FUNCTION

1. CARTESIAN PRODUCT OF TWO SETS :

The cartesian product of two non-empty sets A & B is the set of all possible ordered pair of the form (a, b) where the first entry comes from set A & second comes from set B.

A   B = {(a, b) | a ∈ A, b ∈ B}

e.g. A = {1, 2, 3}   B = {p, q}

A   B = {(1, p)(1, q), (2, p), (2, q), (3, p), (3, q)}

Note :

(i) If either A or B is the null set, then A   B will also be empty set, i.e. A   B = ϕ

(ii) If n(A) = p & n(B) = q, then n(A   B) = p   q, where n(X) denotes the number of elements in set X.

2. RELATION :

A relation R from a non-empty set A to non-empty set B is subset of cartesian product A   B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A   B. The second element is called the image of the first element.

Note :

(i) The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

(ii) The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R.

(iii) Range subset of co-domain.

3. FUNCTION :

A relation R from a set A to a set B is called a function if each element of A has unique image in B.
It is denoted by the symbol.

$f : A \to B$ or $A \xrightarrow{f} B$

which reads 'f ' is a function from A to B 'or' f maps A to B,

If an element a ∈ A is associated with an element b ∈ B, then b is called 'the f image of a' or 'image of a under f 'or' the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

$b = f(a)$ or $f : a \to b$ or $f : (a, b)$

Thus a function 'f ' from a set A to a set B is a subset of A   B in which each 'a' belonging to A appears in one and only one ordered pair belonging to f.

Representation of Function :

(a) Ordered pair : Every function from $A \to B$ satisfies the following conditions :

(i) $f \subset A \times B$ (ii) $\forall$ a ∈ A there exist b ∈ B and (iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

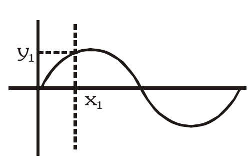
(b) Formula based (uniformly/nonuniformly) : e.g.

(i) $f : R \to R$, $y = f(x) = 4x$, $f(x) = x^2$ (uniformly defined)

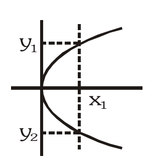
(ii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$ (non-uniformly defined)

(iii) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(c) Graphical representation :



Graph (1)



Graph (2)

Graph(1) represent a function but graph(2) does not represent a function.

Note :

(i) If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.

(ii) Every function is a relation but every relation is not necessarily a function.

4. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION :

Let $f : A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of f images of all the elements of A is known as the range of f.

Thus : Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

Note :

(i) It should be noted that range is a subset of co-domain.

(ii) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range

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Illustration 1 : Find the Domain of the following function :

(i) $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$ (ii) $y = \log_{(x-4)}(x^2 - 11x + 24)$

(iii) $f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

Solution : (i) $\sin x \geq 0$ and $16 - x^2 \geq 0 \Rightarrow 2n\pi \leq x \leq (2n+1)\pi$ and $-4 \leq x \leq 4$

$\therefore$ Domain is $[-4, -\pi] \cup [0, \pi]$

(ii) $y = \log_{(x-4)}(x^2 - 11x + 24)$

Here 'y' would assume real value if,

$x - 4 > 0$ and $\neq 1$, $x^2 - 11x + 24 > 0 \Rightarrow x > 4$ and $\neq 5$, $(x - 3)(x - 8) > 0$

$\Rightarrow x > 4$ and $\neq 5$, $x < 3$ or $x > 8 \Rightarrow x > 8 \Rightarrow$ Domain (y) = $(8, \infty)$

(iii) We have $f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

f(x) is defined if $-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1 > 0$

or if $\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) < -1$ or if $\left(1 + \frac{1}{\sqrt[4]{x}}\right) > (1/2)^{-1}$

or if $1 + \frac{1}{\sqrt[4]{x}} > 2$ or if $\frac{1}{\sqrt[4]{x}} > 1$ or if $x^{1/4} < 1$ or if $0 < x < 1$

$\therefore$ D(f) = (0, 1)

Illustration 2 : Find the range of following functions :

(i) $f(x) = \frac{1}{8 - 3\sin x}$ (ii) $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

(iii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

Solution : (i) $f(x) = \frac{1}{8 - 3\sin x}$

$-1 \leq \sin x \leq 1$

$\therefore$ Range of $f = \left[\frac{1}{11}, \frac{1}{5}\right]$

(ii) Let $y = \log_2\left(\dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

$\Rightarrow \quad 2^y = \sin\left(x - \dfrac{\pi}{4}\right) + 3 \quad \Rightarrow \quad -1 \le 2^y - 3 \le 1$

$\Rightarrow \quad 2 \le 2^y \le 4 \quad \Rightarrow \quad y \quad \Rightarrow \quad [1, 2]$

(iii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

$1 \le 16\sin^2 x + 1 \le 17$

$\therefore \quad 0 \le \log_2(16\sin^2 x + 1) \le \log_2 17$

$\therefore \quad 2 - \log_2 17 \le 2 - \log_2(16\sin^2 x + 1) \le 2$

Now consider $0 < 2 - \log_2(16\sin^2 x + 1) \le 2$

$\therefore \quad -\infty < \log_{\sqrt{2}}[2 - \log_2(16\sin^2 x + 1)] \le \log_{\sqrt{2}} 2 = 2$

$\therefore \quad$ the range is $(-\infty, 2]$

Do yourself - 1 :

(i) Find the domain of following functions :

(a) $y = 1 - \log_{10}x$ (b) $y = \sqrt{5 - 2x}$ (c) $y = \dfrac{1}{\sqrt{x^2 - 4x}}$ (d) $y = \dfrac{1}{\sqrt{|x| - x}}$

(ii) Find the range of the following function :

(a) $f(x) = \sin 3x$ (b) $f(x) = \dfrac{1}{3 - \cos x}$ (c) $f(x) = e^{-3x}$

(d) $f(x) = \cos\left(2x + \dfrac{\pi}{4}\right)$ (e) $f(x) = 2\sin\left(2x + \dfrac{\pi}{4}\right)$ (f) $f(x) = \dfrac{x^2}{x^4 + 1}$

(iii) Given 'n' real numbers $a_1, a_2, \ldots, a_n$. Determine the value of x at which the function $f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2$ takes on the minimum value.

5. IMPORTANT TYPES OF FUNCTION :

(a) Polynomial function :

If a function 'f' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note :

(i) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, a \neq 0$

(ii) There are two polynomial functions, satisfying the relation; $f(x). f(1/x) = f(x) + f(1/x)$. They are (1) $f(x) = x^n + 1$ & (2) $f(x) = 1 - x^n$, where n is a positive integer.

(iii) Domain of a polynomial function is R

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(b) Algebraic function :

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

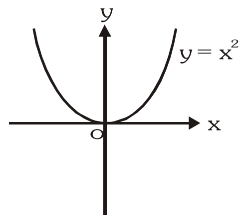
Examples : $f(x) = \sqrt{x^2 + 1}$; $g(x) = \dfrac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$

If y is an algebraic function of x, then it satisfies a polynomial equation of the form $P_0(x)y^n + P_1(x)y^{n-1} + \ldots + P_{n-1}(x)y + P_n(x) = 0$, where 'n' is a positive integer and $P_0(x), P_1(x), \ldots$ are polynomial in x.

Note that all polynomial functions are Algebraic but the converse in not true. A functions that is not algebraic is called **TRANSCEDENTAL** function.

Basic algebraic function :

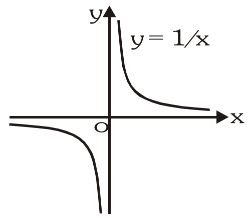
(i) $y = x^2$



Domain : R

Range : $R^+ \cup \{0\}$ or $[0, \infty)$

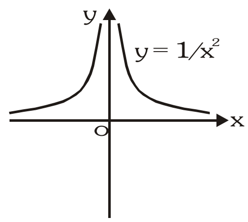
(ii) $y = \dfrac{1}{x}$



Domain : $R - \{0\}$ or R_0

Range : $R - \{0\}$

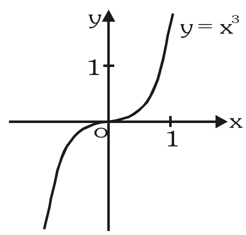
(iii) $y = \dfrac{1}{x^2}$



Domain : R_0

Range : R^+ or $(0, \infty)$

(iv) $y = x^3$



Domain : R

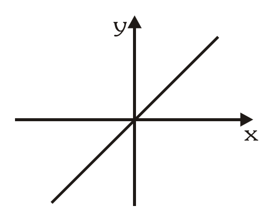
Range : R

(c) Rational function :

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials & $h(x) \neq 0$, **Domain** : $R - \{x \mid h(x) = 0\}$

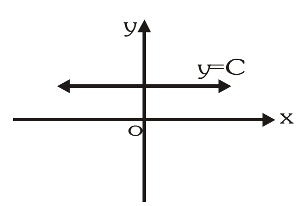
Any rational function is automatically an algebraic function.

(d) Identity function :



The function $f : A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.

(e) Constant function :

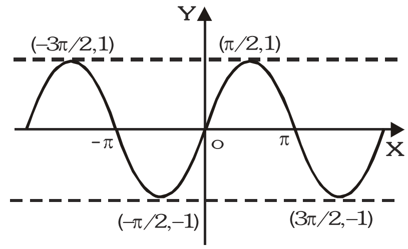


$f : A \to B$ is said to be constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c, \ \forall \ x \in A, c \in B$ is constant function. Note that the range of a constant function is a singleton set.

Domain : R **Range** : {C}

(f) Trigonometric functions :

(i) **Sine function**

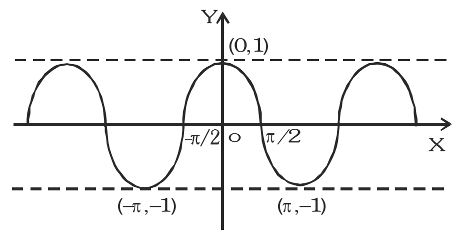


f(x) = sin x

Domain : R

Range : [–1, 1], period 2π

(ii) **Cosine function**

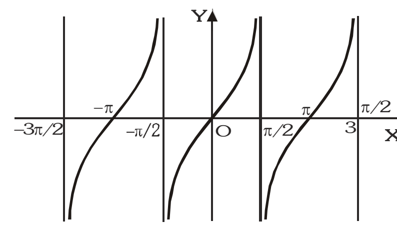


f(x) = cos x

Domain : R

Range : [–1, 1], period 2π

(iii) **Tangent function**

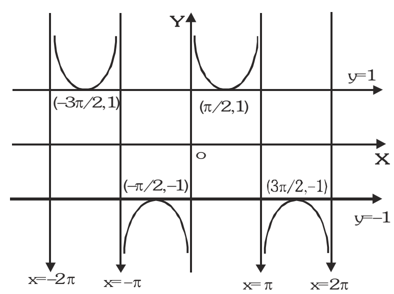


f(x) = tan x

Domain : $R-\left\{x \mid x=\frac{(2n+1)\pi}{2}, n \in I\right\}$

Range : R , period π

(iv) **Cosecant function**

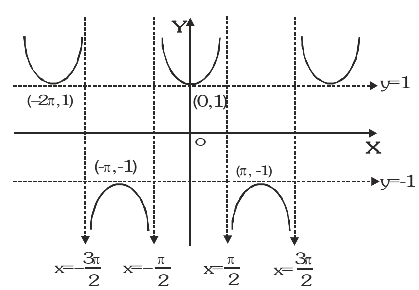


f(x) = cosec x

Domain : R – {x|x = nπ, n ∈I}

Range : R – (–1, 1), period 2π

(v) **Secant function**

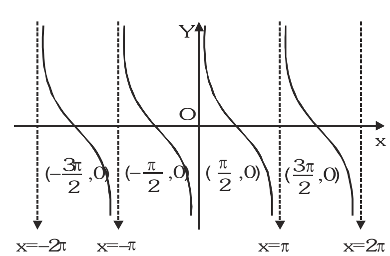


f(x) = sec x

Domain : R – {x|x = (2n + 1) π/2 : n ∈ I}

Range : R – (–1, 1), period 2π

(vi) **Cotangent function**



f(x) = cot x

Domain : R – {x|x = nπ, n ∈I}

Range : R, period π

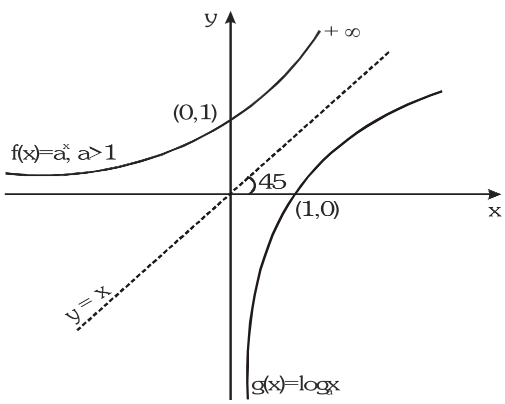
(g) Exponential and Logarithmic Function :

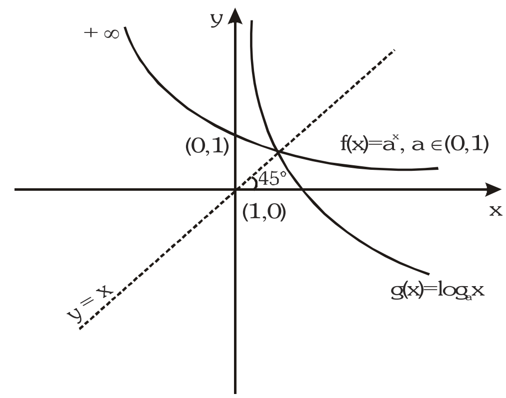
A function $f(x) = a^x (a > 0)$, $a \neq 1$, $x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

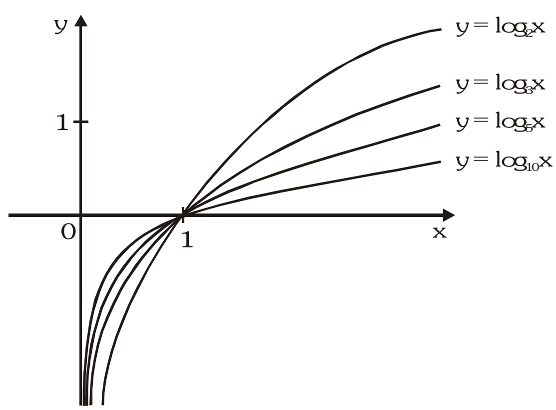
Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (If functions are mirror image of each other about the line y = x)

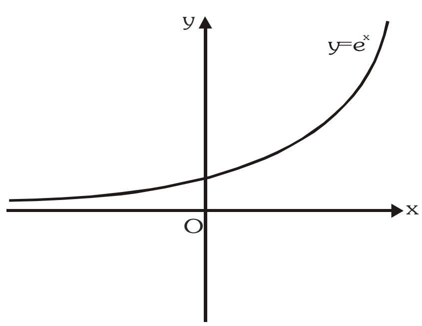
Domain of a^x is R **Range** R^+

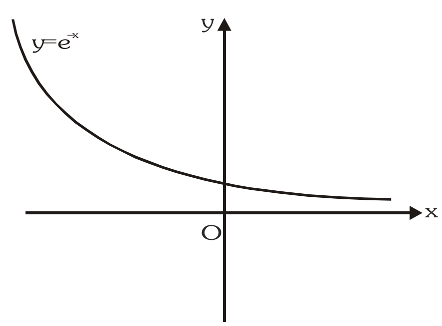
Domain of $\log_a x$ is R^+ **Range** R











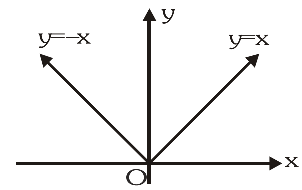
Note-1 : $f(x) = a^{1/x}$, $a > 0$ **Domain** : $R - \{0\}$ **Range** : $R^+ - \{1\}$

Note-2 : $f(x) = \log_x a = \dfrac{1}{\log_a x}$ **Domain** : $R^+ - \{1\}$ **Range** : $R - \{0\}$

$(a > 0)$ $(a \neq 1)$

(h) **Absolute value function :**

The absolute value (or modulus) of a real number x (written |x|) is a non negative real number that satisfies the conditions.



$$|x| = \begin{cases} x & \text{if} \quad x \ge 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

The properties of absolute value function are

(i) the inequality $|x| \le \alpha$ means that $-\alpha \le x \le \alpha$; if $\alpha > 0$

(ii) the inequality $|x| \ge \alpha$ means that $x \ge \alpha$ or $x \le -\alpha$ if $\alpha > 0$

(iii) $|x \pm y| \le |x| + |y|$

(iv) $|x \pm y| \ge ||x| - |y||$

(v) $|xy| = |x|.|y|$

(vi) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $(y \ne 0)$

Note : $f(x) = \frac{1}{|x|}$, **Domain** : $R - \{0\}$, **Range** : R^+

Illustration 3 : Determine the values of x satisfying the equality.
$|(x^2 + 4x + 9) + (2x - 3)| = |x^2 + 4x + 9| + |2x - 3|$.

Solution : The equality $|a + b| = |a| + |b|$ is valid if and only if both summands have the same sign,
$\because$ $x^2 + 4x + 9 = (x + 2)^2 + 5 > 0$ at any values of x, the equality is satisfied at those values of x at which $2x - 3 \ge 0$, i.e. at $x \ge \frac{3}{2}$.

Illustration 4 : Determine the values of x satisfying the equality $|x^4 - x^2 - 6| = |x^4 - 4| - |x^2 + 2|$.

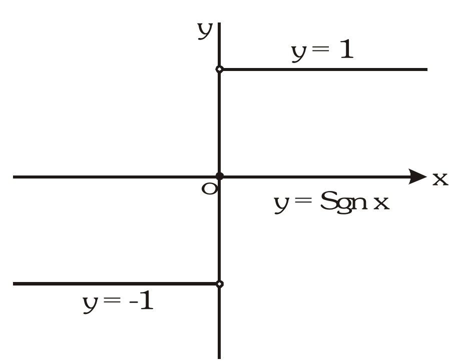
Solution : The equality $|a - b| = |a| - |b|$ holds true if and only if a and b have the same sign and $|a| \ge |b|$.
In our case the equality will hold true for the value of x at which $x^4 - 4 \ge x^2 + 2$.
Hence $x^2 - 2 \ge 1$; $|x| \ge \sqrt{3}$.

Do yourself - 2 :

(i) Determine the values of x satisfying the equality $|x^2 - 8| = |3x^2 - 5| - |2x^2 + 3|$.

(i) **Signum function :**

Signum function y = sgn (x) is defined as follows



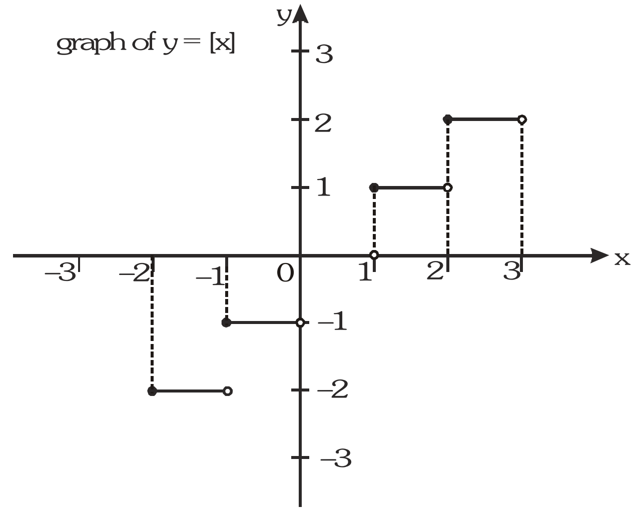
$$y = \begin{cases} \frac{|x|}{x}, x \ne 0 \\ 0, x = 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

Domain : R

Range : {–1, 0, 1}

(j) Greatest integer or step up function :

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :



x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1

Domain : R

Range : I

Properties of greatest integer function :

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

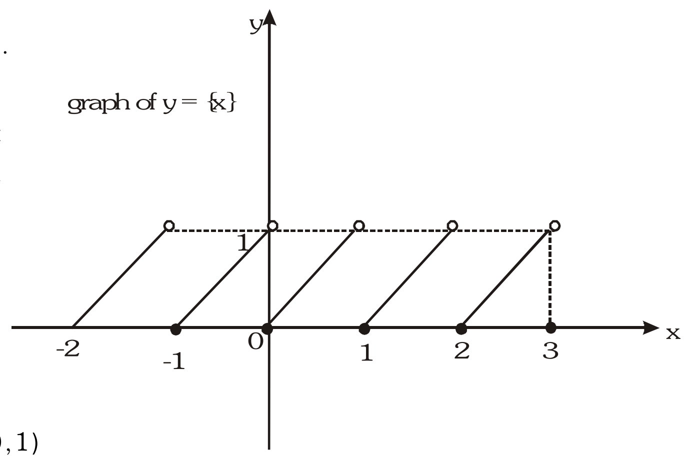
(ii) $[x + m] = [x] + m$ if m is an integer.

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Note : $f(x) = \dfrac{1}{[x]}$ **Domain :** $R - [0, 1)$ **Range :** $\{x \mid x = \frac{1}{n}, n \in I_0\}$

(k) Fractional part function :

It is defined as : $g(x) = \{x\} = x - [x]$ e.g. the fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.



x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

Domain : R

Range : [0,1)

Note : $f(x) = \dfrac{1}{\{x\}}$ **Domain :** $R - I$ **Range :** $(1, \infty)$

Properties of Fractional part function :

(i) $0 \le \{x\} < 1$ (ii) $\{[x]\} = [\{x\}] = 0$ (iii) $\{\{x\}\} = \{x\}$

(iv) $\{x+m\} = \{x\}$, $m \in I$ (v) $\{x\} + \{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$

Illustration 5 : If y = 2[x] + 3 & y = 3[x – 2] + 5 then find [x + y] where [.] denotes greatest integer function.

Solution : $y = 3[x - 2] + 5 = 3[x] - 1$

so $3[x] - 1 = 2[x] + 3$

$[x] = 4 \Rightarrow 4 \le x < 5$

then $y = 11$

so $x + y$ will lie in the interval $[15, 16)$

so $[x + y] = 15$

Illustration 6 : Find the value of $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+......\left[\frac{1}{2}+\frac{2946}{1000}\right]$ where [.] denotes greatest integer function ?

Solution :

$$\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+......\left[\frac{1}{2}+\frac{499}{1000}\right]+\left[\frac{1}{2}+\frac{500}{1000}\right]+......\left[\frac{1}{2}+\frac{1499}{1000}\right]+\left[\frac{1}{2}+\frac{1500}{1000}\right]+......$$

$$+\left[\frac{1}{2}+\frac{2499}{1000}\right]+\left[\frac{1}{2}+\frac{2500}{1000}\right]+......\left[\frac{1}{2}+\frac{2946}{1000}\right]$$

$= 0 + 1 \quad 1000 + 2 \quad 1000 + 3 \quad 447 = 3000 + 1341 = 4341$ **Ans.**

Illustration 7 : Find the domain $f(x) = \dfrac{1}{\sqrt{|[|x|-5]|-11}}$ where [.] denotes greatest integer function.

Solution : $|[|x|-5]| > 11$

so $[|x|-5] > 11$ or $[|x|-5] < -11$

$[|x|] > 16$ $\quad$ $[|x|] < -6$

$|x| \ge 17$ or $|x| < -6$ (Not Possible)

$\Rightarrow$ $x \le -17$ or $x \ge 17$

so $x \in (-\infty, -17] \cup [17, \infty)$

Illustration 8 : Find the range of $f(x) = \dfrac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Solution : $y = \dfrac{x-[x]}{1+x-[x]} = \dfrac{\{x\}}{1+\{x\}}$

$\therefore \quad \dfrac{1}{y} = \dfrac{1}{\{x\}} + 1 \Rightarrow \dfrac{1}{\{x\}} = \dfrac{1-y}{y} \quad \Rightarrow \quad \{x\} = \dfrac{y}{1-y}$

$0 \le \{x\} < 1 \Rightarrow 0 \le \dfrac{y}{1-y} < 1$

Range = $[0, 1/2)$

Illustration 9 : Solve the equation $|2x - 1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and { . } denotes fractional part function.

Solution : We are given that, $|2x - 1| = 3[x] + 2\{x\}$

Let, $2x - 1 \le 0$ i.e. $x \le \dfrac{1}{2}$. The given equation yields.

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow \quad 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \quad \Rightarrow \quad 1 - 5[x] = 4\{x\} \Rightarrow \{x\} = \dfrac{1-5[x]}{4}$

$\Rightarrow \quad 0 \le \dfrac{1-5[x]}{4} < 1 \Rightarrow 0 \le 1 - 5[x] < 4 \Rightarrow -\dfrac{3}{5} < [x] \le \dfrac{1}{5}$

Now, [x] = 0 as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

$\Rightarrow \quad \{x\} = \frac{1}{4} \Rightarrow x = \frac{1}{4}$ which is less than $\frac{1}{2}$, Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \frac{1}{2}$

$\Rightarrow \quad 2x - 1 = 3[x] + 2\{x\} \quad \Rightarrow \quad 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow \quad [x] = -1 \Rightarrow -1 \le x < 0$ which is not a solution as $x > \frac{1}{2}$

$\Rightarrow \quad x = \frac{1}{4}$ is the only solution.

Do yourself - 3 :

(i) Let {x} & [x] denotes the fraction and integral part of a real number x respectively, then match the column.

	Column-I		Column-II
(A)	$[x^2] > 3$	(p)	$x \in [2, 4)$
(B)	$[x]^2 - 5[x] + 6 = 0$	(q)	$x \in (-\infty, -2] \cup [2, \infty)$
(C)	$x = \{x\}$	(r)	$x \in \{0\}$
(D)	$\{x\} = [x]$	(s)	$x \in (-\infty, -5)$
(E)	$[x] < -5.2$	(t)	$x \in \{-2\}$
(F)	$1 + x = \text{sgn}(x)$	(u)	$x \in [0, 1)$

6. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, f + g, f − g, (f. g) & (f/g) as follows :

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x \mid g(x) = 0\}$

7. EQUAL OR IDENTICAL FUNCTION :

Two function f & g are said to be equal if :

(a) The domain of f = the domain of g

(b) The range of f = range of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

e.g. $f(x) = \frac{1}{x}$ & $g(x) = \frac{x}{x^2}$ are identical functions.

Illustration 10 : The functions $f(x) = \log(x - 1) - \log(x - 2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) [1, 2] (B) $[2, \infty)$ (C) $(2, \infty)$ (D) $(-\infty, \infty)$

Solution : Since $f(x) = \log(x - 1) - \log(x - 2)$.

Domain of f(x) is $x > 2$ or $x \in (2, \infty)$ .....(i)

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined if $\frac{x-1}{x-2} > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ .....(ii)

From (i) and (ii), $x \in (2, \infty)$. **Ans. (C)**

Do yourself - 4 :

(i) Are the following functions identical ?

(a) $f(x) = \dfrac{x}{x^2}$ & $\phi(x) = \dfrac{x^2}{x}$ (b) $f(x) = x$ & $\phi(x) = \sqrt{x^2}$ (c) $f(x) = \log_{10}x^2$ & $\phi(x) = 2\log_{10}|x|$

8. CLASSIFICATION OF FUNCTIONS :

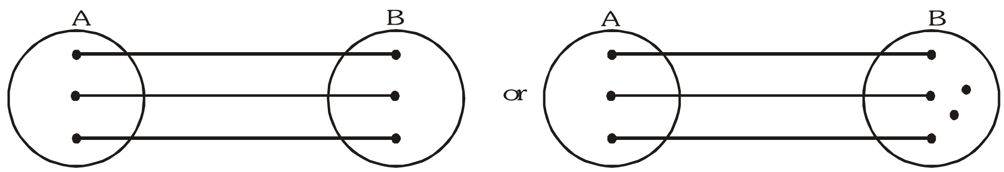
(a) One-One function (Injective mapping) :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Note: (i) Any continuous function which is entirely increasing or decreasing in whole domain is one-one.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

Diagramatically an injective mapping can be shown

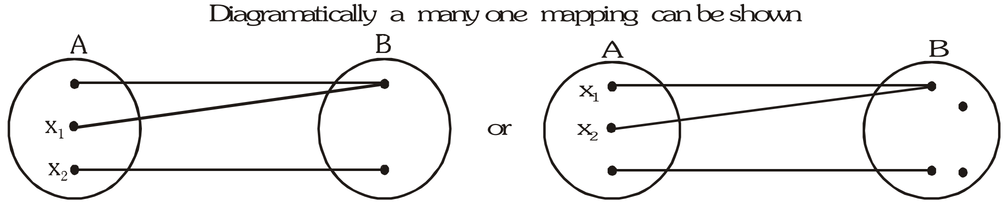


(b) Many-one function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one if $\exists\ x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Diagramatically a many one mapping can be shown

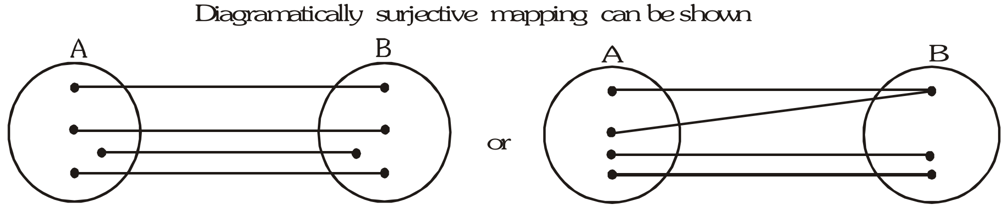


Note : If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

(c) Onto function (Surjective mapping) :

If the function $f : A \to B$ is such that each element in B (co-domain) is the 'f' image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective if $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$

Diagramatically surjective mapping can be shown

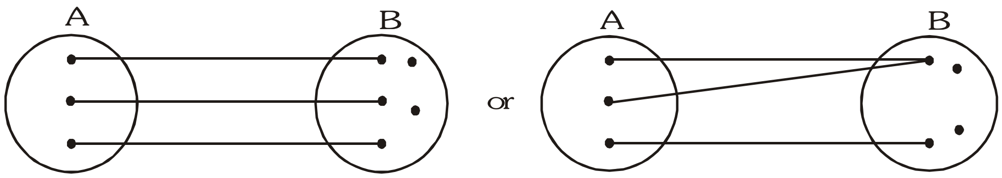


Note that : If range = co-domain, then f(x) is onto.

(d) Into function :

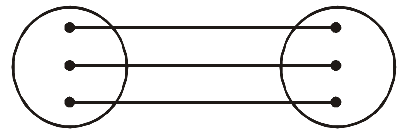
If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Diagramatically into function can be shown

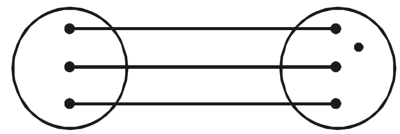


Thus a function can be one of these four types :

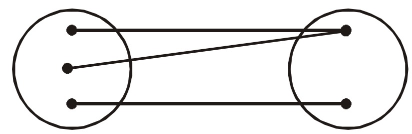
(i) one-one onto (injective & surjective)



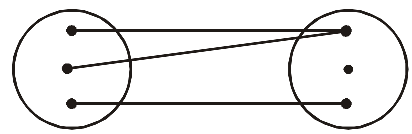
(ii) one-one into (injective but not surjective)



(iii) many-one onto (surjective but not injective)



(iv) many-one into (neither surjective nor injective)



Note : (i) If 'f' is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one and rest are many one.

(iii) $f : R \to R$ is a polynomial

(a) Of even degree, then it will neither be injective nor surjective.

(b) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

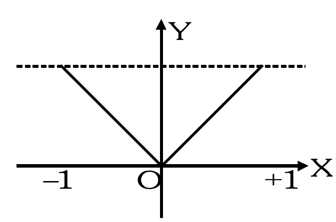
Illustration11 : Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f : A \to B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(a) $f(x) = |x|$ (b) $f(x) = x|x|$ (c) $f(x) = x^3$

(d) $f(x) = [x]$ (e) $f(x) = \sin\dfrac{\pi x}{2}$

Solution : (a) $f(x) = |x|$

Graphically ;



Which shows many one, as the straight line is parallel to x-axis and cuts at two points. Here range for $f(x) \in [0, 1]$

Which is clearly subset of co-domain i.e., $[0, 1] \subseteq [-1,1]$

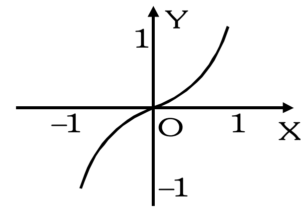
Thus, into.

Hence, function is many-one-into

$\therefore$ Neither injective nor surjective

(b) $f(x) = x|x| = \begin{Bmatrix} -x^2, & -1 < x < 0 \\ x^2, & 0 \le x < 1 \end{Bmatrix}$,

Graphically,



The graph shows f(x) is one-one, as the straight line parallel to x-axis cuts only at one point.

Here, range

$f(x) \in [-1, 1]$

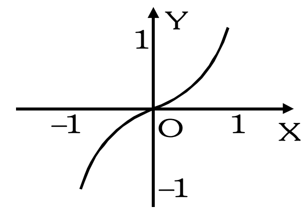
Thus, range = co-domain

Hence, onto.

Therefore, f(x) is one-one onto or (Bijective).

(c) $f(x) = x^3$,

Graphically;



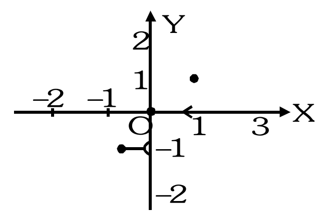
Graph shows f(x) is one-one onto

(i.e. Bijective)

[as explained in above example]

(d) $f(x) = [x]$,

Graphically;



Which shows f(x) is many-one, as the straight line parallel to x-axis meets at more than one point.

Here, range

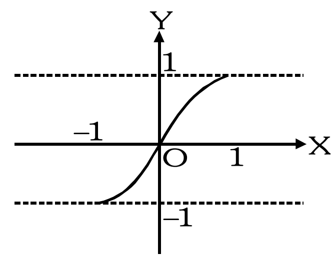
$f(x) \in \{-1, 0, 1\}$

which shows into as range $\subseteq$ co-domain

Hence, many-one-into

(e) $f(x) = \sin\frac{\pi x}{2}$

Graphically;



Which shows f(x) is one-one and onto as range = co-domain.

Therefore, f(x) is bijective.

Illustration 12 : Let $f : R \to R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then f is

(A) injective (B) surjective (C) bijective (D) None of these

Solution : We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \neq -2$

Also, f is not onto as $f(x) \geq 0 \ \forall \ x \in R$

$\therefore$ range of f = $(0, \infty) \subset R$ **Ans.(D)**

Illustration 13 : Let $f(x) = \frac{x^2 + 3x + a}{x^2 + x + 1}$, where $f : R \to R$. Find the value of parameter 'a' so that the given function is one-one.

Solution : $f(x) = \frac{x^2 + 3x + a}{x^2 + x + 1}$

$$f'(x) = \frac{(x^2 + x + 1)(2x + 3) - (x^2 + 3x + a)(2x + 1)}{(x^2 + x + 1)^2} = \frac{-2x^2 + 2x(1 - a) + (3 - a)}{(x^2 + x + 1)^2}$$

Let, $g(x) = -2x^2 + 2x(1 - a) + (3 - a)$

g(x) will be negative if $4(1 - a)^2 + 8(3 - a) < 0$

$\Rightarrow 1 + a^2 - 2a + 6 - 2a < 0 \Rightarrow (a - 2)^2 + 3 < 0$

which is not possible. Therefore function is not monotonic.

Hence, no value of a is possible.

Do yourself - 5 :

(i) Is the function $f : N \to N$ (the set of natural numbers) defined by $f(x) = 2x + 3$ surjective ?

(ii) Let $A = R - \{3\}$, $B = R - \{1\}$ and let $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$. Check whether the function f(x) is bijective or not.

(iii) The function $f : R \to R$, where R is the set of all real numbers defined by $f(x) = 2x + 3$ is -

(A) f is both one-one and onto

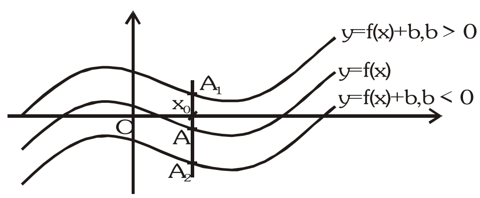
(B) f is one-one but not onto

(C) f is onto but not one-one

(D) f is neither one-one nor onto

9. BASIC TRANSFORMATIONS ON GRAPHS :

(i) Drawing the graph of $y = f(x) + b$, $b \in R$, from the known graph of $y = f(x)$



It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x). $y|_{x=x_0} = f(x_0)$.

The corresponding point on f(x) + b would be $f(x_0) + b$.

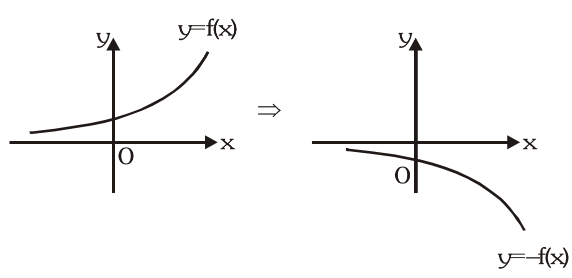
For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

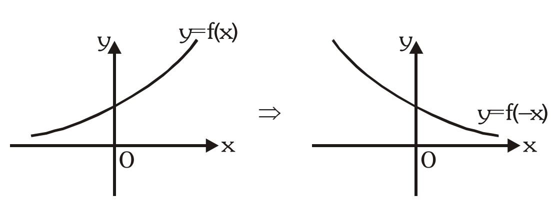
(ii) Drawing the graph of $y = -f(x)$ from the known graph of $y = f(x)$

To draw $y = -f(x)$, take the image of the curve $y = f(x)$ in the x-axis as plane mirror.



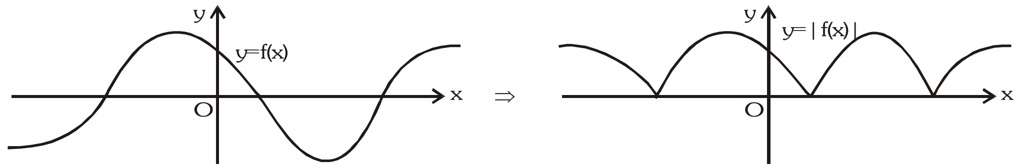
(iii) Drawing the graph of $y = f(-x)$ from the known graph of $y = f(x)$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y-axis as plane mirror.



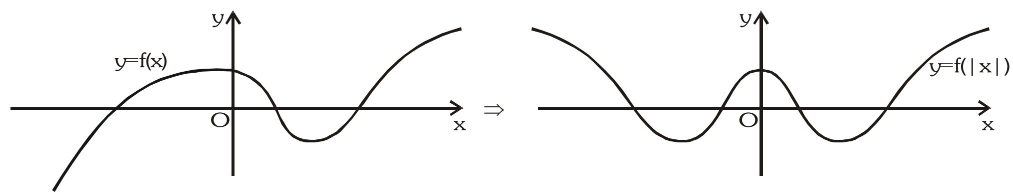
(iv) Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of f(x) and |f(x)| would coincide if $f(x) \geq 0$ and for the portions where $f(x) < 0$ graph of |f(x)| would be image of y = f(x) in x-axis.



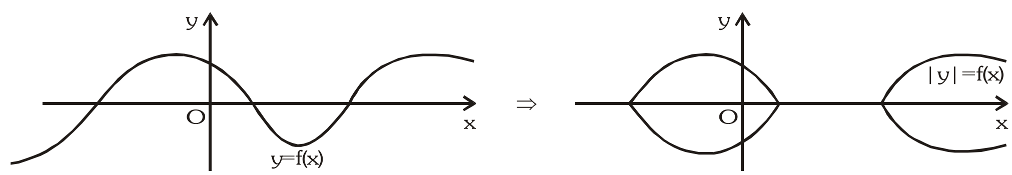
(v) Drawing the graph of $y = f(|x|)$ from the known graph of $y = f(x)$

It is clear that, $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$. Thus f(|x|) would be a even function, graph of f(|x|) and f(x) would be identical in the first and the fourth quadrants (ax $x \geq 0$) and as such the graph of f(|x|) would be symmetric about the y-axis (as (|x|) is even).

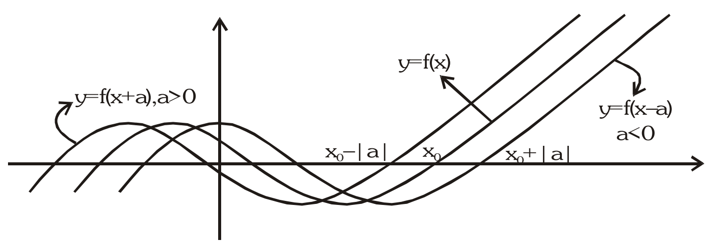


(vi) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where f(x) is non-negative and will be reflected about the x-axis only in those regions.



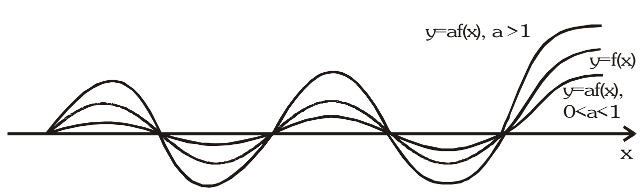
(vii) Drawing the graph of $y = f(x + a)$, $a \in R$ from the known graph of $y = f(x)$



(i) If $a > 0$, shift the graph of f(x) through 'a' units towards left of f(x).

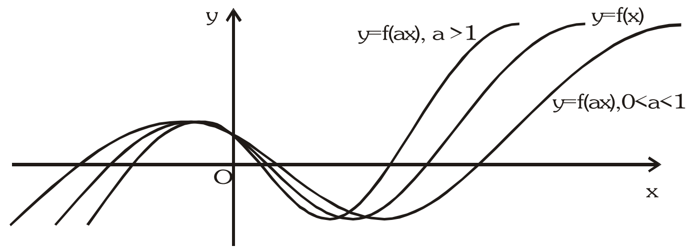
(ii) If $a < 0$, shift the graph of f(x) through 'a' units towards right of f(x).

(viii) Drawing the graph of $y = af(x)$ from the known graph of $y = f(x)$



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

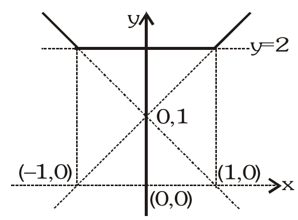
(ix) Drawing the graph of y = f(ax) from the known graph of y = f(x).



Let us take any point $x_0 \in$ domain of f(x). Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and f (x) will stretch by $\frac{1}{a}$ units along the y-axis and if $a > 1$, $x < x_0$, then f(x) will compress by 'a' units along the y-axis.

Illustration 14 : Find f(x) = max {1 + x, 1 − x, 2}.



Solution : From the graph it is clear that

$$f(x) = \begin{cases} 1-x\ ; & x < -1 \\ 2\ ; & -1 \le x \le 1 \\ 1+x\ ; & x > 1 \end{cases}$$

Illustration 15 : Draw the graph of y = |2 −|x −1||.

Solution :

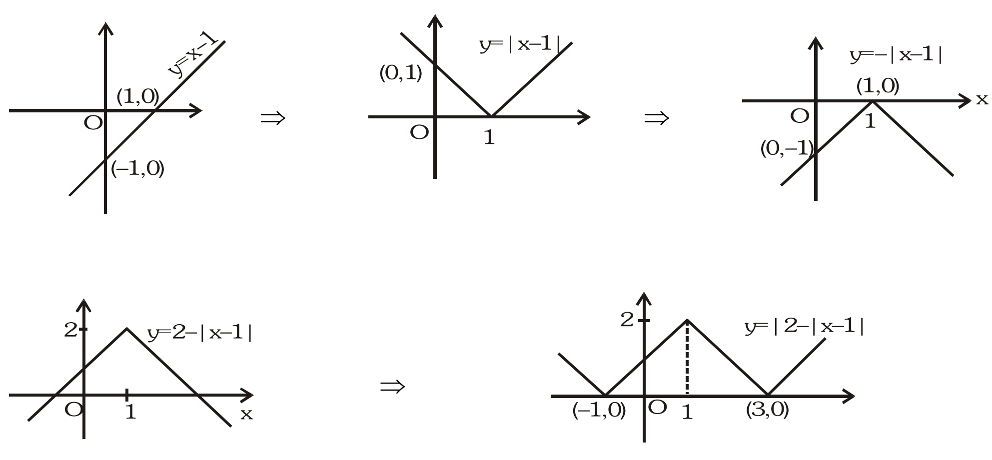
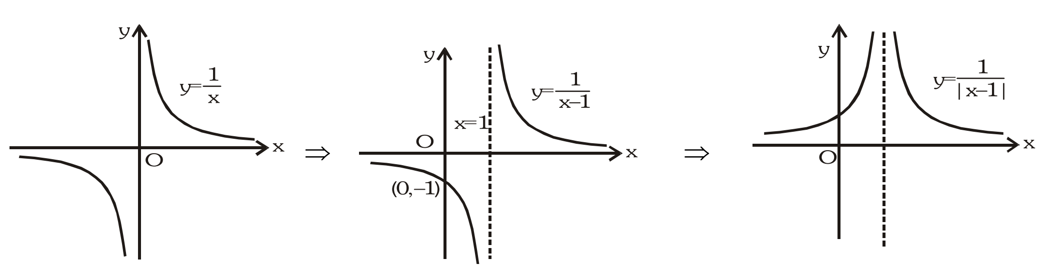


Illustration 16 : Draw the graph of $y = 2 - \frac{4}{|x-1|}$

Solution :



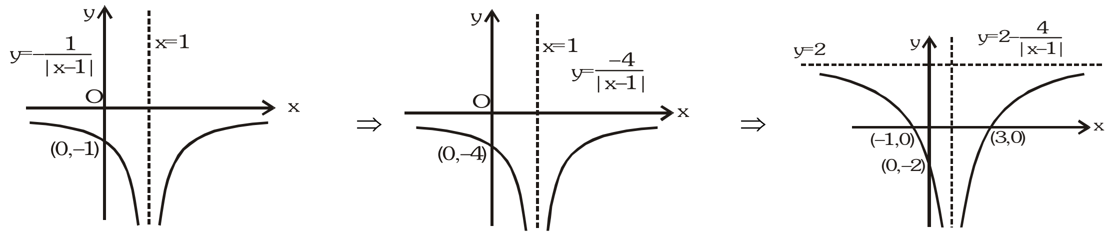
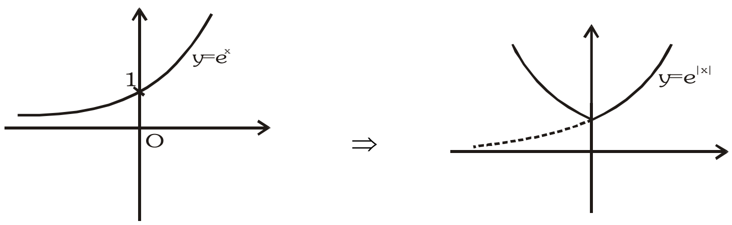


Illustration 17 : Draw the graph of $y = |e^{|x|} - 2|$

Solution :



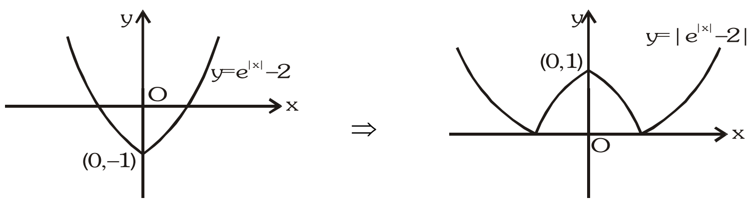
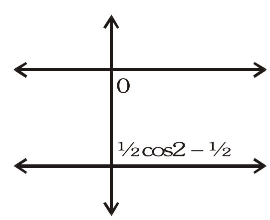


Illustration 18 : Draw the graph of $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$.

Solution : $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$

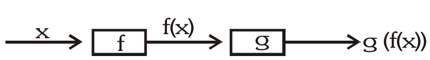
$$= \frac{1}{2}\left[\cos(2x+2) + \cos 2\right] - \frac{1}{2}\left[\cos(2x+2) + 1\right]$$

$$= \frac{1}{2}\cos 2 - \frac{1}{2} < 0 \ .$$



10. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION:

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f & g.

Diagramatically 

Thus the image of every $x \in A$ under the function gof is the g-image of f-image of x.

Note that gof is defined only if $\forall \ x \in A$, f(x) is an element of the domain of 'g' so that we can take its g-image. Hence in gof(x) the range of 'f' must be a subset of the domain of 'g'.

Properties of composite functions:

(a) In general composite of functions is not commutative i.e. $gof \neq fog$.

(b) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(c) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Illustration 19 : If f be the greatest integer function and g be the modulus function, then $(gof)\left(-\frac{5}{3}\right) - (fog)\left(-\frac{5}{3}\right) =$

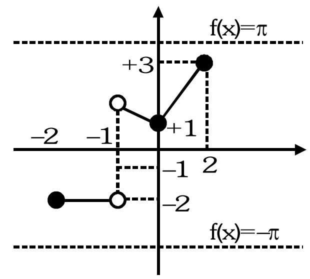
(A) 1 (B) –1 (C) 2 (D) 4

Solution : Given $(gof)\left(\frac{-5}{3}\right) - (fog)\left(\frac{-5}{3}\right) = g\left\{f\left(\frac{-5}{3}\right)\right\} - f\left\{g\left(\frac{-5}{3}\right)\right\} = g(-2) - f\left(\frac{5}{3}\right) = 2 - 1 = 1$ **Ans.(A)**

Illustration 20 : Find the domain and range of h(x) = g(f(x)), where

$f(x) = \begin{cases} [x], & -2 \le x \le -1 \\ |x|+1, & -1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} [x], & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, [.] denotes the greatest integer function.

Solution : $h(x) = g(f(x)) = \begin{cases} [f(x)], & -\pi \le f(x) < 0 \\ \sin(f(x)), & 0 \le f(x) \le \pi \end{cases}$



From graph of f(x), we get

$h(x) = \begin{cases} [[x]], & -2 \le x \le -1 \\ \sin(|x|+1), & -1 < x \le 2 \end{cases}$

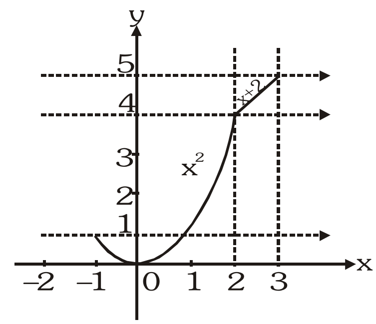
$\Rightarrow$ Domain of h(x) is [–2, 2]

and Range of h(x) is {–2, 1} ∪ [sin3, 1]

Illustration 21 : Let $f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$, find (fog)

Solution : $f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Here, g(x) becomes the variable that means we should draw the graph.



It is clear that g(x) ≤ 1 ; ∀ x ∈ [–1, 1]

and 1 < g(x) ≤ 2 ; ∀ x ∈ (1, $\sqrt{2}$]

$\Rightarrow f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$

Do yourself - 6 :

(i) $f(x) = x^3 - x$ & g(x) = sin 2x, find

(a) f(f(1)) (b) f(f(–1)) (c) $f\left(g\left(\frac{\pi}{2}\right)\right)$

(d) $f\left(g\left(\frac{\pi}{4}\right)\right)$ (e) g(f(1)) (f) $g\left(g\left(\frac{\pi}{2}\right)\right)$

(ii) If $f(x) = \begin{cases} x+1; & 0 \le x < 2 \\ |x|; & 2 \le x < 3 \end{cases}$, then find fof(x).

11. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$ then f(x, y) is homogeneous function of degree n.

Illustration 22 : Which of the following function is not homogeneous ?

(A) $x^3 + 8x^2y + 7y^3$ (B) $y^2 + x^2 + 5xy$ (C) $\frac{xy}{x^2+y^2}$ (D) $\frac{2x-y+1}{2y-x+1}$

Solution : It is clear that (D) does not have the same degree in each term. **Ans. (D)**

---

12. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.

Do yourself - 7 :

(i) Find the boundness of the function $f(x) = \frac{x^2}{x^4+1}$

13. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

Illustration 23 : Which of the following function is implicit function ?

(A) $y = \frac{x^2+e^x+5}{\sqrt{1-\cos^{-1}x}}$ (B) $y = x^2$ (C) $xy - \sin(x+y) = 0$ (D) $y = \frac{x^2\log x}{\sin x}$

Solution : It is clear that in (C) y is not clearly expressed in x. **Ans. (C)**

Do yourself - 8 :

(i) Which of the following function is implicit function ?

(A) $xy - \cos(x+y) = 0$ (B) $y = x^3$

(C) $y = \log(x^2 + x + 1)$ (D) $y = |x|$

(ii) Convert the implicit form into the explicit function :

(a) $xy = 1$ (b) $x^2y = 1$.

14. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A \ \& \ y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) | (x, f(x)) \in f\}$.

Properties of inverse function :

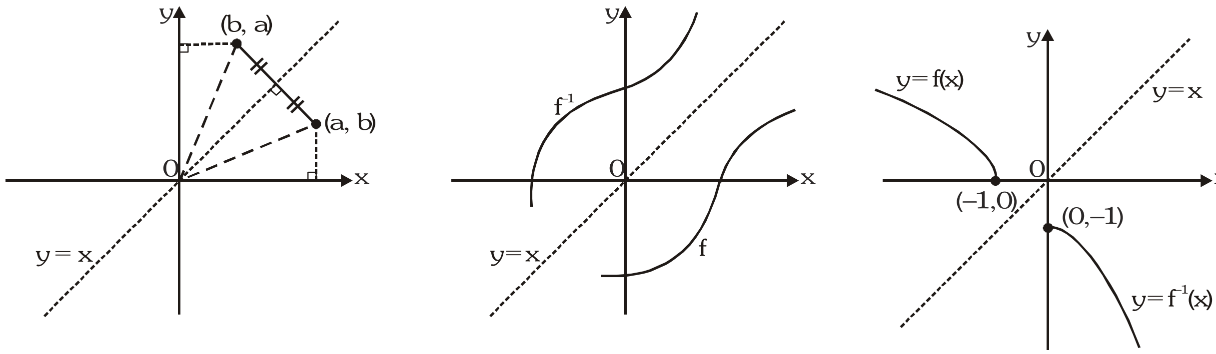
(a) The inverse of a bijection is unique.

(b) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then fog $= I_B$ and gof $= I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If fof = I, then f is inverse of itself.

(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

(e) Since f(a) = b if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line y = x.



The graph of f^{-1} is obtained by reflecting the graph of f about the line y = x.

Drawing the graph of $y = f^{-1}(x)$ from the known graph of y = f(x)

For drawing the graph of $y = f^{-1}(x)$ we have to first of all find the interval in which the function is bijective (invertible). Then take the reflection of y = f(x) (within the invertible region) about the line y = x. The reflected part would give us the graph of $y = f^{-1}(x)$.

e.g. let us draw the graph of $y = \sin^{-1}x$. We know that y = f(x) = sinx is invertible if f : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\Rightarrow$ the inverse mapping would be $f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

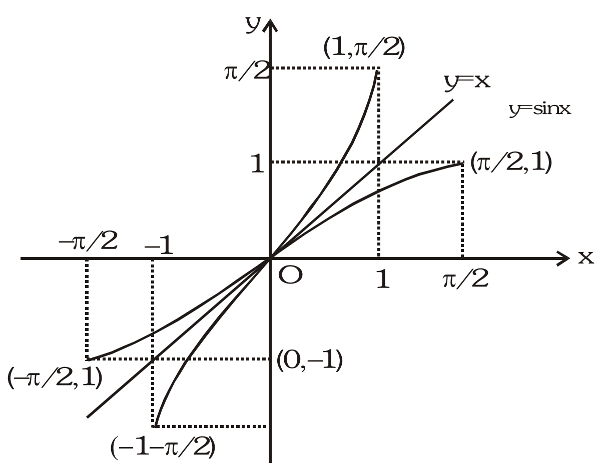


Illustration 24 : Let f : R → R be defined by $f(x) = (e^x - e^{-x})/2$. Is f(x) invertible ? If so, find its inverse.

Solution : Let us check for invertibility of f(x) :

(a) One-One :

Let $x_1, x_2 \in R$ and $x_1 < x_2$

$\Rightarrow$ $e^{x_1} < e^{x_2}$ (Because base e > 1) .... (i)

Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$

$\Rightarrow$ $e^{-x_2} < e^{-x_1}$ (Because base e > 1) .... (ii)

(i) + (ii) $\Rightarrow$ $e^{x_1} + e^{-x_2} < e^{x_2} + e^{-x_1}$

$\Rightarrow$ $\frac{1}{2}\left(e^{x_1} - e^{-x_1}\right) < \frac{1}{2}\left(e^{x_2} - e^{-x_2}\right)$ $\Rightarrow f(x_1) < f(x_2)$ i.e. f is one-one.

(b) Onto :

As x tends to larger and larger values so does f(x) and

when $x \to \infty$, $f(x) \to \infty$.

Similarly as $x \to -\infty$, $f(x) \to -\infty$ i.e. $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore f(x) is onto.

Since f(x) is both one-one and onto, f(x) is invertible.

(c) To find f^{-1} :

Let f^{-1} be the inverse function of f, then by rule of identity $fof^{-1}(x) = x$

$$\frac{e^{f^{-1}(x)} - e^{-f^{-1}(x)}}{2} = x \Rightarrow e^{2f^{-1}(x)} - 2xe^{f^{-1}(x)} - 1 = 0$$

$$\Rightarrow \quad e^{f^{-1}(x)} = \frac{2x \pm \sqrt{4x^2+4}}{2} \Rightarrow e^{f^{-1}(x)} = x \pm \sqrt{1+x^2}$$

Since $e^{f^{-1}(x)} > 0$, hence negative sign is ruled out and

Hence $e^{f^{-1}(x)} = x + \sqrt{1+x^2}$

Taking logarithm, we have $f^{-1}(x) = \ell n(x + \sqrt{1+x^2})$.

Illustration 25 : Find the inverse of the function $f(x) = \begin{cases} x; < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x}; \ x > 4 \end{cases}$

Solution : Given $f(x) = \begin{cases} x; < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x}; \ x > 4 \end{cases}$

Let $f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$

$$\therefore \quad x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ \dfrac{y^2}{64}, & \dfrac{y^2}{64} > 4 \end{cases}$$

$$\Rightarrow \quad f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ \dfrac{y^2}{64}, & y > 16 \end{cases}$$

Hence $f^{-1}(x) = \begin{cases} x; < 1 \\ \sqrt{x}; 1 \le x \le 16 \\ \dfrac{x^2}{64}; x > 16 \end{cases}$ **Ans.**

Do yourself - 9 :

(i) Let $f : [-1, 1] \to [-1, 1]$ defined by $f(x) = x|x|$, find $f^{-1}(x)$.

15. ODD & EVEN FUNCTIONS :

If a function is such that whenever 'x' is in it's domain '−x' is also in it's domain & it satisfies

f(−x) = f(x) it is an even function

f(−x) = −f(x) it is an odd function

Note :

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many − one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '−x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(v) The only function which is defined on the entire number line & even and odd at the same time is f(x) = 0

f(x)	g(x)	f (x) + g(x)	f(x) − g(x)	f (x) . g(x)	f (x)/g(x)	(go$f$) (x)	($f$og)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustration 26 : Which of the following functions is (are) even, odd or neither :

(i) $f(x) = x^2\sin x$ (ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$ (iii) $f(x) = \log\left(\frac{1-x}{1+x}\right)$

(iv) f(x) = sinx − cosx (v) $f(x) = \frac{e^x+e^{-x}}{2}$

Solution :

(i) $f(-x) = (-x)^2\sin(-x) = -x^2\sin x = -f(x)$. Hence f(x) is odd.

(ii) $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$

$= \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$. Hence f(x) is odd.

(iii) $f(-x) = \log\left(\frac{1-(-x)}{1+(-x)}\right) = \log\left(\frac{1+x}{1-x}\right) = -f(x)$. Hence f(x) is odd

(iv) f(−x) = sin(−x) − cos(−x) = −sinx − cosx.

Hence f(x) is neither even nor odd.

(v) $f(-x) = \frac{e^{-x}+e^{-(-x)}}{2} = \frac{e^{-x}+e^{x}}{2} = f(x)$. Hence f(x) is even

Illustration 27 : Identify the given functions as odd, even or neither :

(i) $f(x) = \frac{x}{e^x-1} + \frac{x}{2} + 1$ (ii) f(x + y) = f(x) + f(y) for all x, y ∈ R

Solution : (i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$

Clearly domain of f(x) is R ~ {0}. We have,

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2} + 1 = \frac{-e^x.x}{1-e^x} - \frac{x}{2} + 1 = \frac{(e^x - 1 + 1)x}{(e^x - 1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1 = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Replacing x, y by zero, we get $f(0) = 2f(0) \Rightarrow f(0) = 0$

Replacing y by –x, we get $f(x) + f(-x) = f(0) = 0 \Rightarrow f(x) = -f(-x)$

Hence f(x) is an odd function.

Do yourself - 10 :

(i) Which of the following functions is (are) even, odd or neither :

(a) $f(x) = x^3 \sin 3x$ (b) $f(x) = \dfrac{e^{x^2} + e^{-x^2}}{2x}$

(c) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (d) $f(x) = x^2 + 2^x$

16. PERIODIC FUNCTION :

A function f(x) is called periodic if there exists a least positive number T(T >0) called the period of the function such that f(x + T) = f(x), for all values of x within the domain of f(x).

e.g. The function sinx & cosx both are periodic over 2π & tan x is periodic over π.

Note : For periodic function :

(i) f(T) = f(0) = f(–T), where 'T' is the period.

(ii) Inverse of a periodic function does not exist.

(iii) Every constant function is periodic, but its period is not defined.

(iv) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. f(x) = |sin x| + |cos x|.

(v) If f(x) has period p, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(vi) If f(x) has period T then f(ax + b) has a period T/|a| (a ≠ 0).

Illustration 28 : Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$ (ii) $f(x) = x - [x - b],\ b \in R$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$ (iv) $f(x) = \tan \dfrac{\pi}{2}[x]$

(v) $f(x) = \cos(\sin x) + \cos(\cos x)$ (vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\text{cosec}\, x)}$

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi x|}$

Solution : (i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \text{cosec}(3x - 5)$

Period of $e^{\ell n \sin x} = 2\pi$, $\tan^3 x = \pi$

$\text{cosec}\ (3x - 5) = \dfrac{2\pi}{3}$

∴ Period = 2π

(ii) $f(x) = x - [x - b] = b + \{x - b\}$

$\therefore$ Period = 1

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\dfrac{\pi}{2}$. Hence f(x) is periodic with π as its period

(iv) $f(x) = \tan\dfrac{\pi}{2}[x]$

$\tan\dfrac{\pi}{2}[x + T] = \tan\dfrac{\pi}{2}[x] \Rightarrow \dfrac{\pi}{2}[x + T] = n\pi + \dfrac{\pi}{2}[x]$

$\therefore$ T = 2

$\therefore$ Period = 2

(v) Let f(x) is periodic then $f(x + T) = f(x)$

$\Rightarrow$ $\cos(\sin(x + T)) + \cos(\cos(x + T)) = \cos(\sin x) + \cos(\cos x)$

If x = 0 then $\cos(\sin T) + \cos(\cos T)$

$= \cos(0) + \cos(1) = \cos\left(\cos\dfrac{\pi}{2}\right) + \cos\left(\sin\dfrac{\pi}{2}\right)$

On comparing $T = \dfrac{\pi}{2}$

(vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)} = \dfrac{(1+\sin x)(1+\cos x)\sin x}{\cos x(1+\sin x)(1+\cos x)}$

$\Rightarrow$ $f(x) = \tan x$

Hence f(x) has period π.

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Period of $x - [x] = 1$

Period of $|\cos\pi x| = 1$

Period of $|\cos 2\pi x| = \dfrac{1}{2}$

....

Period of $|\cos n\pi x| = \dfrac{1}{n}$

So period of f(x) will be L.C.M. of all period = 1

Illustration 29 : Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$ (ii) $f(x) = \sin\dfrac{\pi x}{\sqrt{2}} + \cos\dfrac{\pi x}{\sqrt{3}}$ (iii) $f(x) = \sin\dfrac{\pi x}{\sqrt{3}} + \cos\dfrac{\pi x}{2\sqrt{3}}$

Solution :

(i) Period of $e^{x-[x]} = 1$

period of $\sin x = 2\pi$

$\because$ L.C.M. of rational and an irrational number does not exist.

$\therefore$ not periodic.

(ii) Period of $\sin\dfrac{\pi x}{\sqrt{2}} = \dfrac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\dfrac{\pi x}{\sqrt{3}} = \dfrac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ L.C.M. of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\frac{\pi x}{\sqrt{3}} = \frac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\frac{\pi x}{2\sqrt{3}} = \frac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

$\because$ L.C.M. of two similar irrational number exist.

$\therefore$ Periodic with period = $4\sqrt{3}$ **Ans.**

Do yourself - 11 :

(i) Find the periods (if periodic) of the following functions.

(a) $f(x) = 2^{\ell n(\cos x)} + \tan^3 x$.

(b) $f(x) = \sin(\sin x) + \sin(\cos x)$

(c) $f(x) = e^{x-[x]}$, [.] denotes greatest integer function

(d) $f(x) = \left|\sin\frac{x}{2}\right| + \left|\cos\frac{x}{2}\right|$

17. GENERAL :

If x, y are independent variables, then :

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\, x$

(b) $f(xy) = f(x)\,.\,f(y) \Rightarrow f(x) = x^n,\ n \in R$ or $f(x) = 0$

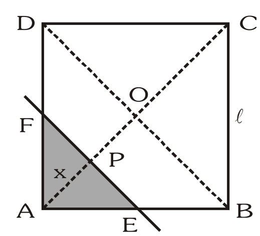
(c) $f(x + y) = f(x)\,.\,f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Miscellaneous Illustration :

Illustration 30 : ABCD is a square of side ℓ. A line parallel to the diagonal BD at a distance 'x' from the vertex A cuts two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x. Find this area at $x = 1/\sqrt{2}$ and at x = 2, when $\ell = 2$.

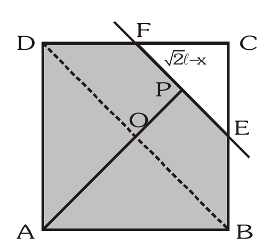
Solution : There are two different situations



Case-I : when x = AP ≤ OA, i.e., $x \le \frac{\ell}{\sqrt{2}}$

$$ar(\Delta AEF) = \frac{1}{2}x.2x = x^2 \qquad (\because\ PE = PF = AP = x)$$

Case-II : when x = AP > OA, i.e., $x > \frac{\ell}{\sqrt{2}}$ but $x \le \sqrt{2}\ell$



ar(ABEFDA) = ar(ABCD) – ar(ΔCFE)

$$= \ell^2 - \frac{1}{2}\left(\sqrt{2}\ell - x\right).2\left(\sqrt{2}\ell - x\right) \qquad [\because\ CP = \sqrt{2}\ell - x]$$

$$= \ell^2 - \left(2\ell^2 + x^2 - 2\sqrt{2}\ell x\right) = 2\sqrt{2}\ell x - x^2 - \ell^2$$

$\therefore$ the required function s(x) is as follows :

$$s(x) = \begin{cases} x^2, & 0 \le x \le \frac{\ell}{\sqrt{2}} \\ 2\sqrt{2}\,\ell x - x^2 - \ell^2, & \frac{\ell}{\sqrt{2}} < x \le \sqrt{2}\ell \end{cases} \ ; \quad \text{area of } s(x) = \begin{cases} \frac{1}{2} & \text{at } x = \frac{1}{\sqrt{2}} \\ 8(\sqrt{2}-1) & \text{at } x = 2 \end{cases}$$ **Ans.**

Illustration 31 : If the function f(x) satisfies the functional rule, f(x + y) = f(x) + f(y) ∀ x,y ∈ R & f(1) = 5, then find $\sum_{n=1}^{m} f(n)$ and also prove that f(x) is odd function.

Solution : Here, $f(x + y) = f(x) + f(y)$; put $x = t - 1$, $y = 1$

$f(t) = f(t - 1) + f(1)$ ....(1)

$\therefore$ $f(t) = f(t - 1) + 5$

$\Rightarrow$ $f(t) = \{f(t - 2) + 5\} + 5$

$\Rightarrow$ $f(t) = f(t - 2) + 2(5)$

$\Rightarrow$ $f(t) = f(t - 3) + 3(5)$

....

....

$\Rightarrow$ $f(t) = f\{t - (t - 1)\} + (t - 1)5$

$\Rightarrow$ $f(t) = f(1) + (t - 1)5$

$\Rightarrow$ $f(t) = 5 + (t - 1)5$

$\Rightarrow$ $f(t) = 5t$

$$\therefore \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m}(5n) = 5[1 + 2 + 3 + \ldots.. + m] = \frac{5m(m+1)}{2}$$

Hence, $\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$ .....(i)

Now putting x=0, y=0 in the given function, we have

$f(0 + 0) = f(0) + f(0)$

$\therefore$ $f(0) = 0$

Also putting (–x) for (y) in the given function.

$f(x - x) = f(x) + f(-x)$

$\Rightarrow$ $f(0) = f(x) + f(-x)$

$\Rightarrow$ $0 = f(x) + f(-x)$

$\Rightarrow$ $f(-x) = -f(x)$ ....(ii)

Thus, $\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$ and f(x) is odd function.

ANSWERS FOR DO YOURSELF

1 : (i) (a) $x \in (0, \infty)$ (b) $x \in (-\infty, 5/2]$ (c) $x \in (-\infty, 0) \cup (4, \infty)$ (d) $x \in (-\infty, 0)$

(ii) (a) $[-1, 1]$ (b) $\left[\frac{1}{4}, \frac{1}{2}\right]$ (c) $[0, \infty)$ (d) $[-1, 1]$ (e) $[-2, 2]$ (f) $\left[0, \frac{1}{2}\right]$

(iii) $x = \frac{a_1 + a_2 + \ldots + a_n}{n}$

2 : (i) $x \in (-\infty, -\sqrt{8}) \cup (\sqrt{8}, \infty)$

3 : (i) (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (u), (D) $\rightarrow$ (r), (E) $\rightarrow$ (s), (F) $\rightarrow$ (t)

4 : (i) (a) no (b) no (c) yes

5 : (i) not onto (ii) yes (iii) A

6 : (i) (a) 0 (b) 0 (c) 0 (d) 0 (e) 0 (f) 0 (ii) $\begin{cases} x+2, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ x, & 2 \le x < 3 \end{cases}$

7 : (i) $\left[0, \frac{1}{2}\right]$

8 : (i) A (ii) (a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x^2}$

9 : (i) $f^{-1}(x) = \begin{cases} -\sqrt{-x}, & -1 \le x \le 0 \\ \sqrt{x}, & 0 \le x \le 1 \end{cases}$

10 : (i) (a) even (b) odd (c) odd (d) neither

11 : (i) (a) 2π (b) 2π (c) 1 (d) π