UNIT # 09

PARABOLA, ELLIPSE & HYPERBOLA

PARABOLA

EXERCISE - 01 CHECK YOUR GRASP

2. **Hint** : Distance between directrix and focus is 2a

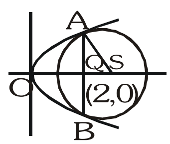
5. Given $(t^2, 2t)$ be one end of focal chord then other end be $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$

length of focal chord

$$= \sqrt{\left(t^2 - \frac{1}{t^2}\right)^2 + \left(2t + \frac{2}{t}\right)^2} = \left(t + \frac{1}{t}\right)^2$$

6. Focus of parabola $y^2 = 8x$ is (2, 0). Equation of circle with centre (2, 0) is

$(x - 2)^2 + y^2 = r^2$



AB is common chord

Q is mid point i.e. (1, 0)

$AQ^2 = y^2$ where $y^2 = 8 \quad 1 = 8$

$\therefore r^2 = AQ^2 + QS^2 = 8 + 1 = 9$

so circle is $(x - 2)^2 + y^2 = 9$

10. Since QR is focal chord so vertex of Q is $(at_1^2, 2at_1)$ and R is $(at_2^2, 2at_2)$

$$\text{area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix}$$

$$A = \frac{1}{2} \left| 2a^2t_1^2 t_2 - 2a^2t_1t_2^2 \right|$$

$$A = \frac{a}{2} \left| 2at_1 - 2at_2 \right| \qquad [t_1t_2 = -1]$$

13. Let the point be (h, k)

Now equation of tangent to the parabola $y^2 = 4ax$ whose slope is m is

$$y = mx + \frac{a}{m}$$

as it passes through (h, k)

$$\therefore k = mh + \frac{a}{m} \Rightarrow m^2h - mk + a = 0$$

It has two roots m_1, $2m_1$

$$\therefore m_1 + 2m_1 = \frac{k}{h},\ 2m_1.m_1 = \frac{a}{h}$$

$$m_1 = \frac{k}{3h} \quad \ldots (i)$$

$$m_1^2 = \frac{a}{2h} \quad \ldots (ii) \qquad \text{from (i) \& (ii)}$$

$$\Rightarrow \frac{k^2}{(3h)^2} = \frac{a}{2h} \Rightarrow k^2 = \frac{9a}{2}h$$

Thus locus of point is $y^2 = \frac{9}{2}ax$.

17. Let slope of tangent be m

So equation of tangent is

$$y = mx + \frac{1}{m}$$

Now tangent passes through (–1, 2) so

$\Rightarrow m^2 + 2m - 1 = 0$

$\Rightarrow m = -1 \pm \sqrt{2}$

equation of tangents are

$$y = (-1 + \sqrt{2})x + \frac{1}{-1 + \sqrt{2}} \qquad \ldots..(i)$$

$$y = (-1 - \sqrt{2})x - \frac{1}{1 + \sqrt{2}} \qquad \ldots\ldots(ii)$$

intercept of tangent (i) & (ii) on line x = 2 is

$y_1 = 3\sqrt{2} - 1$ & $y_2 = -3\sqrt{2} - 1$ respectively.

Now $y_1 - y_2$ is $6\sqrt{2}$

18. Equation of directrix of parabola will be the required locus.

21. We know that area of triangle so formed

$$= \frac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a} = \left(\frac{36 - 32}{4}\right)^{3/2} = 2$$

23. Equation of tangent to $y^2 = 4ax$ at $P(x_1, y_1)$ is

$yy_1 = 2a(x + x_1)$

$\Rightarrow 2ax - yy_1 + 2ax_1 = 0 \qquad \ldots (i)$

Let (h, k) be mid point of chord QR.

Then equation of QR is

$ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$

$\Rightarrow -2ax + ky + 2ah - k^2 = 0 \qquad \ldots (ii)$

Clearly (i) and (ii) represents same line.

$$\frac{2a}{-2a} = \frac{-y_1}{k} = \frac{2ax_1}{2ah - k^2}$$

$y_1 = k$ and $2ax_1 = k^2 - 2ah$

$2ax_1 = y_1^2 - 2ah$

$2ax_1 = 4ax_1 - 2ah \Rightarrow x_1 = h$

$\therefore$ mid point of QR is (x_1, y_1)

25. Let $P(x_1\ y_1)$ be point of contact of two parabola. Tangents at P of the two parabolas are

$yy_1 = 2a(x + x_1) - 4a\ell_1$ and

$xx_1 = 2a(y + y_1) - 4a\ell_2$

$\Rightarrow 2ax - yy_1 = 2a\ (2\ell_1 - x_1)$ ... (i)

and $xx_1 - 2ay = 2a\ (y_1 - 2\ell_2)$ ... (ii)

clearly (i) and (ii) represent same line

$\therefore\ \frac{2a}{x_1} = \frac{y_1}{2a} \quad \Rightarrow \quad x_1y_1 = 4a^2$

Hence locus of P is $xy = 4a^2$

EXERCISE - 02 BRAIN TEASERS

1. Let point $P(at^2, 2at)$ on $y^2 = 4ax$

equation of line joining P & vertex

$$y = \frac{2}{t}x \quad ... (1)$$

equation of line which is

perpendicular tangent at P & passing S(a, 0) is

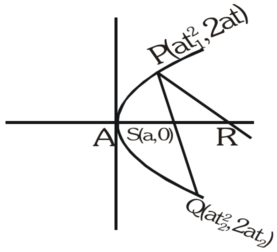
$$y + tx = at \quad ... (2)$$

from (1) & (2) eliminating t

we get the locus of R

$$y^2 + 2x^2 = 2ax$$

4. Let P $(at_1^2\ ,\ 2at_1)$



Relation between t_1 & t_2

$$t_2 = -\ t_1 - \frac{2}{t_1}$$

equation of line PR

$$y - 2at_1 = \frac{2}{t_2}\ (x - at_1^2)$$

Put y = 0 and $t_2 = -\ t_1 - \frac{2}{t_1}$, we get

$R = ((-at_1t_2 + at_1^2),\ 0)$

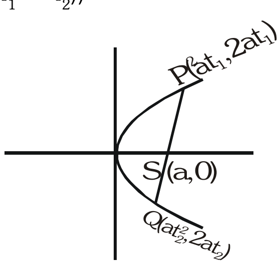
$R = (2a(1 + t_1^2),\ 0)$

Length of PS = $a(1 + t_1^2)$

So AR is twice of PS.

6. Since line passing through focus so $t_1t_2 = -1$

Point of intersection of tangent at P & Q are $(at_1t_2,\ a\ (t_1 + t_2))$



Point of intersection of normal at P & Q

are $(a(t_1^2 + t_2^2 + t_1t_2 + 2)),\ -\ at_1t_2(t_1 + t_2)$

$(x_1,\ y_1) = (-\ a,\ a\ (t_1 + t_2))$

$(x_2,\ y_2) = (a(t_1^2 + t_2^2 - 1),\ a(t_1 + t_2))$

$\Rightarrow \quad y_1 = y_2$

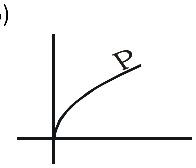
9. The curve $y = \sqrt{x}$ is the part of curve $y^2 = x$

equation of normal at $P\left(\frac{t^2}{4}, \frac{t}{2}\right)$

$$y + tx = \frac{t}{2} + \frac{t^3}{4} \quad .... (1)$$

Since line cut the curve orthogonally

so equation (1) will passes (3, 6)



$$6 + 3t = \frac{t}{2} + \frac{t^3}{4}$$

$t^3 - 10t - 24 = 0$

solving we get t = 4

so equation of line which passes (3, 6) is

$y + 4x = 18$

10. Equation of tangent and normal at P $(at^2,\ 2at)$ on $y^2 = 4ax$ are

$ty = x + at^2$ ... (1)

$y + tx = 2at + at^3$ ... (2)

So $T(-at^2,\ 0)$ & $G(2a + at^2,\ 0)$

equation of circle passing P, T & G is

$(x + at^2)\ (x-(2a + at^2)) + (y - 0)\ (y - 0) = 0$

$x^2 + y^2 - 2ax - at^2\ (2a + at^2) = 0$

equation of tangent on the above circle at

$P(at^2,\ 2at)$ is $at^2x + 2aty - a(x+at^2)-at^2(2a + at^2)=0$

slope of line which is tangent to circle at P

$$m_1 = \frac{a(1-t^2)}{2at} = \frac{1-t^2}{2t}$$

slope of tangent at P, $m_2 = \frac{1}{t}$

$$\therefore\ \tan\theta = \frac{\frac{1-t^2}{2t} - \frac{1}{t}}{1 + \frac{(1-t^2)}{2t^2}} \Rightarrow \tan\ \theta = t$$

$$\Rightarrow \theta = \tan^{-1}t = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$$

12. Let $B = (at_1^2\ ,\ 2at_1)$; $\quad C = (at_2^2\ ,\ 2at_2)$

$A = (at_1t_2,\ a(t_1 + t_2))$

equation of tangent of $y^2 = 4ax$ at $(at^2,\ 2at)$

$ty = x + at^2$ ... (1)

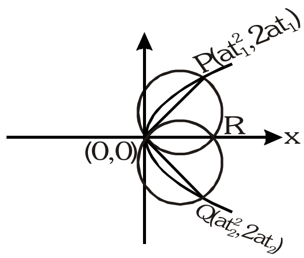
$$p_1 = \left|\frac{at(t_1+t_2) - at_1t_2 - at^2}{\sqrt{1+t^2}}\right| = \frac{|\ a(t_1-t)(t-t_2)\ |}{\sqrt{1+t^2}}$$

$$p_2 = \frac{|2att_1 - at_1^2 - at^2|}{\sqrt{1+t^2}} = \frac{a(t-t_1)^2}{\sqrt{1+t^2}}$$

$$p_3 = \frac{|2att_2 - at_2^2 - at^2)|}{\sqrt{1+t^2}} = \frac{a(t-t_2)^2}{\sqrt{1+t^2}}$$

$\Rightarrow p_1^2 = p_2 p_3$. Hence p_2, p_1, p_3 in G.P.

13.



Let $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

so $\tan\theta_1 = \frac{1}{t_1}$ & $\tan\theta_2 = \frac{1}{t_2}$

$\cot\theta_1 + \cot\theta_2 = t_1 + t_2$ ... (i)

equation of circle with (0, 0) & $(at_1^2, 2at_1)$ as end points of diameter is

$x(x - at_1^2) + y(y - 2at_1)$ so

$S_1 : x^2 + y^2 - at_1^2 x - 2at_1 y = 0$ ... (ii)

similarly other circle is

$S_2 : x^2 + y^2 - at_2^2 x - 2at_2 y = 0$ ... (iii)

equation of OR will be $S_1 - S_2 = 0$

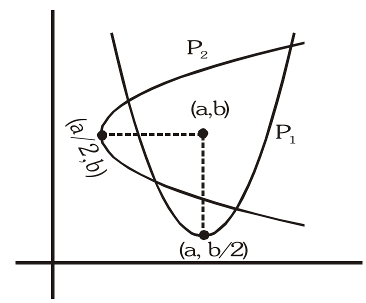
$a(t_2^2 - t_1^2)x + 2a(t_2 - t_1)y = 0$

$$y = -\left(\frac{t_1+t_2}{2}\right)x$$

$$\tan\phi = -\frac{t_1+t_2}{2}$$

from (i) $\cot\theta_1 + \cot\theta_2 = -2\tan\phi$

14. $P_1 \equiv (x-a)^2 = 4.\frac{b}{2}\left(y - \frac{b}{2}\right)$



$\Rightarrow x^2 - 2ax + a^2 - 2yb + b^2 = 0$

Similarly

$P_2 \equiv y^2 - 2ax - 2by + a^2 + b^2 = 0$

Common chord is $P_1 - P_2 = 0$

$\Rightarrow x^2 - y^2 = 0 \Rightarrow (x+y)(x-y) = 0$

slope will be 1, –1

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

Match the column :

1. (A) Equation of normal at $(t^2, 2t)$ on $y^2 = 4x$

$y + tx = 2t + t^3$ using homogenization

$$y^2 = \frac{4x(y+tx)}{(2t+t^3)}$$

for making 90 , coeff. x^2 + coeff. $y^2 = 0$

$$1 - \frac{4}{2+t^2} = 0$$

$t^2 = 2$

(B) Point on $y^2 = 4x$

whose parameter are 1, 2, 4

(1, 2), (4, 4), (16, 8)

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 4 & 4 & 1 \\ 16 & 8 & 1 \end{vmatrix} = 6$$

(C) Equation of normal is $y = mx - 2am - am^3$

since it passes through $\left(\frac{11}{4}, \frac{1}{4}\right)$.

$\therefore$ so we get $4m^3 - 3m + 1 = 0$. Value of m are –1, 1/2, 1/2, so 2 normals can be drawn.

(D) Equation of normal at $(at_1^2, 2at_1)$ to $y^2 = 4ax$

$y + t_1 x = 2at_1 + at_1^3$ ... (i)

If it again meet the curve again at $(at_2^2, 2at_2)$

then $t_2 = -t_1 - \frac{2}{t_1}$

so $t_1 = 1$, & $t_2 = t$

$\Rightarrow t = -1 - 2 = -3$

$|t - 1| = |-3 - 1| = 4$

2. (A) Required area $= \frac{S_1^{3/2}}{2|a|} = \frac{(4)^{3/2}}{2} = 4$

(B) $(x-2)^2 + (y-3)^2 = \left(\frac{3x+4y-6}{5}\right)^2$

$$\sqrt{(x-2)^2 + (y-3)^2} = \frac{3x+4y-6}{5}$$

focus, is (2,3) & directrix is $3x + 4y - 6 = 0$

distance between focus and directrix is

$$2a = \frac{6+12-6}{5} = \frac{12}{5}$$

$\Rightarrow$ Length of Latus Rectum $= 4a = \frac{24}{5}$

(C) $x^2 = y + 4$ $\therefore$ its focus $(0, \frac{-15}{4})$

Let point on $x^2 = y + 4$ is

$(x_1, x_1^2 - 4)$

$x_1^2 + (x_1^2 - 4 + \frac{15}{4})^2 = \frac{625}{16}$

$x_1^2 + x_1^4 + \frac{1}{16} - \frac{x_1^2}{2} = \frac{625}{16}$

$x_1^4 + \frac{x_1^2}{2} = 39$

$2x_1^4 + x_1^2 - 78 = 0$

$(x_1^2 - 6)(2x_1^2 + 13) = 0$

$x_1 = \pm\sqrt{6}$

$x_1^2 = 6 \Rightarrow x_1^2 - 4 = 2$

so point are $(\pm\sqrt{6}, 2)$

& $a + b = 6 + 2 = 8$

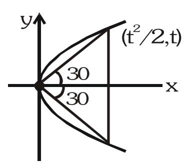
(D) $(y - 1)^2 = 2(x + 2)$

vertex is $(-2, 1)$

so equation is $(y - 1)^2 = 2(x + 2) \Rightarrow Y^2 = 2X$

Let point on $Y^2 = 2X$ is $\left(\frac{1}{2}t^2, t\right)$

From fig. $\tan 30 = \frac{2}{t}$



$\Rightarrow t = 2\sqrt{3}$

so point on parabola is $(6, 2\sqrt{3})$.

But when vertex change, distance (or length of side of equilateral triangle) remain same

$\therefore$ length of side $= \sqrt{(6)^2 + (2\sqrt{3})^2} = 4\sqrt{3}$.

Assertion & Reason :

3. Let $P_1\ (at_1^2, 2at_1)$ & $Q_1\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$

$P_2\ (at_2^2, 2at_2)$ & $Q_2\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$

on $y^2 = 4ax$

equation of P_1P_2 :

$(t_1 + t_2)y = 2x + 2at_1t_2$ ...(i)

equation of Q_1Q_2

$-(t_1 + t_2)y = 2x\, t_1t_2 + 2a$ ...(ii)

add (i) & (ii)

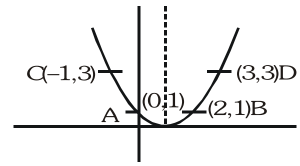
$x = -a$ which is directrix of $y^2 = 4ax$

Locus of point of intersection of tangent is directrix. In case of parabola director circle is directrix

Comprehension # 1

Axis of parabola is bisector of parallel chord A B & CD are parallel chord.

so axis $x = 1$



equation of parabola is

$(x - 1)^2 = ay + b$

It passing (0, 1) & (3, 3)

so $1 = a + b$ ...(1)

$4 = 3a + b$ ...(2)

from (1) & (2)

$a = \frac{3}{2}$ & $b = -\frac{1}{2}$

$(x - 1)^2 = \frac{3}{2}(y - \frac{1}{3})$

1. Vertex $(1, \frac{1}{3})$

2. $a = \frac{3}{8}$

directrix of $x^2 = 4ay$ is $y = -a$

$y - \frac{1}{3} = -\frac{3}{8}$

$\Rightarrow y = \frac{1}{3} - \frac{3}{8}$

$y + \frac{1}{24} = 0$

3. Let parametric point on $y^2 = 4ax$ are $A(t_1)$, $B(t_2)$, $C(t_3)$ and $D(t_4)$

So $t_1 + t_2 = 2 = t_3 + t_4$

Equation of circle passing through OAB is

$x^2 + y^2 + 2gx + 2fy + c = 0$

fourth point $M(t_5)$ putting the value $(t^2, 2t)$ in circle we get four degree equation. In this equation

$t_1 + t_2 + t_5 + 0 = 0 \Rightarrow t_5 = -2$

Similarly circle passing through OCD & fourth point $N(t_6)$ we have $t_1 + t_2 + t_6 + 0 = 0 \Rightarrow t_6 = -2$

It mean both point M and N are same

so common point $(at^2, 2at) \Rightarrow (4, -4)$

4. Parabola $y^2 = 4ax$

$P(t_1) = (at_1^2, 2at_1)$ & $Q(t_2) = (at_2^2, 2at_2)$

Given $t_1t_2 = K$

equation of chord PQ

$(t_1 + t_2)y = 2x + 2at_1t_2$

so $(t_1 + t_2)y = 2x + 2ak$

$$\left(\frac{t_1+t_2}{2}\right) y = x + ak$$

$[L_2 = \lambda L_1 \text{ Type}]$

so $y = 0$ & $x = -ak$

fixed point $(-a k, 0)$

8. $x^2 = y$ ... (1)

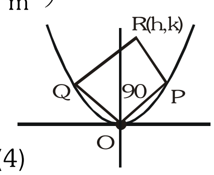
Let equation of OP $y = mx$ ... (2)

equation of OQ $y = \frac{-1}{m}x$ ... (3)

from (1) & (2) we get $P(m, m^2)$

from (1) & (3) we get $Q\left(\frac{-1}{m}, \frac{1}{m^2}\right)$

equation of PR



$$y - m^2 = -\frac{1}{m}(x - m)$$

$$y + \frac{1}{m}x = m^2 + 1 \quad ...(4)$$

equation of QR is

$$y - \frac{1}{m^2} = m\left(x + \frac{1}{m}\right)$$

$$y - mx = 1 + \frac{1}{m^2} \quad ...(5)$$

Locus of R solving (4) & (5) & eliminating m

we get $x^2 = y - 2$

9. $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

point $(\sin\alpha, \cos\alpha)$ not lie out side

$2y^2 + x - 2 = 0$

$\Rightarrow$ $2\cos^2\alpha + \sin\alpha - 2 \le 0$

$2 - 2\sin^2\alpha + \sin\alpha - 2 \le 0$

$\sin\alpha(2\sin\alpha - 1) \ge 0$

$\sin\alpha \le 0$ or $\sin\alpha \ge \frac{1}{2}$

$\alpha \in \left[\pi, \frac{3\pi}{2}\right]$ or $\alpha \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

11. $y = mx + c$ touch $y^2 = 8(x + 2)$

$\therefore$ $(mx + c)^2 = 8(x + 2)$

$m^2x^2 + x(2mc - 8) + c^2 - 16 = 0$ ...(i)

line touch the parabola so D = 0 of equation (i)

$4(mc - 4)^2 - 4m^2(c^2 - 16) = 0$

$m^2c^2 - 8mc + 16 - m^2c^2 + 16m^2 = 0$

$2m^2 - mc + 2 = 0$

Since m is real $D \ge 0$

$c^2 - 16 \ge 0$

$c \in (-\infty, -4] \cup [4, \infty)$

14. Let point on $y^2 = 4ax$ be $P(at^2, 2at)$

equation of tangent of P

$ty = x + at^2$ ... (1)

It intersect the directrix $x = -a$ ... (2)

point of intersection of (1) & (2)

is $A\left(-a, a\left(t - \frac{1}{t}\right)\right)$

Let mid point of PA is (h, k)

$2h = at^2 - a$ ... (3)

$2k = 2at + a\left(t - \frac{1}{t}\right)$ ... (4)

from (3) & (4) eliminating t & replace $h \to x$ & $y \to k$ we get

$y^2(2x + a) = a(3x + a)^2$

17. Let point on parabola $y^2 = 4ax$ is $P(at^2, 2t)$

Given $at^2 = 4a \Rightarrow t = \pm 2$

taking positive t = 2

P(4a, 4a)

equation of tangent at P is $2y = x + 4a$

If intersect x-axis at T then T(–4a, 0)

Normal at (4a, 4a) meet again parabola at

$Q(at_2^2, 2at_2)$ (using $t_2 = -t_1 - \frac{2}{t_1} = -3$)

$\therefore$ Q(9a, – 6a)

Now P(4a, 4a), T(–4a, 0), Q(9a, –6a)

$$PT = \sqrt{(4a+4a)^2 + (4a)^2} = \sqrt{80a^2}$$

$$PQ = \sqrt{(4a-9a)^2 + (4a+6a)^2} = \sqrt{125a^2}$$

$$\frac{PT}{PQ} = \sqrt{\frac{80a^2}{125a^2}} = \frac{4}{5}$$

19. Let point be (h, k)

Equation of normal at $(am^2, 2am)$

$y + mx = 2am + am^3$

$k = mh - 2am - am^3$

$am^3 + m(2a - h) + k = 0$ ... (1)

Its slope is m_1, m_2 & m_3

$$m_1.m_2.m_3 = \frac{-k}{a}$$

$m_3 = \frac{k}{a}$ Put in (1) [Given $m_1m_2 = -1$]

$\Rightarrow y^2 = a(x - 3a)$

21. Equation of normal at $(am^2, 2am)$

on $y^2 = 4ax$

$y + mx = 2am + am^3$ ... (1)

It cuts x − axis at y = 0 i.e. $(2a + am^2, 0)$

Let middle point (h, k)

$2h = am^2 + 2a + am^2$

$h = am^2 + a$ & $k = am$ ... (2)

from (1) & (2)

$$h = a\frac{k^2}{a^2} + a$$

Locus $y^2 = a(x - a)$

vertex (a, 0) L.R. = a

23. Let $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

on $y^2 = 4ax$

co-ordinate of $T(at_1t_2, a(t_1 + t_2))$ which is point of intersection of tangent at P & Q

equation of PQ which is normal at P

$y + t_1x = 2at_1 + at_1^3$ ... (1)

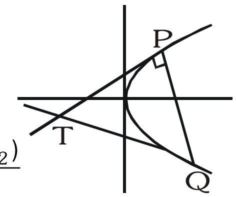
equation of PQ is

$(t_1 + t_2)y = 2x + 2at_1t_2$ ... (2)

equation (1) & (2) are same

Compare slope $\dfrac{2}{t_1 + t_2} = -t_1$

$\Rightarrow t_1^2 + t_1t_2 = -2$



Now mid point of TP

$$x = \frac{at_1^2 + at_1t_2}{2} = \frac{a(t_1^2 + t_1t_2)}{2}$$

$$x = \frac{a(-2)}{2} = -a$$

x = −a which is directrix

Hence TP bisect the directrix

24. Normal at $P(am^2, 2am)$ on $y^2 = 4ax$

$y + mx = 2am + am^3$ ... (1)

$G(2a + am^2, 0)$

Equation of QG is $x = 2a + am^2$

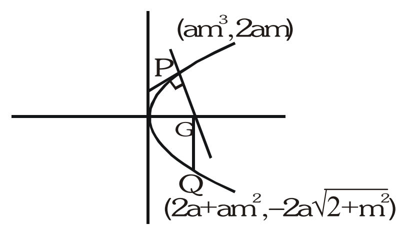
Solving with parabola we get

$$y = \pm 2a\sqrt{2 + m^2}$$

$QG^2 - PG^2 =$

$4a^2(2 + m^2) - (am^2 - am^2 - 2a)^2 - (2am)^2$

$= 4a^2$ which is constant



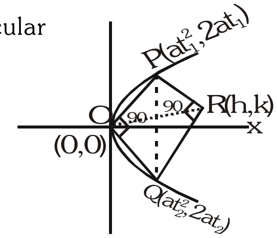
EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. Let parabola $y^2 = 4ax$

Let $P(at_1^2, 2at_1)$, & $Q(at_2^2, 2at_2)$

OP & OQ are perpendicular

$$\Rightarrow \frac{2}{t_1} \cdot \frac{2}{t_2} = -1$$



$t_1t_2 = -4$

Now diagonals of a rectangle bisect each other

$\dfrac{h}{2} = \dfrac{at_1^2 + at_2^2}{2} \Rightarrow h = a(t_1^2 + t_2^2)$ ...(1)

$\dfrac{k}{2} = \dfrac{2at_1 + at_2}{2} \Rightarrow k = 2a(t_1 + t_2)$ ...(2)

$$\frac{k^2}{4a^2} = t_1^2 + t_2^2 + 2t_1t_2$$

$$\frac{k^2}{4a^2} = \frac{h}{a} - 8$$

Required locus is $y^2 = 4a(x - 8a)$

3. Equation of tangent of

$y^2 = 4ax$ in slope form at(x_1, y_1) is

$y_1 = mx_1 + \dfrac{a}{m}$ ... (1)

equation of normal at $(2bt_1, bt_1^2)$ on $x^2 = 4by$

$x + t_1y = 2bt_1 + bt_1^3$

It passes through (x_1, y_1)

$\therefore x_1 + t_1y_1 = 2bt_1 + bt_1^3$ ... (2)

(1) & (2) are same equation so compare

$$\frac{1}{t_1} = -\frac{m}{1} = \frac{a}{m(2bt_1 + bt_1^3)}$$

$t_1m = -1$

$-m^2t_1(2b + bt_1^2) = a$

$\Rightarrow m(2b + bt_1^2) = a$ ... (3)

Put $m = -\dfrac{1}{t_1}$ in equation (3)

$2b + bt_1^2 = -at_1$

$bt_1^2 + at_1 + 2b = 0$

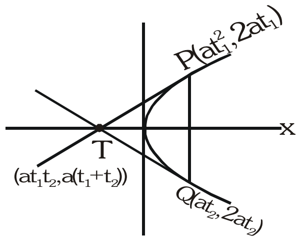
t_1 will be real

$a^2 > 8b^2$

4. Let parabola $y^2 = 4ax$

point on parabola $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

Point of intersection of tangent at P & Q is T $(at_1t_2, a(t_1 + t_2))$



Normal at P & Q meet again in the parabola so relation between t_1t_2

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$t_1t_2 = 2$

equation of line perpendicular to TP & passing through mid point of TP is

$2y - a(3t_1 + t_2) = -t_1(2x - a(2 + t_1^2))$ ... (1)

$2y + 2xt_1 = a(3t_1 + t_2) + at_1(2 + t_1^2)$

similar equation of passing mid point of

TQ and $\perp$ to TQ

$2y + 2xt_2 = a(3t_2 + t_1) + at_2(2 + t_2^2)$ ... (2)

from (1) & (2) & using $t_1t_2 = 2$

Eliminating t_1 & t_2 we get the locus of circumcentre

$2y^2 = a(x - a)$

6. Let $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

on $y^2 = 4ax$

equation of chord of PQ

$(t_1 + t_2)\, y = 2x + 2at_1t_2$ ... (1)

Point on x-axis is $K(-at_1t_2, 0)$

$PK^2 = (at_1^2 + at_1t_2)^2 + 4a^2t_1^2$

$= a^2t_1^2((t_1 + t_2)^2 + 4)$

$QK^2 = a^2t_2^2((t_1 + t_2) + 4)$

$$\frac{1}{PK^2}+\frac{1}{QK^2}=\frac{1}{a^2t_1^2((t_1+t_2)^2+4)}+\frac{1}{a^2t_2^2((t_1+t_2)^2+4)}$$

$$= \frac{t_2^2+t_1^2}{a^2t_1^2t_2^2((t_1+t_2)^2+4)}$$

$$= \frac{t_1^2+t_2^2}{a^2t_1^2t_2^2((t_1^2+t_2^2+2t_1t_2+4)}$$

$= \frac{1}{PK^2}+\frac{1}{QK^2}$ will be independent of K

$\Rightarrow \frac{t_1^2+t_2^2}{a^2t_1^2t_2^2(t_1^2+t_2^2+2t_1t_2+4)} \Rightarrow t_1t_2 = -2$

so fixed point K (2a, 0)

9. Let parabolais $y^2 = 4ax$

equation of normal at $(am^2, 2am)$

$y + mx = 2am + am^3$

it passes through (h, k)

$am^3 + m(2a - h) - k = 0$

its root are m_1 m_2 & m_3

$\Sigma m_1 = 0, \quad \Sigma m_1m_2 = \frac{2a-h}{a}$

$m_1\, m_2\, m_3 = \frac{k}{a}$

let equation of circle be

$x^2 + y^2 + 2gx + 2fy + C = 0$

It passes $(am^2, 2am)$

$a^2m^4 + 4a^2m^2 + 2agm^2 + 4afm + C = 0$

$a^2m^4 + m^2(4a^2 + 2ag) + 4afm + C = 0$

its roots m_1, m_2, m_3 & m_4

$m_1 + m_2 + m_3 + m_4 = 0,$

$\because \quad m_1+m_2 + m_3 = 0$

$\Rightarrow m_4 = 0 \Rightarrow$ circle passes (0, 0)

$m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1 + m_1m_3 + m_2m_4$

$= \frac{4a^2+2ag}{a^2}$

$\Rightarrow \frac{2a-h}{a} = \frac{4a^2+2ag}{a^2}$

$\Rightarrow 2a - h = 4a + 2g$

$\Rightarrow g = \frac{-h-2a}{2}$

$m_1m_2m_3 + m_2m_3m_4 + m_3m_4m_1 + m_4m_1m_2 = \frac{-4af}{a^2}$

$\Rightarrow \frac{k}{a} = \frac{-4af}{a^2}$

$\Rightarrow f = \frac{-k}{4}$

equation of circle

$$x^2 + y^2 - (h + 2a)x + \frac{k}{2}y = 0$$

3. It is a fundamental theorem.

4. Given parabolas are

$y^2 = 4ax$ ..... (i)

$x^2 = 4ay$ ..... (ii)

Putting the value of y from (ii) in (i), we get

$$\frac{x^2}{16a^2} = 4ax \Rightarrow x(x^3 - 64a^3) = 0 \Rightarrow x = 0, 4a$$

from (ii), y = 0, 4a. Let A ≡ (0, 0); B ≡ (4a, 4a)

Since, given line 2bx + 3cy + 4d = 0 passes through A and B,

∴ d = 0 and 8ab + 12ac = 0

⇒ 2b + 3c = 0. (∵ a ≠ 0)

Obviously, $d^2 + (2b + 3c)^2 = 0$

5. $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$

$$\Rightarrow \quad y = \frac{a^3}{3}\left[x^2 + \frac{x}{2} \times \frac{3}{a} \times \frac{2}{2}\right] - 2a$$

$$\Rightarrow \quad y = \frac{a^3}{3}\left[\left(x + \frac{3}{4a}\right)^2\right] - \frac{3a}{16} - 2a$$

$$\Rightarrow \quad y + \frac{35a}{16} = \frac{4a^3}{12}\left(x + \frac{3}{4a}\right)^2$$

∴ Vertices will be (α, β)

So that $\alpha = -\frac{3}{4a}$ and $\beta = -\frac{35a}{16}$

or $\alpha\beta = \left(\frac{-3}{4a}\right) \times \left(\frac{-35a}{16}\right) = \frac{105}{64}$

∴ Required locus will be $xy = \frac{105}{64}$

6. Point must be on the directrix of the parabola

Hence the point is (–2, 0)

8. Locus of point of intersection of perpendicular tangent is directrix of the parabola.

so x = –1

9. tangent of slope m of $y^2 = 4\sqrt{5}\,x$

is $y = mx + \frac{\sqrt{5}}{m}$

also tangent to $\frac{x^2}{5/2} + \frac{y^2}{5/2} = 1$

$$\Rightarrow \quad \frac{5}{m^2} = \frac{5}{2}m^2 + \frac{5}{2}$$

⇒ $2 = m^4 + m^2$ ⇒ $m^4 + m^2 - 2 = 0$

$m = \pm 1$

which satisfy $m^4 - 3m^2 + 2 = 0$

which gives $y = x + \sqrt{5}$ as tangent

So I & II both are true.

1. (a) The parabola is $y^2 = 4.\frac{k}{4}\left(x-\frac{8}{k}\right)$

Putting y = Y, $x - \frac{8}{k}$ = X,

the equation $Y^2 = 4.\frac{k}{4}.X$

$\therefore$ The directrix is $X + \frac{k}{4} = 0$,

i.e. $x - \frac{8}{k} + \frac{k}{4} = 0$

But x − 1 = 0 is the directrix.

So, $\frac{8}{k} - \frac{k}{4} = 1 \Rightarrow k = -8, 4$

(b) Any normal is $y + tx = 6t + 3t^2$. It is identical with x + y = k if $\frac{t}{1} = \frac{1}{1} = \frac{6t+3t^3}{k}$

$\therefore$ t = 1 and $1 = \frac{6+3}{k} \Rightarrow k = 9$

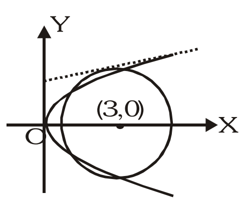
Aliter : y = −x + k

$\therefore$ $c = -[2am + am^3]$

$\Rightarrow c = -[6(-1) + 3(-1)^3]$

$\therefore$ $c = \pm 9$

2. (a) Any tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$.



It touches the circle, if $3 = \left|\frac{3m+\frac{1}{m}}{\sqrt{1+m^2}}\right|$

or $9(1 + m^2) = \left(3m+\frac{1}{m}\right)^2$

or $\frac{1}{m^2} = 3$, $\therefore m = \pm\frac{1}{\sqrt{3}}$

For the common tangent to be above the x-axis, $m = \frac{1}{\sqrt{3}}$

$\therefore$ Common tangent is,

$y = \frac{1}{\sqrt{3}}x + \sqrt{3} \Rightarrow \sqrt{3}\,y = x + 3$

3. $\alpha = \frac{at^2+a}{2}$, $\beta = \frac{2at+0}{2} \Rightarrow 2\alpha = at^2 + a$, at = β

$\therefore$ $2\alpha = a.\frac{\beta^2}{a^2} + a$ or $2a\alpha = \beta^2 + a^2$

$\therefore$ The locus is $y^2 = \frac{4a}{2}\left(x-\frac{a}{2}\right) = 4b(x - b)$, $\left(b=\frac{a}{2}\right)$

$\therefore$ Directrix is (x − b) + b = 0 or x = 0

4. The given curves are

$y^2 = 8x$ ... (1)

and $xy = -1$ ... (2)

If m is the slope of tangent to (1), then equation of tangent is

$y = mx + 2/m$.

If this tangent is also a tangent to (2), then

$x\left(mx+\frac{2}{m}\right) = -1$

$\Rightarrow$ $mx^2 + \frac{2}{m}x + 1 = 0$

$\therefore$ $m^2x^2 + 2x + m = 0$

We should get repeated roots for this equation (conditions of tangency)

$\Rightarrow$ D = 0

$\therefore$ $(2)^2 - 4m^2 . m = 0$

$\Rightarrow$ $m^3 = 1$

$\Rightarrow$ m = 1

Hence required tangent is y = x + 2.

6. Let P be the point (h, k). Then equation of normal to parabola $y^2 = 4x$ from point (h, k), if m is the slope of normal, is $y = mx - 2m - m^3 = 0$

As it passes through (h, k), therefore

$mh - k - 2m - m^3 = 0$

or $m^3 + (2 - h)m + k = 0$ ... (1)

which is cubic in m, giving three values of m say m_1, m_2 and m_3. Then $m_1m_2m_3 = -k$ (from equation) but given that $m_1m_2 = \alpha$

$\therefore$ We get $m_3 = -\frac{k}{\alpha}$

But m_3 must satisfy equation (1)

$\therefore$ $\frac{-k^3}{\alpha^3} + (2 - h)\left(\frac{-k}{\alpha}\right) + k = 0$

$\Rightarrow$ $k^2 - 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$

$\therefore$ Locus of P(h, k) is $y^2 = \alpha^2x + (\alpha^3 - 2\alpha^2)$

But ATQ, locus of P is a part of parabola $y^2 = 4x$, therefore comparing the two, we get $\alpha^2 = 4$ and $\alpha^3 - 2\alpha^2 = 0$

$\Rightarrow$ α = 2

8. The given equation of parabola is

$$y^2 - 2y - 4x + 5 = 0 \quad \dots (1)$$

$\Rightarrow$ $(y - 1)^2 = 4(x - 1)$

Any parametric point on this parabola is

$P\ (t^2 + 1,\ 2t + 1)$

Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} - 4 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{y-1}$$

$\therefore$ Slope of tangent to (1) at pt.

$$P(t^2 + 1,\ 2t + 1) \text{ is } m = \frac{2}{2t} = \frac{1}{t}$$

$\therefore$ Equation of tangent at $P(t^2 + 1,\ 2t + 1)$ is

$$y - (2t + 1) = \frac{1}{t}(x - t^2 - 1)$$

$\Rightarrow$ $yt - 2t^2 - t = x - t^2 - 1$

$\Rightarrow$ $x - yt + (t^2 + t - 1) = 0 \quad \dots (2)$

Now direction of given parabola is

$(x - 1) = -1 \quad \Rightarrow \quad x = 0$

Tangent to (2) meets directrix at $Q\left(0, \frac{t^2+t-1}{t}\right)$

Let pt. R be (h, k)

ATQ R divides the line joining QP in the ratio

$\frac{1}{2}$: 1 i.e. 1 : 2 externally.

$$\therefore (h,\ k) = \left[\frac{1(1+t^2)-0}{-1}, \frac{t+2t^2-2t^2-2t+2}{-t}\right]$$

$$\Rightarrow \quad h = -(1 + t^2) \text{ and } k = \frac{t-2}{t}$$

$$\Rightarrow \quad t^2 = -1 - h \text{ and } \quad t = \frac{2}{1-k}$$

Eliminating t we get $\left(\frac{2}{1-k}\right)^2 = -1 - h$

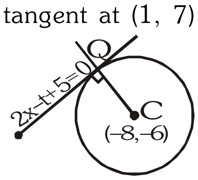
$\Rightarrow$ $4 = -(1 - k)^2\ (1 - h)$

$\Rightarrow$ $(h - 1)\ (k - 1)^2 + 4 = 0$

$\therefore$ locus of R(h, k) is, $(x - 1)\ (y - 1)^2 + 4 = 0$

9. The given curve is $y = x^2 + 6$

Equation of tangent at (1, 7) is



$$\frac{1}{2}(y + 7) = x.1 + 6$$

$\Rightarrow$ $2x - y + 5 = 0 \quad \dots (1)$

ATQ this tangent (1) touches the circle

$x^2 + y^2 + 16x + 12y + C = 0$

at Q. (centre of circle (– 8, – 6)).

Then equation of CQ which is perpendicular to (1) and passes through (– 8, – 6) is

$$y + 6 = -\frac{1}{2}(x + 8)$$

$\Rightarrow$ $x + 2y + 20 = 0 \quad \dots (2)$

Now Q is pt. of intersection of (1) and (2) i.e. $x = -6,\ y = -7$

$\therefore$ Req. pt. is (– 6, – 7).

13. Without loss of generality we can assume the square ABCD with its vertices A(1, 1), B(–1, 1), C(–1, –1), D(1, –1)

P to be the point (0, 1) and Q as ($\sqrt{2}$, 0)

$$\text{Then, } \frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$$

$$= \frac{1+1+5+5}{2[(\sqrt{2}-1)^2+1]+2((\sqrt{2}+1)^2+1)} = \frac{12}{16} = 0.75$$

14. Let C' be the said circle touching C_1 and L, so that C_1 and C' are on the same side of L. Let us draw a line T parallel to L at a distance equal to the radius of circle C_1, on opposite side of L.

Then for N, centre of circle C', MN = NO

$\Rightarrow$ N is equidistant from a line and a point

$\Rightarrow$ locus of N is a parabola.

15. Since S is equidistant from A and line BD, it traces a parabola. Clearly AC is the axis, A(1, 1) is the focus and $T_1\left(\frac{1}{2}, \frac{1}{2}\right)$ is the vertex of parabola,

$$AT_1 = \frac{1}{\sqrt{2}}.$$

$$T_2T_3 = \text{latus rectum of parabola} = 4 \quad \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$\therefore \text{Area } (\Delta T_1\ T_2\ T_3) = \frac{1}{2} \quad 2\sqrt{2} = \frac{1}{2} = 1 \text{ sq. units.}$$

16. $$\frac{Ar\Delta PQS}{Ar\Delta PQR} = \frac{\frac{1}{2}QP \times ST}{\frac{1}{2}PQ \times TR} = \frac{ST}{TR} = \frac{2}{8} = \frac{1}{4}$$

17. For ΔPRS,

$Ar(\Delta PRS) = \Delta = \frac{1}{2}\ \ SR\ \ PT = \frac{1}{2}\ \ 10\ \ 2\sqrt{2}$

$\therefore\quad \Delta = 10\sqrt{2}$, $a = PS = 2\sqrt{3}$,

$b = PR = 6\sqrt{2}$, $c = SR = 10$

$\therefore$ radius of circumference

$= R = \frac{abc}{4\Delta} = \frac{2\sqrt{3}\times 6\sqrt{2}\times 10}{4\times 10\sqrt{2}}\ 3\sqrt{3}$

18. Radius of incircle

$= \frac{\text{area of } \Delta PQR}{\text{semi perimeter of } \Delta PQR} = \frac{\Delta}{s}$

We have $a = PR = 6\sqrt{2}$, $b = QP = PR = 6\sqrt{2}$

$c = PQ = 4\sqrt{2}$

and $\Delta = \frac{1}{2}\ \ PQ\ \ TR = 16\sqrt{2}$

$\therefore\quad s = \frac{6\sqrt{2}+6\sqrt{2}+4\sqrt{2}}{2} = 8\sqrt{2}$

$\therefore\quad r = \frac{16\sqrt{2}}{8\sqrt{2}} = 2$

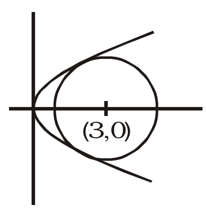
20. $C_1 : y^2 = 4x$

$C_2 : x^2 + y^2 - 6x + 1 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0 \quad \Rightarrow x = 1$

$y = \pm 2$

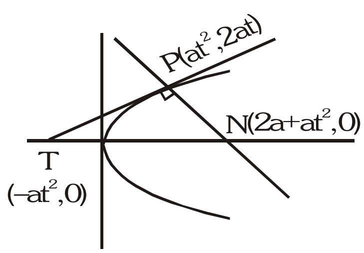


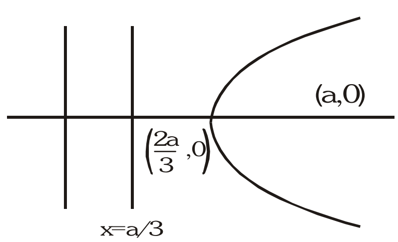
so the curves touches each other at two points (1, 2) & (1, –2)

21. $3h = 2a + at^2$

$3k = 2at$

$3h = 2a + \frac{a\cdot 9k^2}{4a^2}$





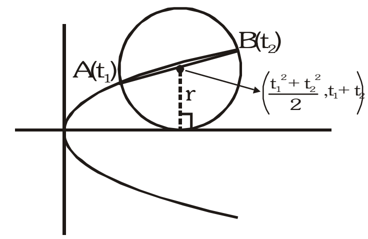
$y^2 = \frac{4a}{9}(3x - 2a)$

$y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$

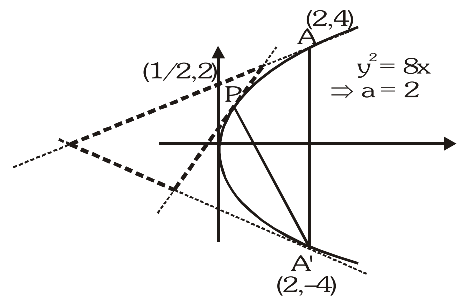
22. $t_1 + t_2 = r$

$\frac{2}{r} = \frac{2}{t_1 + t_2}$

similarly $-\frac{2}{r}$ is also possible



23.



Δ_1 = area of Δ PAA' $= \frac{1}{2}.8.\frac{3}{2} = 6$

$\Delta_2 = \frac{1}{2}(\Delta_1)$

(Using property : Area of triangle formed by tangents is always half of original triangle)

$\Rightarrow\quad \frac{\Delta_1}{\Delta_2} = 2$

24. Let P be (h, k)

on using section formula $P\left(\frac{x}{4}, \frac{y}{4}\right)$

$\therefore\quad h = \frac{x}{4}$ and $k = \frac{y}{4}$

$\Rightarrow\quad x = 4h$ and $y = 4k$

$\because\quad (x, y)$ lies on $y^2 = 4x$

$\therefore\quad 16k^2 = 16h \quad \Rightarrow \quad k^2 = h$

Locus of point P is $y^2 = x$.

25. Equation of normal is $y = mx - 2m - m^3$

It passes through the point (9, 6) then

$6 = 9m - 2m - m^3$

$\Rightarrow\quad m^3 - 7m + 6 = 0$

$\Rightarrow\quad (m - 1)(m - 2)(m + 3) = 0$

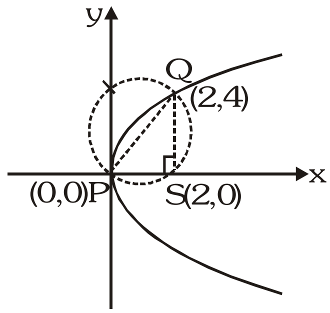
$\Rightarrow\quad m = 1, 2, -3$

Equations of normals are

$y - x + 3 = 0$, $y + 3x - 33 = 0$

& $y - 2x + 12 = 0$

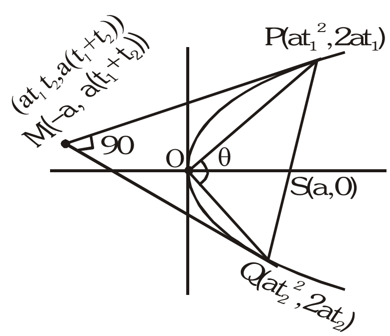
26. Focus of parabola S(2,0) points of intersection of given curves : (0,0) and (2,4).



Area (ΔPSQ) $= \frac{1}{2}.2.4 = 4$ sq. units

Paragraph for Question 27 and 28

27. Single tangent at the extrimities of a focal



chord will intersect on directrix.

$\therefore$ M(–a, $a(t_1 + t_2)$)

lies on y = 2x + a

$a(t_1 + t_2) = -2a + a \quad \Rightarrow \quad t_1 + t_2 = -1$

$\& \quad t_1t_2 = -1$

$$\tan\theta = \left(\frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1t_2}}\right) = \left(\frac{2(t_2 - t_1)}{3}\right)$$

$\because \quad (t_2 - t_1)^2 = (t_2 + t_1)^2 - 4t_1t_2 = 5$

$t_2 - t_1 = \pm\sqrt{5}$

$\therefore \quad \tan\theta = \pm\frac{2\sqrt{5}}{3}$

but θ is obtuse because O is the interior point of the circle for which PQ is diameter.

$\therefore \quad \tan\theta = \frac{-2\sqrt{5}}{3}$

28. Length of focal chord

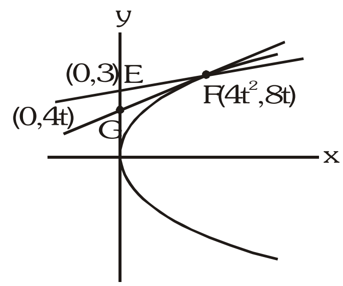
PQ $= a(t_1 - t_2)^2$

$= a[(t_1 + t_2)^2 - 4t_1t_2]$

$= a[1 + 4] = 5a$

29. Let F($4t^2$, 8t)

where $0 \le 8t \le 6 \quad \Rightarrow \quad 0 \le t \le \frac{3}{4}$



$\Delta EFG = \frac{1}{2}(3 - 4t)4t^2$

$\Delta = (6t^2 - 8t^3)$

$\frac{d\Delta}{dt} = 12t - 24t^2 = 0 \quad \begin{cases} t = 0\,(\text{minima}) \\ t = \frac{1}{2}\,(\text{maxima}) \end{cases}$

– + –

0 1/2

$\Rightarrow \quad m = \frac{8t - 3}{4t^2 - 0} = \frac{4 - 3}{1} = 1$

$(\Delta EFG)_{max} = \frac{6}{4} - 1 = \frac{1}{2}$

$y_0 = 8t = 4 \ \& \ y_1 = 4t = 2$