}$ or $|\vec{y}|^2 \vec{z} - (\vec{y} \,.\, \vec{z})\vec{y} = \vec{b} \quad \vec{y}$ or $|\vec{y}|^2 \vec{z} = \vec{b} \quad \vec{y} + \vec{y}$ {by (iii)}

or $z = \dfrac{1}{|\vec{y}|^2}[\vec{b} \times \vec{y} + \vec{y}]$

where $\vec{y}$ is given by (v) .... (vi)

Results (v) and (vi) give the values of $\vec{x}, \vec{y}$ and $\vec{z}$ in terms of $\vec{a}, \vec{b}$ and γ. **Ans.**

ANSWERS FOR DO YOURSELF

1 :	(i) 7	(ii) A
3 :	(i) (a) $\vec{a}, \vec{d}$; $\vec{b}, \vec{x}, \vec{z}$; $\vec{c}, \vec{y}$ (b) $\vec{b}, \vec{x}$; $\vec{a}, \vec{d}$; $\vec{c}, \vec{y}$ (c) $\vec{a}, \vec{y}, \vec{z}$ (d) $\vec{b}, \vec{z}$; $\vec{x}, \vec{z}$	
	(ii) $\dfrac{1}{4}$	(iii) A,B,C
4 :	(i) $\dfrac{12\vec{a} - 13\vec{b}}{5}$, $-5\vec{b}$	(iii) $\dfrac{3}{7}\hat{i} - \dfrac{6}{7}\hat{j} + \dfrac{2}{7}\hat{k}$
6 :	(i) $\dfrac{\pi}{6}$ (ii) -15 (iii) 1	(iv) $\dfrac{2}{7}, \dfrac{2}{49}(3\hat{i} + 6\hat{j} + 2\hat{k})$ and $\dfrac{190\hat{i} - 110\hat{j} + 45\hat{k}}{49}$
	(v) $-\dfrac{2}{21}\hat{i} + \dfrac{32}{21}\hat{j} - \dfrac{8}{21}\hat{k}$ & $\dfrac{16}{21}\hat{i} - \dfrac{4}{21}\hat{j} - \dfrac{20}{21}\hat{k}$	
7 .	(i) $\vec{r} = \dfrac{2}{3}\left(\hat{i} + \hat{j} - 2\hat{k}\right)$	
8 :	(ii) $-\hat{i} - \hat{j} + 2\hat{k}$	(iii) $-3\hat{i} + 6\hat{j} + 6\hat{k}$
9 :	(i) $\dfrac{1}{\sqrt{6}}$	
10 :	(i) ± 1 (ii) 0 (iii) 7	(iv) Right handed system (v) C
11 :	(i) 62, $92\hat{i} + 102\hat{j} + 32\hat{k}$	
12 :	(i) 3 (ii) $\dfrac{2\hat{i} + \hat{k}}{3}, \dfrac{-8\hat{i} + 3\hat{j} - 7\hat{k}}{3}, \dfrac{-7\hat{i} + 3\hat{j} - 5\hat{k}}{3}$	(iii) linearly dependent.
13 :	(i) 20 units	