ELLIPSE

EXERCISE - 01 CHECK YOUR GRASP

3. Given $\frac{x}{3}$ = cost + sint & $\frac{y}{4}$ = cost –sint

Squaring these two,

$\Rightarrow \frac{x^2}{9}$ =1 + 2costsint .....(i)

$\frac{y^2}{16}$ = 1 – 2 sint cost .....(ii)

Adding (i) & (ii)

$\frac{x^2}{9}+\frac{y^2}{16}=2 \quad \Rightarrow \frac{x^2}{18}+\frac{y^2}{32}=1$

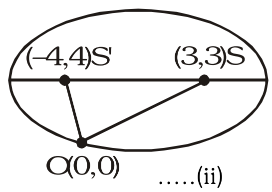
5. Here S is (3, 3) & S' is (–4, 4).

$\Rightarrow$ SS' = $\sqrt{50}$ = 2ae ..(i)

Now OS + OS' = 2a

$3\sqrt{2}+4\sqrt{2}=2a$

$7\sqrt{2}=2a$ .....(ii)



From (i) & (ii)

$e=\frac{5}{7}$

6. Since major axis is along y-axis.

$\therefore$ Equation of tangent is x = my + $\sqrt{b^2m^2+a^2}$

slope of tangent = $\frac{1}{m}=\frac{-4}{3} \Rightarrow m=\frac{-3}{4}$

Hence equation of tangent is 4x + 3y = 24

or $\frac{x}{6}+\frac{y}{8}=1$

Its intercepts on the axes are 6 and 8.

Area (ΔAOB) = $\frac{1}{2}$ 6 8 = 24 sq.unit.

7. Let any tangent of ellipse is

$\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1$

Let it meets axes at $A\left(\frac{4}{\cos\theta},0\right)$ & $B\left(0,\frac{3}{\sin\theta}\right)$

Let mid point of AB is (h, k) then

$2h=\frac{4}{\cos\theta}, \quad 2k=\frac{3}{\sin\theta}$

Since $\cos^2\theta + \sin^2\theta = 1$

$\therefore \frac{16}{4h^2}+\frac{9}{4k^2}=1$

$\Rightarrow 16k^2 + 9h^2 = 4h^2k^2$

Hence locus is $16y^2 + 9x^2 = 4x^2y^2$.

9. Let equations of tangent to the two ellipses are

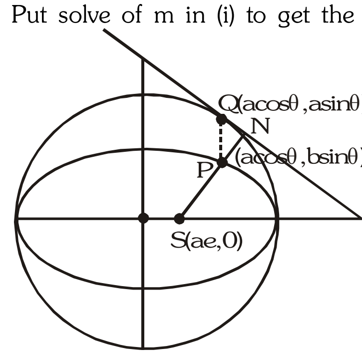
$y = mx \pm\sqrt{(a^2+b^2)m^2+b^2}$ ......(i)

$y = mx \pm\sqrt{a^2m^2+a^2+b^2}$ ......(ii)

On solving (i) and (ii) we get $m=\pm\frac{a}{b}$

Put solve of m in (i) to get the answer.

11.



Equation of tangent at P

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ ....(i)

Equation of tangent at Q

$x\cos\theta + y\sin\theta = a$

$\Rightarrow \frac{x\cos\theta}{a}+\frac{y}{a}\sin\theta=1$ ....(ii)

(i) – (ii)

$y\sin\theta\left(\frac{1}{b}-\frac{1}{a}\right)=0$

$\Rightarrow$ y = 0 [$\sin\theta \neq 0$; $a \neq b$]

13. Positive end of latus rectum is (ae, $\frac{b^2}{a}$)

$\therefore$ Equation of normal is

$\frac{a^2x}{ae}-\frac{b^2ay}{b^2}=a^2e^2$

$\Rightarrow x - ey - e^3a = 0$

15. Equation of normal at P ($3\cos\theta$, $\sin\theta$) is

$3x\sec\theta - y\operatorname{cosec}\theta=8$ .....(i)

Now equation of diameter through Q is

$3y\cos\theta + x\sin\theta = 0$ .....(ii)

Solving (i) & (ii) we get intersection point R,

$\left(\frac{12}{5}\cos\theta,\frac{-4}{5}\sin\theta\right)$

Let (h, k) be mid point of PR then

$2h = \frac{27}{5}\cos\theta,\ 2k = \frac{1}{5}\ \sin\theta.$

Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \frac{h^2}{(2.7)^2} + \frac{k^2}{(0.1)^2} = 1$

$\therefore$ Locus is ellipse.

17. $e = \sqrt{1 - \frac{3}{5}} = \sqrt{\frac{2}{5}}$

$\therefore\ S_1 = (\sqrt{2}\ ,\ 0),\ S_2 = (-\sqrt{2}\ ,\ 0)$

Equation of tangent is $y = mx + \sqrt{5m^2 + 3}$

$S_1F_1 = \left|\frac{-\sqrt{2}m - \sqrt{5m^2 + 3}}{\sqrt{1 + m^2}}\right|$

$S_2F_2 = \left|\frac{\sqrt{2}m - \sqrt{5m^2 + 3}}{\sqrt{1 + m^2}}\right|$

Now $(S_1F_1)\ (S_2F_2) = \frac{5m^2 + 3 - 2m^2}{(1 + m^2)} = 3$.

20. Given slope of common tangent $m = \frac{1}{2}$.

Equation of general tangent to $y^2 = 4x$ is

$y = mx + \frac{1}{m}$ ......(i)

$\Rightarrow\ y = \frac{1}{2}x + 2 \quad \left[\because m = \frac{1}{2} \text{ in given equation}\right]$

On comparing with given equation,
we get $k = 4$

Equation of tangent of $\frac{x^2}{a^2} + \frac{y^2}{3} = 1$ is

$y = mx \pm \sqrt{a^2m^2 + 3}$ ......(ii)

On comparing (i) & (ii)

$\frac{1}{m} = \pm\sqrt{a^2m^2 + 3}$ ....(iii)

$\Rightarrow\ a^2 = 4 \ \Rightarrow\ a = \pm 2$ ....(iv)

Using (iii) & (iv) we get $m = \pm\frac{1}{2}$.

So equation of other common tangent is
$x + 2y + 4 = 0.$

EXERCISE - 02 BRAIN TEASERS

1. Equation of tangent of ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$ ....(i)

Given equation is $x - 2y + 4 = 0$ ....(ii)

Since (i) & (ii) are same, comparing them, we get

$m = \frac{1}{2}\ \&\ \sqrt{a^2m^2 + b^2} = 2$

$\Rightarrow 4.\frac{1}{4} + b^2 = 4$

$\Rightarrow b = \pm\sqrt{3}$

Equation of tangent of parabola

$y = mx + \frac{1}{m}$ ....(iii)

By (i) & (iii)

$\frac{1}{m^2} = a^2m^2 + b^2$

on solving it we get $m = \pm\ \frac{1}{2}$

with $m = -\frac{1}{2}$ we get $x + 2y + 4 = 0$

which is other equation of common tangent.

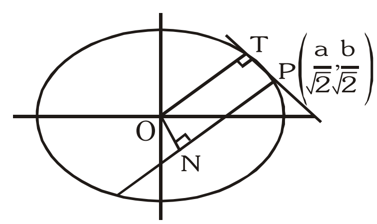
3. Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of tangent at P $\left(a\cos\frac{\pi}{4},\ b\sin\frac{\pi}{4}\right)$ is

$\frac{x}{a} + \frac{y}{b} = \sqrt{2}$

Equation of normal at P is

$\sqrt{2}ax - \sqrt{2}by = a^2 - b^2$



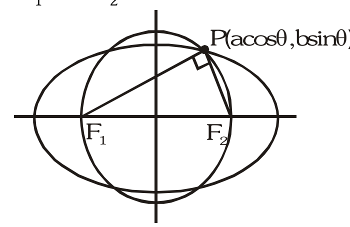
Now $OT = \left|\frac{-\sqrt{2}ab}{\sqrt{a^2 + b^2}}\right|$

and $ON = \left|\frac{-(a^2 - b^2)}{\sqrt{2}\sqrt{a^2 + b^2}}\right|$

Area of rectangle $= OT.\ ON = \frac{(a^2 - b^2)ab}{a^2 + b^2}$

6. Co-ordinate of point $P(a\cos\theta,\ b\sin\theta)$.

Also, $PF_1 + PF_2 = 17$ .....(i)



Given $\frac{1}{2}PF_1.PF_2 = 30 \ \Rightarrow PF_1.PF_2 = 60$ .....(ii)

From (i) & (ii) $PF_1 = 5\ \&\ PF_2 = 12$

$\therefore\ (F_1F_2)^2 = (PF_1)^2 + (PF_2)^2$

$= 5^2 + 12^2 \quad \Rightarrow F_1F_2 = 13$

7. Equation of normal at P is

$ax\sec\theta - by\,\text{cosec}\theta = a^2 - b^2$ ....(i)

$Q \equiv \left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$, $R \equiv \left(0, -\frac{a^2-b^2}{b}\sin\theta\right)$

Let middle point of QR be S(h,k).

$2h = \frac{a^2-b^2}{a}.\cos\theta$; $2k = -\frac{a^2-b^2}{b}.\sin\theta$

$2h = ae^2\cos\theta$ $\quad 2k = -\frac{a^2e^2}{b}\sin\theta$

$\cos\theta = \frac{2h}{ae^2}$ ....(ii) $\quad \sin\theta = \frac{-2bk}{a^2e^2}$ ....(iii)

Square & add (ii) & (iii),

$\frac{4h^2}{a^2e^4} + \frac{4b^2k^2}{a^4e^4} = 1$

$\Rightarrow \frac{x^2}{\left(\frac{ae^2}{2}\right)^2} + \frac{y^2}{\left(\frac{a^2e^2}{2b}\right)^2} = 1 \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$

where , $A = \frac{ae^2}{2}$ & $B = \frac{a^2e^2}{2b} = \frac{ae^2}{2}.\frac{a}{b}$

B > A

$e' = 1 - \frac{A^2}{B^2} = 1 - \frac{a^2e^4.4b^2}{4.a^4e^4}$

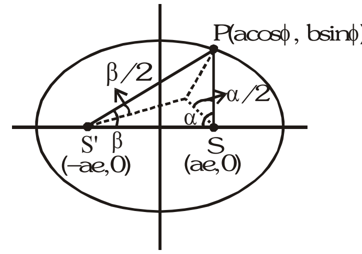
$= 1 - \frac{b^2}{a^2} = e^2 \Rightarrow e' = e$

10. By definition of ellipse

PS + PS' = 2a if a > b

PS + PS' = 2b if a < b

and SS' = 2ae



Now by sine rule in $\Delta PSS'$

$\frac{SP}{\sin\beta} = \frac{S'P}{\sin\alpha} = \frac{SS'}{\sin[\pi - (\alpha+\beta)]}$

or $\frac{SP + S'P}{\sin\beta + \sin\alpha} = \frac{SS'}{\sin(\alpha+\beta)}$

or $\frac{2a}{\sin\beta + \sin\alpha} = \frac{2ae}{\sin(\alpha+\beta)}$

or $\frac{1}{e} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$

$\Rightarrow \frac{1-e}{1+e} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ [By C & D]

14. (a) $x_2 = x_1r$, $x_3 = x_1r^2$ and so

$y_2 = y_1r$, $y_3 = y_1r^2$

$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_1r & y_1r & 1 \\ x_1r^2 & y_1r^2 & 1 \end{vmatrix}$

$[R_3 \to R_3 - rR_2, R_2 \to R_2 - rR_1]$

$= \begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1-r \\ 0 & 0 & 1-r \end{vmatrix} = 0$

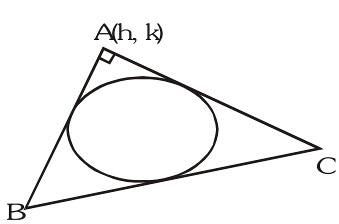
Hence points lie on a line i.e. they are collinear.

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

Fill in the blanks :

2. From the definition of director circle, locus of point is the director circle of the ellipse,

i.e. $x^2 + y^2 = a^2 + b^2$



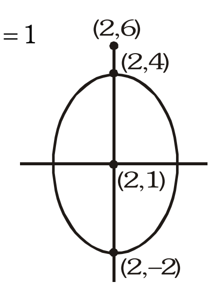
Match the column :

1. (A) $9(x^2 - 4x + 4) + 8(y^2 - 2y + 1) = 28 + 36 + 8$

$\Rightarrow 9(x-2)^2 + 8(y-1)^2 = 72$

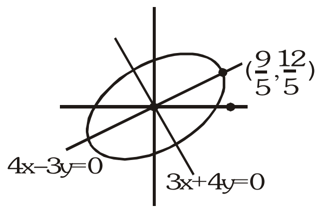
$\Rightarrow \frac{(x-2)^2}{8} + \frac{(y-1)^2}{9} = 1$

Minimum distance of (2, 6) from the ellipse is 2 & maximum distance of (2, 6) from the ellipse is 8



(B) $\dfrac{(3x+4y)^2}{225}+\dfrac{(4x-3y)^2}{100}=1$

$$\frac{\left(\frac{3x+4y}{5}\right)^2}{9}+\frac{\left(\frac{4x-3y}{5}\right)^2}{4}=1$$



Point $\left(\frac{9}{5},\frac{12}{5}\right)$ lie on line $4x - 3y = 0$

$\therefore$ Minimum distance = 0

Maximum distance = 6

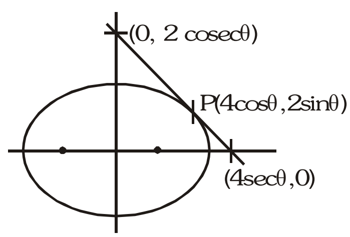
(C) $C_1 : x^2 + y^2 = 3$

$C_2 : x^2 + y^2=6$

$C_3 : x^2 + y^2=12$

GM of 3, 6, 12 is $(3.6.12)^{1/3} = 6$

(D) $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$



equation of tangent at P is

$$\frac{x\cos\theta}{4}+\frac{y\sin\theta}{2}=1$$

$x\cos\theta + 2y\sin\theta = 4$

Area of triangle

$$\Delta = \frac{1}{2}.4\sec\theta.2\,\text{cosec}\theta=\frac{8}{\sin 2\theta}$$

minimum area = 8 when $\sin 2\theta = 1$

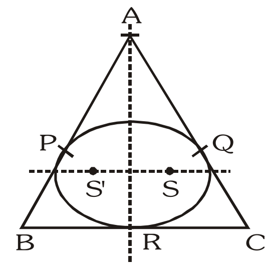
Assertion & Reason :

3. $x^2 + y^2 + xy = 1$

Replacing x by –x & y by –y we get the same equation.

$\therefore$ Centre of conic is (0, 0) and every chord passing through the centre is bisected by the point. Hence st. I & st. II both are true & st I explains st. II.

Comprehension # 1



$2ae = 4$

$2a = 6$

$e = 2/3$

$b^2= a^2(1-e^2)$

$= 9(1-\frac{4}{9})$

$= 5$

$\therefore$ Equation of ellipse is $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$

1. If $\angle BAC = 90$ then locus of A is the director circle of the ellipse.

$x^2 + y^2 = 14$

2. Let A be (h, k) then chord of contact PQ is

$$\frac{hx}{9}+\frac{ky}{5}=1$$

Homogenizing the equation of ellipse

$$\frac{x^2}{9}+\frac{y^2}{5}=\left(\frac{hx}{9}+\frac{ky}{5}\right)^2$$

$$x^2\left(\frac{h^2}{81}-\frac{1}{9}\right)+y^2\left(\frac{k^2}{25}-\frac{1}{5}\right)+\frac{2hk}{45}xy=0$$

coefficient of x^2 + coefficient of $y^2 = 0$

$$\frac{h^2}{81}-\frac{1}{9}+\frac{k^2}{25}-\frac{1}{5}=0 \Rightarrow 25x^2 + 81y^2 = 630$$

3. Chord of contact of A(h,k) is

$$\frac{hx}{9}+\frac{ky}{5}=1 \qquad ......(1)$$

$$\frac{x}{3}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{\sqrt{5}}.\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right) \qquad .....(2)$$

Comparing (1) & (2)

$$\frac{h}{3\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sqrt{5}\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{1}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\frac{h}{3\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sqrt{5}\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)=\frac{h}{2\sqrt{3}} \quad ; \quad \sin\left(\frac{\alpha+\beta}{2}\right)=\frac{\sqrt{3}k}{2\sqrt{5}}$$

$$\Rightarrow \frac{x^2}{12}+\frac{3y^2}{20}=1 \qquad \Rightarrow 5x^2 + 9y^2= 60$$

7. $\frac{x^2}{25}+\frac{y^2}{16}=1$

Since point $(7-\frac{5}{4}\alpha,\ \alpha)$ lies inside the ellipse,

$\therefore\ S_1 < 0$

$\Rightarrow 16\ (7-\frac{5}{4}\alpha)^2 + 25.\alpha^2 < 400$

$\Rightarrow (28-5\alpha)^2 + 25\alpha^2 < 400$

$\Rightarrow 50\alpha^2 - 280\alpha + 384 < 0$

$\Rightarrow 25\alpha^2 - 140\alpha + 192 < 0$

$\Rightarrow \alpha \in \left(\frac{12}{5},\frac{16}{5}\right)$

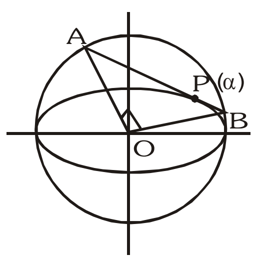
10. Equation of auxiliary circle is $x^2 + y^2 = a^2$ ....(1)

Equation of tangent at point P $(a\cos\alpha,\ b\sin\alpha)$

is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$ ....(2)

Equation of pair of lines OA, OB is obtained by homogenous equation of (1) with the help of (2)

$\therefore\quad x^2 + y^2 = a^2\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)^2$



$\Rightarrow (1-\cos^2\alpha)x^2 - \frac{2xy\,a\sin\alpha\cos\alpha}{b} + y^2\left(1-\frac{a^2}{b^2}\sin\alpha\right)$

But $\angle$ AOB = 90

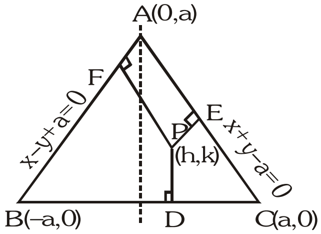
$\therefore$ Coeff. of x^2 + coeff. of y^2 = 0

$1-\cos^2\alpha + 1 - \frac{a^2}{b^2}\sin^2\alpha = 0$

$\Rightarrow\quad 1=\frac{a^2-b^2}{b^2}\sin^2\alpha \quad \Rightarrow 1=\frac{a^2e^2}{a^2(1-e^2)}\sin^2\alpha$

$\Rightarrow\quad e = (1+\sin^2\alpha)^{-1/2}$

12.



$(PD)^2 = \frac{1}{2}PE.PF$

$k^2 = \frac{1}{2}\left|\frac{h+k-a}{\sqrt{2}}\right|\left|\frac{h-k+a}{\sqrt{2}}\right|$

$4k^2 = -(h+k-a)(h-k+a)$

$4k^2 = -\{h^2-(k-a)^2\}$

$4k^2 = -\{h^2-k^2+2ak-a^2\}$

$h^2+3k^2+2ak-a^2=0$

$\therefore$ Locus of (h, k) is

$x^2 + 3y^2 + 2ay - a^2 = 0$

$x^2 + 3(y^2 + \frac{2}{3}ay + \frac{1}{9}a^2) = a^2 + \frac{a^2}{3}$

$x^2 + 3(y + \frac{1}{3}a)^2 = \frac{4a^2}{3}$

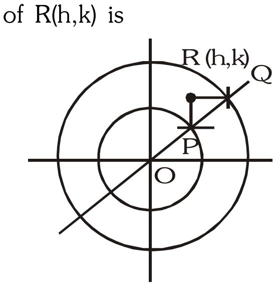
$\frac{x^2}{\frac{4a^2}{3}}+\frac{(y+\frac{1}{3}a)^2}{\frac{4a^2}{9}}=1$

$\therefore\ e = \sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{1}{3}}=\sqrt{\frac{2}{3}}$

13. $P(a\cos\theta, a\sin\theta)$ & $Q(b\cos\theta, b\sin\theta)$

$h = a\cos\theta,\ k = b\sin\theta$

$\therefore$ locus of R(h,k) is



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$b > a$

Foci (0, be) lies on inner circle, then $b^2e^2 = a^2$

$\Rightarrow e^2 = \frac{a^2}{b^2}$

$e^2 = 1-\frac{a^2}{b^2}$

foci lie in the inner circle then

$\frac{a^2}{b^2}=1-\frac{a^2}{b^2}$ [a = be]

$\frac{a^2}{b^2} = \frac{1}{2} \Rightarrow \frac{a}{b} = \frac{1}{\sqrt{2}} = e$

14. Tangent at point $(t^2, 2t)$ on parabola $y^2 = 4x$ is

$ty = x + t^2$ .....(i)

Normal at $(\sqrt{5}\cos\phi, 2\sin\phi)$ on ellipse $4x^2+5y^2=20$

is $\sqrt{5}\,x\sec\phi - 2y\,\text{cosec}\phi = 1$ .....(ii)

(i) & (ii) are same lines, hence by comparing

$$\frac{-\sqrt{5}}{\cos\phi} = \frac{-2}{t\sin\phi} = \frac{1}{t^2}$$

$\Rightarrow \cos\phi = -\sqrt{5}\,t^2$ ..... (iii)

$\sin\phi = \frac{-2t^2}{t}$ ......(iv)

Square & add (iii) & (iv) we get

$t = \pm \frac{1}{\sqrt{5}}, \ t = 0$

when $t = \frac{-1}{\sqrt{5}}$, $\tan\phi = -2 \Rightarrow \phi = \pi - \tan^{-1}2$

$t = \frac{1}{\sqrt{5}}$, $\tan\phi = 2 \Rightarrow \phi = \pi + \tan^{-1}2$

$t = 0$, $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

15. Let point is P $(2\cos\theta, \sin\theta)$.

Equation of tangent is $\frac{x}{2}\cos\theta + \frac{y\sin\theta}{1} = 1$

Equation of normal is $2x\sec\theta - y\,\text{cosec}\theta = 3$

Now tangent and normal meet major axis at

$Q\left(\frac{2}{\cos\theta}, 0\right)$ and $R\left(\frac{3}{2}\cos\theta, 0\right)$ respectively

Given QR = 2

$\Rightarrow \left|\frac{2}{\cos\theta} - \frac{3}{2}\cos\theta\right| = 2$

$\Rightarrow 3|\cos\theta|^2 + 4|\cos\theta| - 4 = 0$

$\Rightarrow |\cos\theta| = \frac{2}{3}, -2$ (reject)

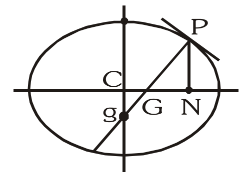
$\Rightarrow \cos\theta = \pm\left(\frac{2}{3}\right)$

16. Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let point is P $(a\cos\theta, b\sin\theta)$.

Equation of normal at P is

$ax\sec\theta - by\,\text{cosec}\theta = a^2 - b^2$



$$\Rightarrow \frac{x}{\frac{(a^2-b^2)\cos\theta}{a}} + \frac{y}{\frac{-(a^2-b^2)\sin\theta}{b}} = 1$$

It meet major and minor axis at

$G\left(\frac{(a^2-b^2)}{a}\cos\theta, 0\right)$ and $g\left(0, \frac{-(a^2-b^2)}{b}\sin\theta\right)$

respectively.

$\therefore (CG)^2 = \left(\frac{a^2-b^2}{a}\right)^2 \cos^2\theta$

and $(Cg)^2 = \left(\frac{a^2-b^2}{b}\right)^2 \sin^2\theta$

$\therefore a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2$

PN is ordinate,

$\therefore$ coordinate of $N(a\cos\theta, 0)$.

$e^2CN = \left(\frac{a^2-b^2}{a^2}\right) a\cos\theta = \left(\frac{a^2-b^2}{a}\right)\cos\theta = CG$

17. For point P, x-coordinate = 3

Given ellipse $9x^2 + 25y^2 = 225$

$9(3)^2 + 25y^2 = 225$

$y = \pm\frac{12}{5}$

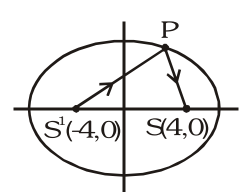
Coordinate of P is $\left(3, \pm\frac{12}{5}\right)$

Now $e = \frac{4}{5}$ & $ae = 4$

so foci is $(\pm 4, 0)$

Now equation of reflected ray (PS) is

$12x + 5y = 48$ or $12x - 5y = 48$



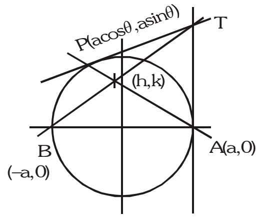
EXERCISE - 04[B] BRAIN STORMING SUBJECTIVE EXERCISE

1. Equation of tangent at P

$x\cos\theta + y\sin\theta = a$

which meets x = a at T

$\therefore$ T (a, a tan $\theta/2$)



Equation of AP $\rightarrow$ $y = -\cot(\theta/2)(x - a)$ ....(1)

Equation of BT $\rightarrow$ $y = \dfrac{\tan(\theta/2)}{2}(x + a)$ ....(2)

From (1) & (2)

$y^2 = -\dfrac{1}{2}(x^2 - a^2)$

$x^2 + 2y^2 = a^2$

5. $A(ae + r_1\cos\theta,\ r_1\sin\theta)$

$B(ae - r_2\cos\theta,\ -r_2\sin\theta)$

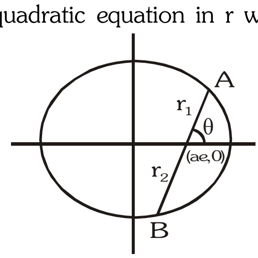
Both points lie on the ellipse

$\therefore \dfrac{(ae + r\cos\theta)^2}{a^2} + \dfrac{r^2\sin^2\theta}{b^2} = 1$

$b^2a^2e^2 + 2ab^2 er\cos\theta + b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$

$r^2(b^2\cos^2\theta + a^2\sin^2\theta) + 2ab^2e\cos\theta r + a^2b^2(e^2-1) = 0$

This is a quadratic equation in r with roots r_1 & $-r_2$.



$|r_1 + r_2| = \sqrt{(r_1 - r_2)^2 + 4r_1r_2}$

$$= \sqrt{\left(\frac{2ab^2e\cos\theta}{b^2\cos^2\theta + a^2\sin^2\theta}\right)^2 - 4.\frac{a^2b^2(e^2-1)}{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{\sqrt{4a^2b^4e^2\cos^2\theta - 4(b^2\cos^2\theta + a^2\sin^2\theta)a^2b^2(e^2-1)}}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{\sqrt{4a^2b^2[b^2e^2\cos^2\theta - (e^2-1)(b^2\cos^2\theta + a^2\sin^2\theta)}}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{2ab^2}{b^2\cos^2\theta + a^2\sin^2\theta} \quad \text{(by putting } e^2 = 1 - \frac{b^2}{a^2}\text{)}$$

9. Equation of tangent at point P $(a\cos\theta,\ b\sin\theta)$

on $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$

foci $F_1 \equiv (ae, 0)$, $F_2 = (-ae, 0)$

and $d = \dfrac{1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}} = \dfrac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

Now $4a^2\left(1 - \dfrac{b^2}{d^2}\right)$

$$= 4a^2\left(1 - \frac{b^2(a^2\sin^2\theta + b^2\cos^2\theta)}{a^2b^2}\right)$$

$$= 4a^2\left(1 - \sin^2\theta - \frac{b^2}{a^2}\cos^2\theta\right)$$

$= 4\cos^2\theta\,(a^2 - b^2)$

$= 4a^2e^2\cos^2\theta = (2ae\cos\theta)^2$

$= [(a - ae\cos\theta) - (a + ae\cos\theta)]^2$

$= (PF_1 - PF_2)^2$

EXERCISE - 05 [A] JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. Foci are $(\pm ae, 0)$. Therefore accoording to the condition, $2ae = 2b$ or $ae = b$

Also, $b^2 = a^2(1 - e^2) \Rightarrow e^2 = (1 - e^2) \Rightarrow e = \dfrac{1}{\sqrt{2}}$

2. Since directrix is parallel to y-axis, hence axes of the ellipse are parallel to x-axis.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (a > b)$$

$$e^2 = 1 - \frac{b^2}{a^2} \Rightarrow \frac{b^2}{a^2} = 1 - e^2 = 1 - \frac{1}{4} \Rightarrow \frac{b^2}{a^2} = \frac{3}{4}$$

Also, one of the directrices is $x = 4$

$$\Rightarrow \frac{a}{e} = 4 \Rightarrow a = 4e = 4.\frac{1}{2} = 2;$$

$$b^2 = \frac{3}{4}a^2 = \frac{3}{4}.4 = 3$$

$\therefore$ Required ellipse is $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

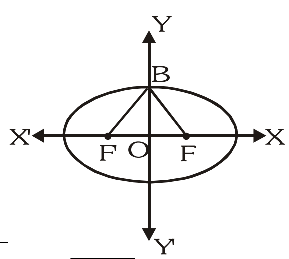
or $3x^2 + 4y^2 = 12$

4. $\angle F'BF = 90$, $F'B \perp FB$

i.e., slope of (F'B) Slope of (FB) = –1

$$\Rightarrow \frac{b}{ae}\ \frac{b}{-ae} = -1,$$

$b^2 = a^2e^2$ ..... (i)



We know that

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2e^2}{a^2}} = \sqrt{1 - e^2}$$

$$e^2 = 1 - e^2,\ 2e^2 = 1,\ e^2 = \frac{1}{2},\ e = \frac{1}{\sqrt{2}}$$

5. Distance between foci = 6 $\Rightarrow ae = 3$

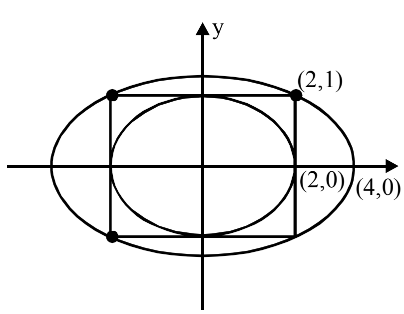
Minor axis = 8 $\Rightarrow 2b = 8 \Rightarrow b = 4 \Rightarrow b^2 = 16$

$\Rightarrow a^2(1 - e^2) = 16 \Rightarrow a^2 - a^2e^2 = 16$

$\Rightarrow a^2 - 9 = 16 \Rightarrow a = 5$

Hence $ae = 3 \Rightarrow e = \dfrac{3}{5}$

7.



Ellipse $x^2 + 4y^2 = 4$ (Given)

Eq^n. of the ellipse (required)

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$$

Ellipse passes through (2, 1)

therefore $\dfrac{4}{16} + \dfrac{1}{b^2} = 1 \Rightarrow b^2 = \dfrac{4}{3}$

$$\frac{x^2}{16} + \frac{y^2}{4/3} = 1 \Rightarrow \frac{x^2}{16} + \frac{3y^2}{4} = 1$$

$\Rightarrow x^2 + 12y^2 = 16$

9. Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

from the given conditions

$a = 4$ and $b = 2$

$\therefore$ Eq of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

or $x^2 + 4y^2 = 16$

10. Let equation of any tangent to $y^2 = 16\sqrt{3}\,x$

be $y = mx + \dfrac{4\sqrt{3}}{m}$ ......(i)

and equation of any tangent to $2x^2 + y^2 = 4$

be $y = mx + \sqrt{2m^2 + 4}$ ......(ii)

but (i) and (ii) are same lines

$$\therefore \frac{4\sqrt{3}}{m} = \sqrt{2m^2 + 4}$$

$\Rightarrow m^4 + 2m^2 - 24 = 0$

$\Rightarrow m^2 = -6, 4$

$\therefore m = \pm 2$

11. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{7}}{4}$$

foci $(\pm ae, 0) \equiv (\pm\sqrt{7}, 0)$

centre of circle is (0, 3)

$x^2 + y^2 - 6y + c = 0$

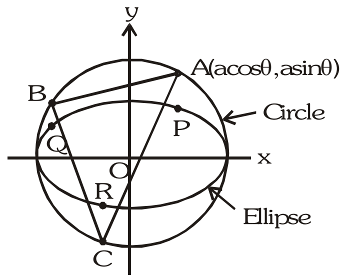
passes through $(\sqrt{7}, 0)$

$7 + 0 - 0 + c = 0$

$c = -7$

So $x^2 + y^2 - 6y - 7 = 0$

1. Let $A \equiv (a\cos\theta, a\sin\theta)$ so the coordinates of
$B \equiv (a\cos(\theta + 2\pi/3),\ a\sin(\theta + 2\pi/3))$
$C \equiv (a\cos(\theta + 4\pi/3),\ a\sin(\theta + 4\pi/3))$



According to the given condition, coordinates of P are $(a\cos\theta, b\sin\theta)$ and that of Q are $(a\cos(\theta+2\pi/3),\ b\sin(\theta + 2\pi/3))$ and that of R are $(a\cos(\theta + 4\pi/3),\ b\sin(\theta + 4\pi/3))$

(It is given that P, Q, R are on the same side of x-axis as A, B and C) Equation of the normal to the ellipse at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

or $ax\sin\theta - by\cos\theta = \frac{1}{2}(a^2 - b^2)\sin 2\theta$ ....(1)

Equation of normal to the ellipse at Q is

$$ax\sin\left(\theta + \frac{2\pi}{3}\right) - by\cos\left(\theta + \frac{2\pi}{3}\right) = \frac{1}{2}(a^2 - b^2)\sin\left(2\theta + \frac{4\pi}{3}\right) \quad ....(2)$$

Equation of normal to the ellipse at R is
$ax\sin(\theta + 4\pi/3) - by\cos(\theta + 4\pi/3)$
$= \frac{1}{2}(a^2 - b^2)\sin(2\theta + 8\pi/3)$ ...(3)

But $\sin(\theta+4\pi/3) = \sin(2\pi+\theta - 2\pi/3) = \sin(\theta-2\pi/3)$
and $\cos(\theta+4\pi/3) = \cos(2\pi+\theta - 2\pi/3) = \cos(\theta-2\pi/3)$
and $\sin(2\theta+8\pi/3) = \sin(4\pi+2\theta-4\pi/3) = \sin(2\theta-4\pi/3)$

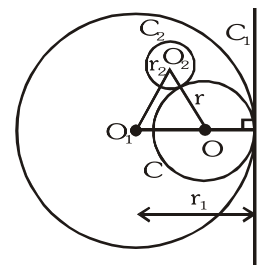
Now (3) can be written as
$ax\sin(\theta-2\pi/3) - by\cos(\theta-2\pi/3)$
$= \frac{1}{2}(a^2 - b^2)\sin(2\theta-4\pi/3)$ ...(4)

For the lines (1), (2) and (4) to be concurrent, we must have determinant.

$$\Delta_1 = \begin{vmatrix} a\sin\theta & -b\cos\theta & \frac{1}{2}(a^2-b^2)\sin 2\theta \\ a\sin\left(\theta + \frac{2\pi}{3}\right) & -b\cos\left(\theta + \frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin(2\theta + 4\pi/3) \\ a\sin\left(\theta - \frac{2\pi}{3}\right) & -b\cos\left(\theta - \frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin(2\theta - 4\pi/3) \end{vmatrix} = 0$$

Thus lines (1), (2) and (4) are concurrent.

2. Let the given circles C_1 and C_2 have centres O_1 and O_2 and radii r_1 and r_2 respectively.
Let the variable circle C touching C_1 internally, C_2 externally have a radius r and centre at O.
Now, $OO_2 = r + r_2$ and $OO_1 = r_1 - r$.



$\Rightarrow OO_1 + OO_2 = r_1 + r_2$
which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$
($\because C_2$ lies inside C_1)
$\Rightarrow$ Locus of O is an ellipse with foci O_1 and O_2.

4. Given tangent is drawn at $(3\sqrt{3}\cos\theta, \sin\theta)$ to $\frac{x^2}{27} + \frac{y^2}{1} = 1$

$\Rightarrow$ Equation of tangent is $\frac{x\cos\theta}{3\sqrt{3}} + \frac{y\sin\theta}{1} = 1$

Thus sum of intercepts $= (3\sqrt{3}\sec\theta + \operatorname{cosec}\theta) = f(\theta)$

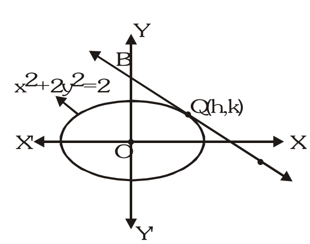
To minimise $f(\theta)$, $f'(\theta) = 0$

$$\Rightarrow f'(\theta) = \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta} = 0$$

$\Rightarrow \sin^3\theta = \frac{1}{3^{3/2}}\cos^3\theta$ or $\tan\theta = \frac{1}{\sqrt{3}}$, i.e. $\theta = \frac{\pi}{6}$

7. Let the point of contact be
$R \equiv (\sqrt{2}\cos\theta,\ \sin\theta)$
Equation of tangent AB is

$$\frac{x}{\sqrt{2}}\cos\theta + y\sin\theta = 1$$



$\Rightarrow$ A $\equiv$ ($\sqrt{2}\sec\theta$, 0); B $\equiv$ (0, $\text{cosec}\,\theta$)

Let the middle point Q of AB be (h, k)

$$\Rightarrow h = \frac{\sec\theta}{\sqrt{2}},\ k = \frac{\text{cosec}\,\theta}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{h\sqrt{2}},\ \sin\theta = \frac{1}{2k} \Rightarrow \frac{1}{2h^2}+\frac{1}{4k^2} = 1,$$

$$\therefore \text{ Required locus is } \frac{1}{2x^2} + \frac{1}{4y^2} = 1$$

Trick : The locus of mid-points of the portion of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepted between axes is $a^2y^2 + b^2x^2 = 4x^2y^2$

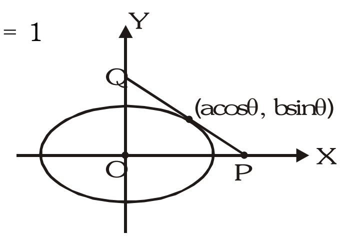
i.e., $\frac{a^2}{4x^2} + \frac{b^2}{4y^2} = 1$ or $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

8. Equation of tangent at (acosθ, bsinθ) is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$P = \left(\frac{a}{\cos\theta}, 0\right)$$

$$Q = \left(0, \frac{b}{\sin\theta}\right)$$



$$\text{Area of OPQ} = \frac{1}{2}\left|\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)\right| = \frac{ab}{|\sin 2\theta|}$$

$\therefore$ $(\text{Area})_{\min}$ = ab

10. Equation of ellipse is $\frac{x^2}{4}+y^2=1$

eccentricity $e = \frac{\sqrt{3}}{2}$

so focus are $(\sqrt{3},0)$ & $(-\sqrt{3},0)$

so end points of latus rectum will be

$$\left(\sqrt{3},\frac{1}{2}\right)\left(\sqrt{3},-1/2\right),\left(-\sqrt{3},1/2\right) \ \& \ \left(-\sqrt{3},-1/2\right)$$

$\because$ $y_1 < 0$ & $y_2 < 0$

Hence coordinates of P & Q will be

$$P\left(\sqrt{3},-\frac{1}{2}\right) \ \& \ Q\left(-\sqrt{3},-\frac{1}{2}\right).$$

So now equation of parabola taking these points as end points of latus rectum.

Focus will be (0, −1/2)

$$4a = 2\sqrt{3} \Rightarrow a = \frac{\sqrt{3}}{2}$$

Hence vertex of the parabolas will be

$$\left(0,-\frac{1}{2}+\frac{\sqrt{3}}{2}\right),\ \left(0,-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$$

so eq. of parabolas will be

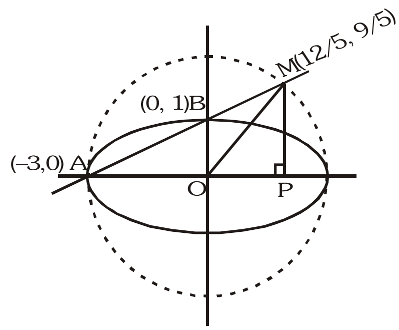
$$x^2 = -2\sqrt{3}\left(y+\frac{1}{2}-\frac{\sqrt{3}}{2}\right) \ \&$$

$$x^2 = 2\sqrt{3}\left(y+\frac{1}{2}+\frac{\sqrt{3}}{2}\right)$$

$$x^2 + 2\sqrt{3}y = 3-\sqrt{3}$$

$$x^2 - 2\sqrt{3}y = 3+\sqrt{3}$$

11. Area of triangle AOM = $\frac{1}{2}$ AO. PM



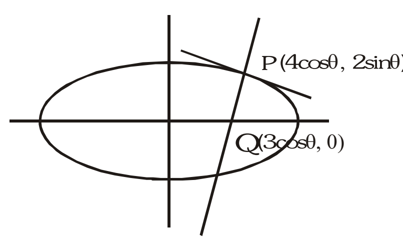
$\Rightarrow$ Eqaution of AM is $y = \frac{1}{3}(x + 3)$

x − 3y + 3 = 0 which is chord of auxiliary circle $x^2 + y^2 = 9$, and PM is ordinate of point M

$\Rightarrow (3y - 3)^2 + y^2 = 9 \Rightarrow y = \frac{9}{5}$ = PM $\Rightarrow$ Area of triangle $= \frac{1}{2} \cdot 3 \cdot \frac{9}{5} = \frac{27}{10}$.

12. $\frac{x^2}{16}+\frac{y^2}{4}=1$

4x secθ −2 y cosecθ = 12



x = 3cosθ

Q ≡ (3cosθ, 0)

2h = 7 cos θ

2k = 2 sinθ

$$\frac{4x^2}{49}+\frac{y^2}{1}=1 \qquad \ldots(i)$$

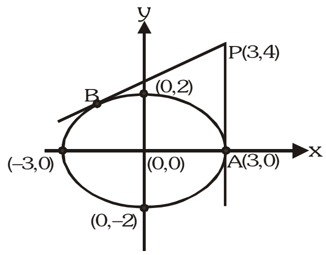
L.R $\Rightarrow x = 2\sqrt{3}$

Putting in (i)

$y = \pm \dfrac{1}{7}$ $\therefore$ $\left(\pm 2\sqrt{3}, \pm\dfrac{1}{7}\right)$

Paragraph for Question 13 to 15

13. As shown in figure, one of the point of contact is (3,0)



Let equation of other tangent,

$$y = mx + \sqrt{9m^2 + 4} \quad \text{as } c > 0$$

It passes through (3,4), so

$$4 = 3m + \sqrt{9m^2 + 4}$$

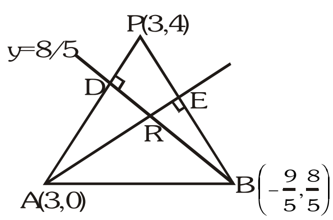
$$(4 - 3m)^2 = 9m^2 + 4$$

Solving, $m = \dfrac{1}{2}$

As we know that point of contact for the tangent given by $\left(-\dfrac{a^2m}{\sqrt{a^2m^2 + b^2}}, \dfrac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$

$\therefore$ Point of contact is $\left(-\dfrac{9}{5}, \dfrac{8}{5}\right)$

14. Equation of line BD : $y = \dfrac{8}{5}$



Equation of line AE : $2x + y = 6$

Now orthocentre R of ΔPAB will be intersection of line BD and line AE.

Solving for R, we get $R \equiv \left(\dfrac{11}{5}, \dfrac{8}{5}\right)$

15. Equation of line AB is $x + 3y = 3$

Now let the point be (h,k)

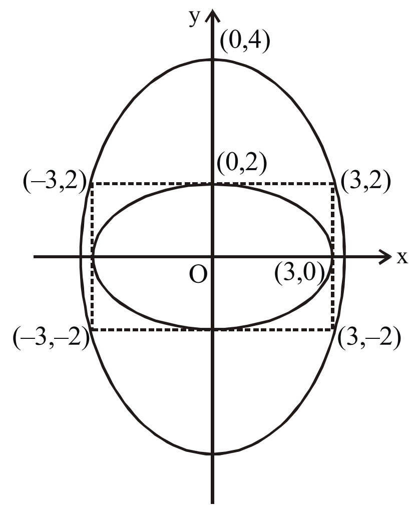
According to question,

$$\left|\frac{h + 3k - 3}{\sqrt{1^2 + 3^2}}\right| = \sqrt{(h-3)^2 + (4-k)^2}$$

After solving, we get

$9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

16. Let equation of E_2 be



$\dfrac{x^2}{a^2} + \dfrac{y^2}{16} = 1$ ($\because$ E_2 passes through (0, 4))

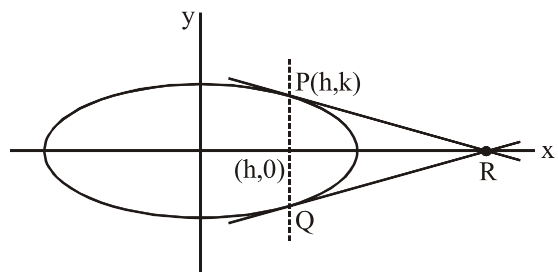
$\because$ E_2 passes through (3,2)

$\therefore$ $\dfrac{9}{a^2} + \dfrac{4}{16} = 1$

$\Rightarrow$ $a^2 = 12$

$\therefore$ $e^2 = 1 - \dfrac{a^2}{16} = 1 - \dfrac{3}{4} \Rightarrow e = \dfrac{1}{2}$

17.



Tangent at P(h, k) is $\dfrac{xh}{4} + \dfrac{ky}{3} = 1$

$\Rightarrow$ $R\left(\dfrac{4}{h}, 0\right)$

$\Delta PQR = k\left(\dfrac{4}{h} - h\right)$

$$= \sqrt{3\left(1 - \frac{h^2}{4}\right)}\left(\frac{4}{h} - h\right)$$

which is a decreasing function in $\left[\dfrac{1}{2}, 1\right]$

$\Rightarrow$ $\Delta_1 = \sqrt{3\left(1 - \dfrac{1}{16}\right)}\left(8 - \dfrac{1}{2}\right) = \dfrac{45\sqrt{5}}{8}$

& $\Delta_2 = \sqrt{3\left(1 - \dfrac{1}{4}\right)}(4 - 1) = \dfrac{9}{2}$

$$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$$