LIMIT

1. INTRODUCTION :

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in f(x). Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function.

2. DEFINITION :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim\limits_{x\to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

This implies if we can make the value of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

3. LEFT HAND LIMIT AND RIGHT HAND LIMIT OF A FUNCTION :

The value to which f(x) approaches, as x tends to 'a' from the left hand side ($x \to a^-$) is called left hand limit of f(x) at x = a. Symbolically, LHL = $\lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h)$.

The value to which f(x) approaches, as x tends to 'a' from the right hand side ($x \to a^+$) is called right hand limit of f(x) at x = a. Symbolically, RHL = $\lim\limits_{x\to a^+} f(x) = \lim\limits_{h\to 0} f(a+h)$.

Limit of a function f(x) is said to exist as, $x \to a$ when $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ = Finite quantity.

Example :

Graph of y = f(x)

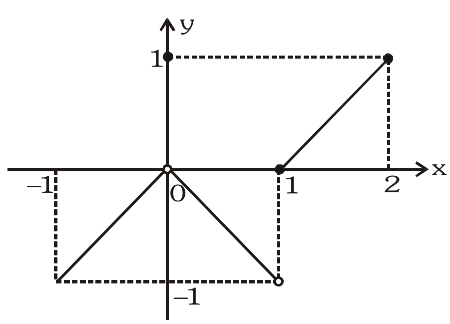


Fig. 1

$$\lim_{x\to -1^+} f(x) = \lim_{h\to 0} f(-1+h) = f(-1^+) = -1$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = f(0^-) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = f(0^+) = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = f(1^-) = -1$$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = f(1^+) = 0$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = f(2^-) = 1$$

$\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 1} f(x)$ does not exist.

Important note :

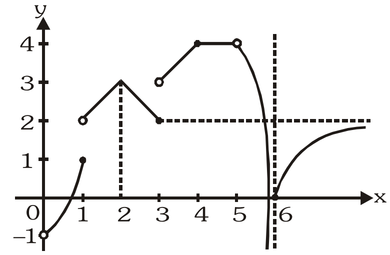
In $\lim\limits_{x\to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

Also it is necessary to note that if f(x) is defined only on one side of '**x = a**', one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of '**a**' both sided limits are to be considered.

As in $\lim\limits_{x\to 1} \cos^{-1} x = 0$, though f(x) is not defined for x >1, even in it's immediate vicinity.

Illustration 1 : Consider the adjacent graph of $y = f(x)$

Find the following :



(a) $\lim\limits_{x\to 0^-} f(x)$ (b) $\lim\limits_{x\to 0^+} f(x)$ (c) $\lim\limits_{x\to 1^-} f(x)$

(d) $\lim\limits_{x\to 1^+} f(x)$ (e) $\lim\limits_{x\to 2^-} f(x)$ (f) $\lim\limits_{x\to 2^+} f(x)$

(g) $\lim\limits_{x\to 3^-} f(x)$ (h) $\lim\limits_{x\to 3^+} f(x)$ (i) $\lim\limits_{x\to 4^-} f(x)$

(j) $\lim\limits_{x\to 4^+} f(x)$ (k) $\lim\limits_{x\to\infty} f(x) = 2$ (l) $\lim\limits_{x\to 6^-} f(x) = -\infty$

Solution : (a) As $x \to 0^-$: limit does not exist (the function is not defined to the left of $x = 0$)

(b) As $x \to 0^+$: $f(x) \to -1 \Rightarrow \lim\limits_{x\to 0^+} f(x) = -1$.

(c) As $x \to 1^-$: $f(x) \to 1 \Rightarrow \lim\limits_{x\to 1^-} f(x) = 1$.

(d) As $x \to 1^+$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to 1^+} f(x) = 2$.

(e) As $x \to 2^-$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$.

(f) As $x \to 2^+$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 2^-} f(x) = 3$.

(g) As $x \to 3^-$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to 3^-} f(x) = 2$.

(h) As $x \to 3^+$: $f(x) \to 3 \Rightarrow \lim\limits_{x\to 3^+} f(x) = 3$.

(i) As $x \to 4^-$: $f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^-} f(x) = 4$.

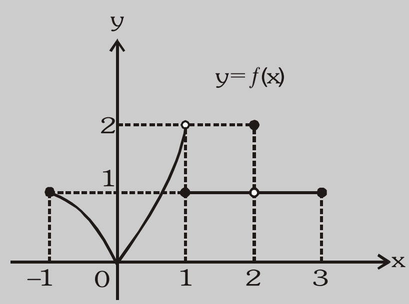
(j) As $x \to 4^+$: $f(x) \to 4 \Rightarrow \lim\limits_{x\to 4^+} f(x) = 4$.

(k) As $x \to \infty$: $f(x) \to 2 \Rightarrow \lim\limits_{x\to \infty} f(x) = 2$.

(l) As $x \to 6^-$,$f(x) \to -\infty \Rightarrow \lim\limits_{x\to 6^-} f(x) = -\infty$ limit does not exist because it is not finite.

Do yourself - 1 :

(i) Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false ?



(a) $\lim\limits_{x\to -1^+} f(x) = 1$ (b) $\lim\limits_{x\to 2} f(x)$ does not exist

(c) $\lim\limits_{x\to 2} f(x) = 2$ (d) $\lim\limits_{x\to 1^-} f(x) = 2$

(e) $\lim\limits_{x\to 1} f(x)$ does not exist (f) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$

(g) $\lim\limits_{x\to c} f(x)$ exists at every $c \in (-1, 1)$

(h) $\lim\limits_{x\to c} f(x)$ exists at every $c \in (1, 3)$

(i) $\lim\limits_{x\to 1^-} f(x) = 0$ (j) $\lim\limits_{x\to 3^+} f(x)$ does not exist.

4. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim\limits_{x\to a} f(x) = l \;\&\; \lim\limits_{x\to a} g(x) = m$. If l & m exists finitely then :

(a) Sum rule : $\lim\limits_{x\to a} \{f(x)+g(x)\} = l + m$

(b) Difference rule : $\lim\limits_{x\to a} \{f(x)-g(x)\} = l - m$

(c) Product rule : $\lim\limits_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers then $\lim\limits_{x\to a}[f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(g) $\lim\limits_{x\to a} f[g(x)] = f\left(\lim\limits_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim\limits_{x\to a} \ell n(g(x)) = \ell n[\lim\limits_{x\to a} g(x)]$

$= \ell n$ (m); provided ℓnx is continuous at x = m, m = $\lim\limits_{x\to a} g(x)$.

5. INDETERMINATE FORMS :

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0\times\infty, 1^{\infty}, 0^0, \infty^0$.

Initially we will deal with first five forms only and the other two forms will come up after we have gone through differentiation.

Note : (i) Here 0,1 are not exact, infact both are aproaching to their corresponding values.

(ii) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(a) $\infty + \infty \to \infty$ (b) $\infty \times \infty \to \infty$ (c) $\infty^{\infty} \to \infty$ (d) $0^{\infty} \to 0$

6. GENERAL METHODS TO BE USED TO EVALUATE LIMITS :

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \ldots\ldots\ldots + a^{n-1})$, $n \in N$

(ii) $x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + \ldots\ldots\ldots + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Illustration 2 : Evaluate : $\lim\limits_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$

Solution : We have

$$\lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right] = \lim_{x\to 2}\left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right] = \lim_{x\to 2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$ **Ans.**

Illustration 3 : $\lim\limits_{x\to 2}\frac{2^x + 2^{3-x} - 6}{2^{-x/2} - 2^{1-x}}$ is equal to

(A) 8 (B) 4 (C) 2 (D) none of these

Solution :

$$\lim_{x\to 2}\frac{2^x + 2^{3-x} - 6}{2^{-x/2} - 2^{1-x}} = \lim_{x\to 2}\frac{\dfrac{\left(2^{2x} + 2^3 - 6.2^x\right)}{2^x}}{\dfrac{1}{2^{x/2}} - \dfrac{2}{2^x}} = \lim_{x\to 2}\frac{2^{2x} - 6.2^x + 8}{2^{x/2} - 2} = \lim_{x\to 2}\frac{\left(2^x - 4\right)\left(2^x - 2\right)}{\left(2^{x/2} - 2\right)}$$

$$= \lim_{x\to 2}\left(2^{\frac{x}{2}} + 2\right)\left(2^x - 2\right) = (2+2).(4-2) = 8$$ **Ans. (A)**

Illustration 4 : Evaluate : $\lim_{x\to 1}\dfrac{x^{P+1}-(P+1)x+P}{(x-1)^2}$

Solution : $\lim_{x\to 1}\dfrac{x^{P+1}-(P+1)x+P}{(x-1)^2}$ $\left(\dfrac{0}{0}\text{form}\right)$

$$=\lim_{x\to 1}\frac{x^{P+1}-Px-x+P}{(x-1)^2}=\lim_{x\to 1}\frac{x(x^P-1)-P(x-1)}{(x-1)^2}$$

Dividing numerator and denominator by (x –1), we get

$$=\lim_{x\to 1}\frac{\dfrac{x(x^P-1)}{x-1}-P}{(x-1)}=\lim_{x\to 1}\frac{(x+x^2+x^3+\ldots+x^P)-P}{(x-1)}$$

$$=\lim_{x\to 1}\frac{(x+x^2+x^3+\ldots+x^P)-(1+1+1+\ldots\ldots\text{upto P times})}{(x-1)}$$

$$=\lim_{x\to 1}\left\{\frac{(x-1)}{(x-1)}+\frac{(x^2-1)}{(x-1)}+\frac{(x^3-1)}{(x-1)}+\ldots\ldots+\frac{(x^P-1)}{(x-1)}\right\}$$

$$= 1 + 2(1)^{2-1} + 3(1)^{3-1} +\ldots\ldots+ P(1)^{P-1} = 1 + 2 + 3 + \ldots\ldots+ P = \frac{P(P+1)}{2}$$ **Ans.**

Do yourself - 2 :

(i) Evaluate : $\lim_{x\to 1}\dfrac{x-1}{2x^2-7x+5}$

(b) **Rationalization or double rationalization :**

Illustration 5 : Evaluate : $\lim_{x\to 1}\dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

Solution : $\lim_{x\to 1}\dfrac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}=\lim_{x\to 1}\dfrac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$

$$=\lim_{x\to 1}\frac{(15-15x)}{(3-3x)}\ \frac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}}=\frac{5}{2}$$ **Ans.**

Illustration 6 : Evaluate : $\lim_{x\to 1}\left(\dfrac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Solution : This is of the form $\dfrac{3-3}{2-2}=\dfrac{0}{0}$ if we put x = 1

To eliminate the $\dfrac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$$\therefore\ \text{Limit}=\lim_{x\to 1}(\sqrt{x^2+8}-\sqrt{10-x^2})\ \frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\ \frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to 1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\ \frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}=\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)1=\frac{2+2}{3+3}=\frac{2}{3}$$ **Ans.**

Do yourself - 3 :

(i) Evaluate : $\lim\limits_{x\to 0}\dfrac{\sqrt{p+x}-\sqrt{p-x}}{\sqrt{q+x}-\sqrt{q-x}}$

(ii) Evaluate : $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, $a \neq 0$

(iii) If $G(x) = -\sqrt{25-x^2}$, then find the $\lim\limits_{x\to 1}\left(\dfrac{G(x)-G(1)}{x-1}\right)$

(c) **Limit when $x \to \infty$:**

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

Illustration 7 : Evaluate : $\operatorname*{Lim}\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$

Solution : $\operatorname*{Lim}\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$, $\left(\dfrac{\infty}{\infty}\text{ form}\right)$

Put $x = \dfrac{1}{y}$

Limit $= \operatorname*{Lim}\limits_{y\to 0}\dfrac{1+y+y^2}{3+2y-5y^2} = \dfrac{1}{3}$ **Ans.**

Illustration 8 : If $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2$, then

(A) a = 1, b = 1 (B) a = 1, b = 2 (C) a = 1, b = –2 (D) none of these

Solution : $\lim\limits_{x\to\infty}\left(\dfrac{x^3+1}{x^2+1}-(ax+b)\right)=2 \Rightarrow \lim\limits_{x\to\infty}\dfrac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}=2$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{x(1-a)-b-\dfrac{a}{x}+\dfrac{(1-b)}{x^2}}{1+\dfrac{1}{x^2}}=2 \Rightarrow 1-a=0,\ -b=2 \Rightarrow a=1,\ b=-2$ **Ans. (C)**

Do yourself - 4 :

(i) Evaluate : $\lim\limits_{n\to\infty}\dfrac{\underline{|n+2}+\underline{|n+1}}{\underline{|n+2}-\underline{|n+1}}$

(ii) Evaluate : $\lim\limits_{n\to\infty}(n-\sqrt{n^2+n})$

(d) **Squeeze play theorem (Sandwich theorem) :**

Statement : If $f(x)\le g(x)\le h(x)$; $\forall x$ in the neighbourhood at x = a and

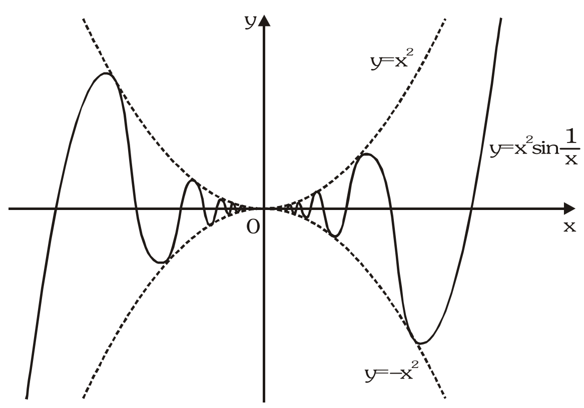
$\operatorname*{Lim}\limits_{x\to a} f(x)=\ell=\operatorname*{Lim}\limits_{x\to a} h(x)$ then $\operatorname*{Lim}\limits_{x\to a} g(x)=\ell$,

Ex.1 $\operatorname*{Lim}\limits_{x\to 0} x^2\sin\dfrac{1}{x}=0$,

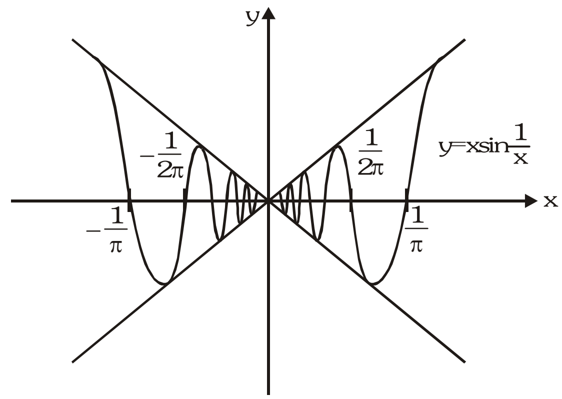
$\because \sin\left(\dfrac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x^2 \le x^2\sin\dfrac{1}{x}\le x^2$

$\Rightarrow \operatorname*{Lim}\limits_{x\to 0} x^2\sin\dfrac{1}{x}=0$ as $\operatorname*{Lim}\limits_{x\to 0}(-x^2)=\operatorname*{Lim}\limits_{x\to 0} x^2=0$



Ex.2 $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$



$\because$ $\sin\left(\dfrac{1}{x}\right)$ lies between –1 & 1

$\Rightarrow -x \le x\sin\dfrac{1}{x} \le x$

$\Rightarrow \text{Lim}_{x\to 0} x\sin\dfrac{1}{x} = 0$ as $\text{Lim}_{x\to 0}(-x) = \text{Lim}_{x\to 0} x = 0$

Illustration 9 : Evaluate : $\lim\limits_{n\to\infty} \dfrac{[x]+[2x]+[3x]+.....[nx]}{n^2}$ Where [.] denotes the greatest integer function.

Solution : We know that $x - 1 < [x] \le x$

$\Rightarrow x + 2x + ...nx - n < \sum\limits_{r=1}^{n}[rx] \le x + 2x + ...+ nx$

$\Rightarrow \dfrac{xn}{2}(n+1) - n < \sum\limits_{r=1}^{n}[rx] \le \dfrac{x.n(n+1)}{2} \Rightarrow \dfrac{x}{2}\left(1+\dfrac{1}{n}\right) - \dfrac{1}{n} < \dfrac{1}{n^2}\sum\limits_{r=1}^{n}[rx] \le \dfrac{x}{2}\left(1+\dfrac{1}{n}\right)$

Now, $\lim\limits_{n\to\infty} \dfrac{x}{2}\left(1+\dfrac{1}{n}\right) = \dfrac{x}{2}$ and $\lim\limits_{n\to\infty} \dfrac{x}{2}\left(1+\dfrac{1}{n}\right) - \dfrac{1}{n} = \dfrac{x}{2}$

Thus, $\lim\limits_{n\to\infty} \dfrac{[x]+[2x]+......+[nx]}{n^2} = \dfrac{x}{2}$ **Ans.**

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7. LIMIT OF TRIGONOMETRIC FUNCTIONS :

$$\text{Lim}_{x\to 0}\frac{\sin x}{x} = 1 = \text{Lim}_{x\to 0}\frac{\tan x}{x} = \text{Lim}_{x\to 0}\frac{\tan^{-1}x}{x} = \text{Lim}_{x\to 0}\frac{\sin^{-1}x}{x}$$ [where x is measured in radians]

(a) If $\text{Lim}_{x\to a} f(x) = 0$, then $\text{Lim}_{x\to a}\dfrac{\sin f(x)}{f(x)} = 1$, e.g. $\text{Lim}_{x\to 1}\dfrac{\sin(\ell nx)}{(\ell nx)} = 1$

Illustration 10 : Evaluate : $\lim\limits_{x\to 0}\dfrac{x^3\cot x}{1-\cos x}$

Solution : $\lim\limits_{x\to 0}\dfrac{x^3\cos x}{\sin x(1-\cos x)} = \lim\limits_{x\to 0}\dfrac{x^3\cos x(1+\cos x)}{\sin x.\sin^2 x} = \lim\limits_{x\to 0}\dfrac{x^3}{\sin^3 x}.\cos x(1+\cos x) = 2$ **Ans.**

Illustration 11 : Evaluate : $\lim\limits_{x\to 0}\dfrac{(2+x)\sin(2+x)-2\sin 2}{x}$

Solution : $\lim\limits_{x\to 0}\dfrac{2(\sin(2+x)-\sin 2)+x\sin(2+x)}{x} = \lim\limits_{x\to 0}\left(\dfrac{2.2.\cos\left(2+\dfrac{x}{2}\right)\sin\dfrac{x}{2}}{x} + \sin(2+x)\right)$

$$= \lim_{x\to 0}\frac{2\cos\left(2+\frac{x}{2}\right)\sin\frac{x}{2}}{\frac{x}{2}} + \lim_{x\to 0}\sin(2+x) = 2\cos 2 + \sin 2$$ **Ans.**

Illustration 12 : Evaluate : $\lim\limits_{n\to\infty} \dfrac{\sin\frac{a}{n}}{\tan\frac{b}{n+1}}$

Solution : As $n \to \infty$, $\frac{1}{n} \to 0$ and $\frac{a}{n}$ also tends to zero

$\sin\frac{a}{n}$ should be written as $\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$

The given limit $= \lim\limits_{n\to\infty}\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right)\cdot\dfrac{a(n+1)}{n.b}$

$= \lim\limits_{n\to\infty}\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right)\cdot\dfrac{a}{b}\left(1+\dfrac{1}{n}\right) = 1 \quad 1 \quad \dfrac{a}{b} \quad 1 = \dfrac{a}{b}$ **Ans.**

Illustration 13 : $\lim\limits_{x\to\infty} x\cos\left(\dfrac{\pi}{4x}\right)\sin\left(\dfrac{\pi}{4x}\right)$ is equal to -

(A) $\pi/2$ (B) $\pi/4$ (C) 1 (D) none of these

Solution : $\lim\limits_{x\to\infty}\dfrac{x}{2}\left(2\sin\dfrac{\pi}{4x}\cos\dfrac{\pi}{4x}\right) = \lim\limits_{x\to\infty}\dfrac{x}{2}\sin\left(\dfrac{\pi}{2x}\right)$

$= \lim\limits_{x\to\infty}\dfrac{\sin\left(\frac{\pi}{2x}\right)}{\frac{\pi}{2x}}\cdot\dfrac{\pi}{4} = \dfrac{\pi}{4} \quad \lim\limits_{y\to 0}\dfrac{\sin y}{y} = \dfrac{\pi}{4}$, where $y = \dfrac{\pi}{2x}$ **Ans. (B)**

Do yourself - 5 :

(i) Evaluate :

(a) $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\tan\beta x}$ (b) $\lim\limits_{x\to y}\dfrac{\sin^2 x - \sin^2 y}{x^2 - y^2}$ (c) $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h) - a^2\sin a}{h}$

(b) Using substitution

$\operatorname{Lim}\limits_{x\to a} f(x) = \operatorname{Lim}\limits_{h\to 0} f(a-h)$ or $\operatorname{Lim}\limits_{h\to 0} f(a+h)$ i.e. by substituting x by a − h or a + h

Illustration 14 : Evaluate : $\operatorname{Lim}\limits_{x\to\frac{\pi}{2}}(\sec x - \tan x)$

Solution : $\operatorname{Lim}\limits_{x\to\frac{\pi}{2}}(\sec x - \tan x); (\infty - \infty \text{ form})$

$\operatorname{Lim}\limits_{x\to\frac{\pi}{2}}\left(\dfrac{1-\sin x}{\cos x}\right); \left(\text{now in } \dfrac{0}{0} \text{ form}\right)$

Put $x = \left(\dfrac{\pi}{2}+h\right)$

$\therefore$ Limit $= \operatorname{Lim}\limits_{h\to 0}\left[\dfrac{1-\sin\left(\frac{\pi}{2}+h\right)}{\cos\left(\frac{\pi}{2}+h\right)}\right] = \operatorname{Lim}\limits_{h\to 0}\left[\dfrac{1-\cos h}{-\sin h}\right]$

$$= \lim_{h\to 0}\left[\frac{2\sin^2\frac{h}{2}}{-2\sin\frac{h}{2}\cos\frac{h}{2}}\right] = \lim_{h\to 0}\left[\frac{\sin\frac{h}{2}}{-\cos\frac{h}{2}}\right] = 0$$ **Ans.**

8. LIMIT USING SERIES EXPANSION :

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x = 1 + \frac{x\,\ell na}{1!} + \frac{x^2\ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + \ldots\ldots\ldots a > 0$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$

(c) $\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ \text{for} -1 < x \le 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots\ldots$

(i) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots\ldots$

(j) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots\ n \in Q$

Illustration 15 : $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x}$

Solution : $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x} \Rightarrow \lim_{x\to 0}\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots..\right) - 2x}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots..\right)}$

$$\Rightarrow \lim_{x\to 0}\frac{2.\frac{x^3}{6} + 2.\frac{x^5}{5!} + \ldots\ldots}{\frac{x^3}{6} + \frac{x^5}{5!}\ldots..} \Rightarrow \lim_{x\to 0}\frac{x^3\left(\frac{1}{3} + \frac{1}{60}x^2 + \ldots..\right)}{x^3\left(\frac{1}{6} + \frac{1}{120}x^2 + \ldots..\right)} = \frac{1/3}{1/6} = 2$$ **Ans.**

Do yourself - 6 :

(i) Evaluate : $\lim_{x\to 0}\frac{x - \sin x}{\sin(x^3)}$

(ii) Evaluate : $\lim_{x\to 0}\frac{x - \tan^{-1} x}{x^3}$

9. LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim_{x\to 0}\frac{a^x - 1}{x} = \ell n\, a\,(a > 0)$ In particular $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$.

In general if $\lim_{x\to a} f(x) = 0$,then $\lim_{x\to a}\frac{a^{f(x)} - 1}{f(x)} = \ell na,\ a > 0$

Illustration 16 : Evaluate : $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{x}}{\tan x - x}$

Solution :

$$\lim_{x\to 0}\frac{e^{\tan x}-e^{x}}{\tan x - x}=\lim_{x\to 0}\frac{e^{x}\times e^{(\tan x - x)}-e^{x}}{\tan x - x}$$

$$=\lim_{x\to 0}\frac{e^{x}(e^{\tan x - x}-1)}{\tan x - x} = \lim_{x\to 0\, y\to 0}\frac{e^{x}(e^{y}-1)}{y} \text{ where } y = \tan x - x \text{ and } \lim_{y\to 0}\frac{e^{y}-1}{y} = 1$$

$= e^{0} \quad 1$ $\qquad$ [as $x \to 0$, $\tan x - x \to 0$]

$= 1 \quad 1 = 1$ **Ans.**

Do yourself - 7 :

(i) Evaluate : $\lim\limits_{x\to a}\dfrac{e^{x}-e^{a}}{x-a}$ $\qquad$ (ii) Evaluate : $\lim\limits_{x\to 0}\dfrac{2^{x}-1}{(1+x)^{1/2}-1}$

(b) (i) $\lim\limits_{x\to 0}(1+x)^{1/x} = e = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}$ (Note : The base and exponent depends on the same variable.)

In general, if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}(1+f(x))^{1/f(x)} = e$

(ii) $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x} = 1$

(iii) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a}\ \phi(x) = \infty$, then ; $\lim\limits_{x\to a}\left[f(x)\right]^{\phi(x)} = e^{k}$ where $k = \lim\limits_{x\to a}\ \phi\ (x)\ [f(x)-1]$

Illustration 17 : Evaluate $\lim\limits_{x\to 1}(\log_3 3x)^{\log_x 3}$

Solution :

$$\lim_{x\to 1}(\log_3 3x)^{\log_x 3} = \lim_{x\to 1}(\log_3 3 + \log_3 x)^{\log_x 3}$$

$$= \lim_{x\to 1}(1+\log_3 x)^{1/\log_3 x} = e \qquad \because \log_b a = \frac{1}{\log_a b}$$ **Ans.**

Illustration 18 : Evaluate : $\lim\limits_{x\to 0}\dfrac{x\ell n(1+2\tan x)}{1-\cos x}$

Solution :

$$\lim_{x\to 0}\frac{x\ell n(1+2\tan x)}{1-\cos x} = \lim_{x\to 0}\frac{x\ell n(1+2\tan x)}{\dfrac{1-\cos x}{x^{2}}.x^{2}}.\frac{2\tan x}{2\tan x} = 4$$ **Ans.**

Illustration 19 : Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{2x^{2}-1}{2x^{2}+3}\right)^{4x^{2}+2}$

Solution : Since it is in the form of 1^{∞}

$$\lim_{x\to\infty}\left(\frac{2x^{2}-1}{2x^{2}+3}\right)^{4x^{2}+2} = e^{\lim\limits_{x\to\infty}\left(\dfrac{2x^{2}-1-2x^{2}-3}{2x^{2}+3}\right)(4x^{2}+2)} = e^{-8}$$ **Ans.**

Illustration 20 : $\lim_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}}, a \neq n\pi, n$ is an integer, equals -

(A) $e^{\cot a}$ (B) $e^{\tan a}$ (C) $e^{\sin a}$ (D) $e^{\cos a}$

Solution : $\lim_{x\to a}\left(\frac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} = \lim_{x\to a}\left(1+\frac{\sin x-\sin a}{\sin a}\right)^{\frac{1}{x-a}} = \lim_{x\to a}\left[\left\{1+\left(\frac{\sin x-\sin a}{\sin a}\right)\right\}^{\frac{\sin a}{\sin x-\sin a}}\right]^{\frac{\sin x-\sin a}{(x-a)\sin a}}$

$= \lim_{e^{x\to a}} \frac{2}{x-a}\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right).\frac{1}{\sin a} = e^{\frac{\cos a}{\sin a}} = e^{\cot a}$ **Ans. (A)**

Illustration 21 : $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} =$

(A) abc (B) $\sqrt{abc}$ (C) $(abc)^{1/3}$ (D) none of these

Solution : $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} = \lim_{x\to 0}\left(1+\frac{a^x+b^x+c^x-3}{3}\right)^{1/x}$

$= \lim_{x\to 0}\left[\left(1+\frac{(a^x-1)}{3}+\frac{(b^x-1)}{3}+\frac{(c^x-1)}{3}\right)^{\frac{3}{(a^x-1)+(b^x-1)+(c^x-1)}}\right]^{\frac{a^x-1+b^x-1+c^x-1}{3x}}$

$= e^{1/3}\lim_{x\to 0}\left[\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right] = e^{1/3\ (\log a + \log b + \log c)} = e^{\log (abc)1/3} = (abc)^{1/3}$ **Ans. (C)**

Do yourself - 8 :

(i) Evaluate : $\lim_{x\to\infty} x\{\ell n(x+a)-\ell n x\}$

(ii) Evaluate : $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$

(iii) Evaluate : $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{pn+q}$

(iv) Evaluate : $\lim_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$

(v) Evaluate : $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$

(c) If $\lim_{x\to a} f(x) = A > 0$ & $\lim_{x\to a} \phi(x) = B$ (a finite quantity) then ; $\lim_{x\to a}[f(x)]^{\phi(x)} = e^{B\, ln\, A} = A^B$

Illustration 22 : Evaluate : $\lim_{x\to\infty}\left(\frac{7x^2+1}{5x^2-1}\right)^{\frac{x^5}{1-x^3}}$

Solution : Here $f(x) = \frac{7x^2+1}{5x^2-1}$, $\phi(x) = \frac{x^5}{1-x^3} = \frac{x^2.x^3}{1-x^3} = \frac{x^2}{\frac{1}{x^3}-1}$

$\therefore \lim_{x\to\infty} f(x) = \frac{7}{5}$ & $\lim_{x\to\infty}\phi(x) \to -\infty$

$\Rightarrow \lim_{x\to\infty}(f(x))^{\phi(x)} = \left(\frac{7}{5}\right)^{-\infty} = 0$ **Ans.**

Do yourself - 9 :

(i) Evaluate : $\lim\limits_{x\to\infty}\left(\dfrac{1+5x^2}{1+3x^2}\right)^{-x^2}$

Miscellaneous Illustrations :

Illustration 23 : Find $\lim\limits_{n\to\infty}\dfrac{\sqrt{1-x_0^2}}{x_1x_2x_3x_4.....x_n}$, where $x_{r+1}=\sqrt{\dfrac{1+x_r}{2}}$, $0\le r\le(n-1)$, $r\in I$, $n\in N$

Solution : Let $x_0=\cos\theta$ then $x_1=\sqrt{\dfrac{1+x_0}{2}}=\cos\dfrac{\theta}{2}$

$$x_2=\sqrt{\frac{1+x_1}{2}}=\cos\frac{\theta}{2^2}\ ,.........x_n=\cos\frac{\theta}{2^n}$$

$$\therefore\quad \text{Limits}=\frac{\sin\theta}{\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}.....\cos\frac{\theta}{2^n}}\ ,\ n\to\infty$$

$$=\lim_{n\to\infty}\frac{\sin\theta}{\sin\theta}2^n\sin\frac{\theta}{2^n}=\lim_{n\to\infty}\theta.\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}=\theta=\cos^{-1}x_0$$ **Ans.**

Illustration 24 : Evaluate : $\lim\limits_{x\to0}\dfrac{\cos^2\{1-\cos^2(1-\cos^2(...........(1-\cos^2(x))))\}}{\sin\left[\pi\left(\dfrac{\sqrt{x+4}-2}{x}\right)\right]}$

Solution : Let $A=\lim\limits_{x\to0}\dfrac{\cos^2\{1-\cos^2(1-\cos^2(...........(1-\cos^2(x))))\}}{\sin\left[\pi\left(\dfrac{\sqrt{x+4}-2}{x}\right)\right]}$

$$=\lim_{x\to0}\frac{\cos^2\{\sin^2(\sin^2(......(1-\cos^2(x))))\}}{\sin\left(\pi\left(\frac{\sqrt{x+4}-2}{x}.\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)\right)}=\frac{\cos^2 0}{1}\lim_{x\to0}\frac{1}{\sin\left(\pi\frac{(x+4-4)}{x\left(\sqrt{x+4}+2\right)}\right)}=\frac{1}{\sin\frac{\pi}{4}}=\sqrt{2}$$ **Ans.**

Illustration 25 : Evaluate the following limits, if exist $\lim\limits_{n\to\infty}n^{-n^2}\left((n+1)\left(n+\dfrac{1}{2}\right)\left(n+\dfrac{1}{2^2}\right)..........\left(n+\dfrac{1}{2^{n-1}}\right)\right)^n$

Solution :

$$\lim_{n\to\infty}n^{-n^2}\left((n+1)\left(n+\frac{1}{2}\right)......\left(n+\frac{1}{2^{n-1}}\right)\right)^n=\lim_{n\to\infty}\left(\frac{(n+1)\left(n+\frac{1}{2}\right)......\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right)^n$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n.\left(\frac{n+\frac{1}{2}}{n}\right)^n.......\left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}......\left(1+\frac{1}{2^{n-1}n}\right)^{\frac{2.^{n-1}n}{2^{n-1}}}$$

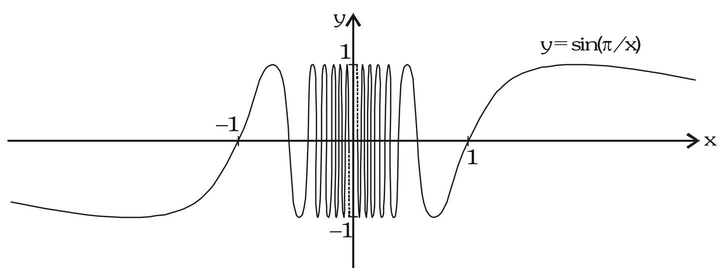
$$=e.e^{\frac{1}{2}}.\ e^{\frac{1}{4}}......=e^{\left(1+\frac{1}{2}+\frac{1}{4}...........\right)}=e^2$$ **Ans.**

Illustration 26 : Evaluate $\lim\limits_{x\to 0}\sin\frac{\pi}{x}$.

Solution : Again the function f(x) = sin(π/x) is undefined at 0. Evaluating the function for some small values of x, we get f(1) = sinπ = 0, $f\left(\frac{1}{2}\right)=\sin 2\pi=0$,

f(0.1) = sin10π = 0, f(0.01) = sin100π = 0.

On the basis of this information we might be tempted to guess that $\lim\limits_{x\to 0}\sin\frac{\pi}{x}=0$ but this time our guess is wrong. Note that although f(1/n) = sinnπ = 0 for any integer n, it is also true that f(x) = 1 for infinitely many values of x that approach 0. [In fact, sin(π/x) = 1 when $\frac{\pi}{x}=\frac{\pi}{2}+2n\pi$ and solving for x, we get x = 2/(4n + 1)]. The graph of f is given in following figure



The dashed line indicate that the values of sin(π/x) oscillate between 1 and –1 infinitely often as x approaches 0. Since the values of f(x) do not approach a fixed number as x approaches 0,

⇒ $\lim\limits_{x\to 0}\sin\frac{\pi}{x}$ does not exist.

ANSWERS FOR DO YOURSELF

1 : (i) (a) T (b) F (c) F (d) T (e) T (f) T (g) T (h) T (i) F (j) T

2 : (i) $-\frac{1}{3}$

3 : (i) $\frac{\sqrt{q}}{\sqrt{p}}$ (ii) $\frac{2}{3\sqrt{3}}$ (iii) $\frac{1}{\sqrt{24}}$

4 : (i) 1 (ii) $-\frac{1}{2}$

5 : (i) (a) $\frac{\alpha}{\beta}$ (b) $\frac{\sin 2y}{2y}$ (c) $2a\sin a + a^2\cos a$

6. (i) $\frac{1}{6}$ (ii) $\frac{1}{3}$

7 : (i) e^a (ii) $2\ell n2$

8 : (i) a (ii) e^2 (iii) e^p (iv) $e^{\frac{1}{2}}$ (v) e^5

9 : (i) 0