SOLUTION OF TRIANGLE

EXERCISE - 01 CHECK YOUR GRASP

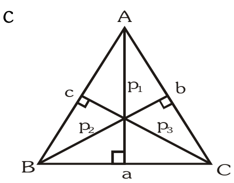
4. $\left(\frac{4R^2\sin^2 A}{\sin A}+\frac{4R^2\sin^2 B}{\sin B}+\frac{4R^2\sin^2 C}{\sin C}\right)\prod \sin\frac{A}{2}$

$= 4R^2(\sin A+\sin B+\sin C)\prod \sin\frac{A}{2}$

$= 16R^2\prod\cos\frac{A}{2}.\prod\sin\frac{A}{2} = 2R^2\sin A.\sin B.\sin C$

$= 2R^2\frac{abc}{8R^3}=\frac{abc}{4R}=\Delta$

5. $\Delta=\frac{1}{2}p_1.a=\frac{1}{2}p_2.b=\frac{1}{2}p_3.c$

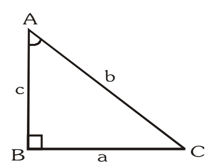


$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}$

$=\frac{a+b-c}{2\Delta}=\frac{2(s-c)}{2\Delta}$

($\because$ a + b + c = 2s)

6. $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}$



$=\frac{1-\frac{c}{b}}{1+\frac{c}{b}}=\frac{b-c}{b+c}$

8. a + b + c = 2s $\Rightarrow$ s = 2a

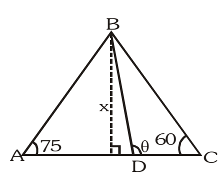
Applying half angle formulae.

$\cot\frac{B}{2}.\cot\frac{C}{2}=\sqrt{\frac{(s)(s-b)}{(s-a)(s-c)}.\frac{(s)(s-c)}{(s-a)(s-b)}}$

$=\frac{s}{s-a}=2$

10. Area of Δ BAD

$=\sqrt{3}$ Area of Δ BCD



$\Rightarrow \frac{1}{2}$ AD x

$=\sqrt{3}\ \frac{1}{2}$ DC x $\Rightarrow \frac{AD}{DC}=\frac{\sqrt{3}}{1}$

Applying m – n theorem

$(\sqrt{3}+1)\cot\theta=\cot 75-\sqrt{3}\cot 60$

$(\sqrt{3}+1)\cot\theta=\frac{\sqrt{3}-1}{\sqrt{3}+1}-1$

$(\sqrt{3}+1)\cot\theta=\frac{-2}{\sqrt{3}+1}$

$\cot\theta=\frac{-2}{(\sqrt{3}+1)^2}=\frac{-2}{4+2\sqrt{3}}=\frac{-1}{2+\sqrt{3}}=-(2-\sqrt{3})$

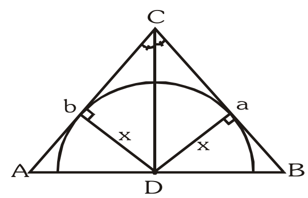
$\cot\theta=\cot 105$

$\therefore\ \theta = 105$ $\qquad\therefore\ \angle ADB = 75$

$\therefore\ \angle ABD = 30$

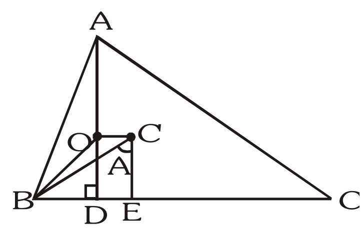
11. Area(ΔADC) $=\frac{1}{2}$b.x , Area(ΔBCD) $=\frac{1}{2}$x.a

$\Rightarrow\ \Delta=\frac{1}{2}x(b+a)$



$\Rightarrow\ x=\frac{2\Delta}{a+b}$

15.



According to question.

OD = C'E

$\Rightarrow$ 2Rcos Bcos C = R cos A

$\Rightarrow$ 2 cos Bcos C = – cos (B + C)

($\because$ cos A = cos (π – (B + C))

$\Rightarrow 2=\frac{-\cos(B+C)}{\cos B.\cos C}\Rightarrow$ tan B. tan C = 3.

16.

$r=\frac{\Delta}{s}$	$R=\frac{abc}{4\Delta}$	$r_1=\frac{\Delta}{s-a}$
$r=\frac{\frac{\sqrt{3}}{4}a^2}{\frac{3a}{2}}=\frac{a}{2\sqrt{3}}$	$R=\frac{a^3}{4.\frac{\sqrt{3}a^2}{4}}$	$r_1=\frac{(\sqrt{3}/4)a^2}{a/2}=\frac{\sqrt{3}}{2}a$
	$R=\frac{a}{\sqrt{3}}=\frac{2a}{2\sqrt{3}}$	$r_1=\frac{3a}{2\sqrt{3}}$

Hence r, R, r_1 are in A.P.

17. Using $r_1=\frac{\Delta}{(s-a)},\ r_2=\frac{\Delta}{(s-b)},\ r_3=\frac{\Delta}{(s-c)}$

we get $\frac{(2s-(a+b))(2s-(b+c))(2s-(c+a))}{\Delta^3}$

$\Rightarrow\ \frac{abc}{\Delta^3}=\frac{KR^3}{(abc)^2}\ \Rightarrow\ \frac{64R^3}{(abc)^2}=\frac{KR^3}{(abc)^2}$

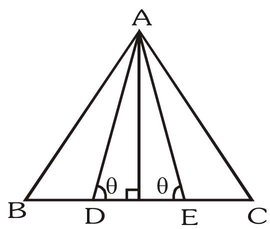
hence K = 64

20. $$\frac{\sum \frac{\Delta}{s-a}}{\sqrt{\sum \frac{\Delta^2}{(s-a)(s-b)}}} = \frac{\sum \frac{1}{s-a}}{\sqrt{\sum \frac{1}{(s-a)(s-b)}}}$$

$$= \frac{\sum \frac{1}{s-a}}{\sqrt{\sum \frac{s(s-c)}{\Delta^2}}} = \sum \frac{1}{(s-a)} \times \frac{\Delta}{s} = \sum \tan \frac{A}{2}$$

$\left(\because \quad r = (s-a)\tan \frac{A}{2} = (s-b)\tan \frac{B}{2} = (s-c)\tan \frac{C}{2}\right)$

24.



Using m – n theorem

$3\cot\theta = 2\cot B - \cot C$ ..... (i)

$3\cot(\pi - \theta) = \cot B - 2\cot C$ ......(ii)

add (i) & (ii)

$\cot B = \cot C$ ..... (iii)

$3\cot\theta = \cot B$ (using (i) & (iii))

$\Rightarrow \quad 3\tan B = \tan\theta$

$3\cot\theta = \cot C$ (using (i) & (iii))

$\Rightarrow \quad 3\tan C = \tan\theta$

Draw a perpendicular line from A joning mid point of BC. It is median as ΔABC is isosceles

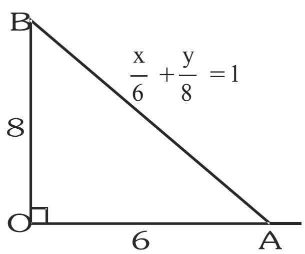
$B = 90 - \frac{A}{2}$

$\Rightarrow \quad 3\cot\frac{A}{2} = \tan\theta \quad \Rightarrow \quad 9\cot^2\frac{A}{2} = \tan^2\theta$

Now $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \dfrac{6\tan\theta}{\tan^2\theta - 9}$

EXERCISE - 02 BRAIN TEASERS

4. Hint :



$4 \sin \frac{C}{2} . \sin \frac{A}{2} . \sin \frac{B}{2}$.

$= 2 \sin \frac{C}{2} \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}$

$= 2 \cos\left(\frac{A+B}{2}\right) .\cos\left(\frac{A-B}{2}\right) - 2\sin^2\frac{C}{2}$

$(\because \ 1 - \cos C = 2\sin^2\frac{C}{2})$

$= \cos A + \cos B + \cos C - 1$

Aliter :

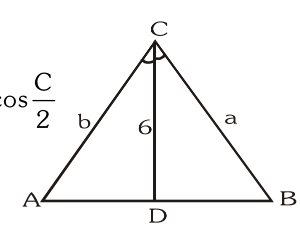
$\prod \sin\frac{A}{2} = \frac{r}{4R}$

$\Rightarrow \quad \prod \sin\frac{A}{2} = \frac{1}{10} \qquad [\because \quad r : R = 2 : 5]$

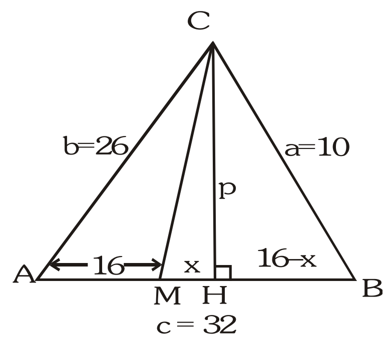
5. $\cos\frac{C}{2} = \frac{1}{3}$

$\Rightarrow \quad \ell(CD) = 6 = \frac{2ba}{b+a}.\cos\frac{C}{2}$

$\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{9}$



6.



In Δ AHC $p^2 + (16+x)^2 = (26)^2$ ...(i)

In Δ BCH $p^2 + (16-x)^2 = (10)^2$ ...(ii)

from (i) & (ii)

$\Rightarrow \quad (16+x)^2 - (16-x)^2 = (26)^2 - (10)^2$

$\Rightarrow \quad (32)(2x) = (36)(16)$

$\Rightarrow \quad x = 9.$

9. Draw parellelogram OBDC.

OE = ED & as

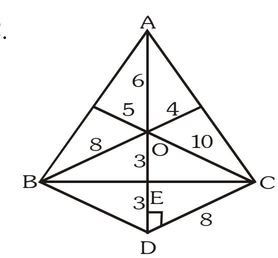
$OC^2 = OD^2 + DC^2$,

ΔODC is a

right angled $\Delta(6,8,10)$

$\Rightarrow$ Area of ΔADC

$= \frac{1}{2} \times 8 \times 12 = 48$



$\Delta EDC = \frac{1}{2} \times 3 \times 8 = 12$

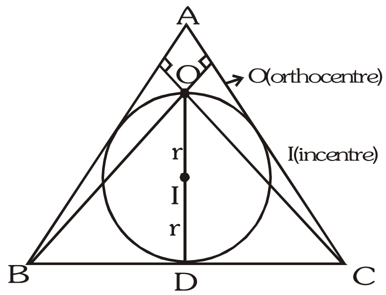
Area ΔAEC = Area ΔADC – Area ΔEDC

= 48 – 12 = 36

Area ΔAEC = 36 $\quad \Rightarrow \quad$ Area ΔABC = 72

10. Hint : $2R .\cos B. \cos C = 2r = OD$

$\Rightarrow \quad R \cos B. \cos C = 4R . \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$



$\Rightarrow \quad \cos^2 B = 4\sin\frac{A}{2} \cdot \sin^2\frac{B}{2} \quad$ (use $B = \frac{\pi}{2} - \frac{A}{2}$)

11. $\cos A + \cos C = 4\sin^2\frac{B}{2}$

$$\Rightarrow \quad \cos\left(\frac{A+C}{2}\right).\cos\left(\frac{A-C}{2}\right) = 2\sin^2\frac{B}{2}$$

$$= 2\cos^2\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{A-C}{2}\right)}{\cos\frac{A+C}{2}} = \frac{2}{1} \quad \Rightarrow \quad \cot\frac{A}{2}.\cot\frac{C}{2} = \frac{2+1}{2-1}$$

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

$$\Rightarrow \quad \cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$$

13. $\frac{\text{Perimeter}_{DEF}}{\text{Perimeter}_{ABC}} = \frac{R(\sin 2A + \sin 2B + \sin 2C)}{a+b+c}$

$$= \frac{4\sin A.\sin B.\sin C}{2(\sin B + \sin A + \sin C)}$$

$$= \frac{4 \times 8(\Pi \sin A/2)(\Pi \cos A/2)}{2 \times 4(\Pi \cos A/2)}$$

$$= \frac{r}{R} \quad \therefore (r = 4R\ \Pi\ \sin\frac{A}{2})$$

15. $(r_1 - r)\ (r_2 - r)\ (r_3 - r)$

$$= \left(\frac{\Delta}{s-a} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{s-b} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{s-c} - \frac{\Delta}{s}\right)$$

$$= \frac{\Delta^3 abc}{s^2.\Delta^2} \qquad \left(R = \frac{abc}{4\Delta}\right)$$

$$= \frac{\Delta}{s^2}\frac{abc}{4\Delta} \quad 4\Delta = 4r^2R$$

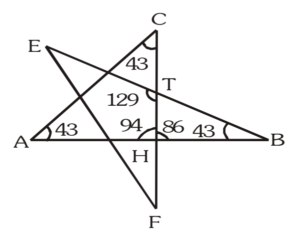
16. $\angle CAB = \angle ABE$

$= \angle\ ACF = 43$

$\angle CHB = 86$ {exterior angle of Δ ACH}

$\angle$ ETH = 129

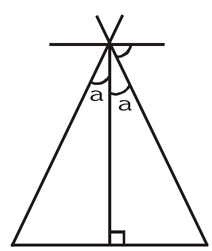
{exterior angle of ΔTHB}



EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

True/False :

1. True



2. False, R sin 2B < 0 is possible in obtuse angle Δ.

3. True, $p_1 = \frac{2\Delta}{a}$, $p_2 = \frac{2\Delta}{b}$, $p_3 = \frac{2\Delta}{c}$

$\frac{2\Delta}{a}, \frac{2\Delta}{b}, \frac{2\Delta}{c}$ are in A.P.

$\therefore$ a, b, c are in H.P.

4. True, $a^4 + b^4 + c^4 - 2a^2c^2 - 2a^2b^2 + b^2c^2 = 0$

$(a^2 - b^2 - c^2)^2 = b^2c^2 \Rightarrow \quad a^2 - b^2 - c^2 = \pm bc$

$a^2 - b^2 - c^2 = \pm bc$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \pm\frac{1}{2}$$

$\angle A = 60°$ or 120

Fill in the blank :

1. tan A = K

tan B = 2K

tan C = 3K

$\because \quad \sum \tan A = \prod \tan A$

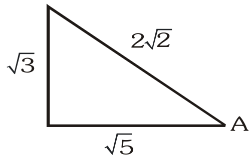
$\Rightarrow \quad 6\ K = 6K^3 \quad \Rightarrow 1 = K^2 \Rightarrow K = 1$

$\therefore \quad \sin A = \frac{1}{\sqrt{2}}$, $\sin B = \frac{2}{\sqrt{5}}$, $\sin C = \frac{3}{\sqrt{10}}$

$\Rightarrow \quad \sin A : \sin B : \sin C = \frac{1}{\sqrt{2}} : \frac{2}{\sqrt{5}} : \frac{3}{\sqrt{10}}$

$= \sqrt{5} : 2\sqrt{2} : 3$

2.



$$\cos A = \frac{\sqrt{5}}{2\sqrt{2}} = \frac{9+c^2-4}{2.3c} \Rightarrow \frac{3\sqrt{5}c}{\sqrt{2}} = 5 + c^2$$

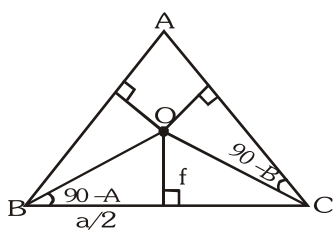
$$\Rightarrow \quad c^2 - \frac{3}{2}\sqrt{10}\,c + 5 = 0$$

$$\Rightarrow \quad 2c^2 - 3c\sqrt{10} + 10 = 0$$

$$\Rightarrow \quad c = \sqrt{10},\ \frac{1}{2}\sqrt{10}$$

$$\therefore \quad k_1 \ \&\ k_2 = 1 \ \&\ \frac{1}{2}$$

3.



$$\tan A = \frac{a}{2f}$$

$$\tan B = \frac{b}{2g}$$

$$\tan C = \frac{c}{2h}$$

Now $\sum \tan A = \prod \tan A$

$$\sum \frac{a}{2f} = \frac{abc}{8fgh}$$

$$\sum \frac{a}{f} = \frac{abc}{4fgh} \Rightarrow K = \frac{1}{4}$$

Match the column :

Use $p_1 = \frac{2\Delta}{a}$, $p_2 = \frac{2\Delta}{b}$, $p_3 = \frac{2\Delta}{c}$

1. (A) $$\frac{3}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \le \sqrt[3]{p_1p_2p_3} \quad (\text{HM} \le \text{GM})$$

(B) $$\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3}$$

$$= \frac{a\cos A + b\cos B + c\cos C}{2\Delta}$$

$$= \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2\Delta}$$

$(\therefore\ a = 2R \sin A)$

$$= \frac{R.4.\sin A.\sin B.\sin C}{2\Delta} = \frac{4R}{2\Delta}.\frac{abc}{8R^3} = \frac{1}{R}$$

(C) $$\frac{b^2}{c}.\frac{2\Delta}{a} + \frac{c^2}{a}.\frac{2\Delta}{b} + \frac{a^2}{b}.\frac{2\Delta}{c}$$

$$= 2\Delta\left(\frac{a^3+b^3+c^3}{abc}\right)$$

Now, $\frac{a^3+b^3+c^3}{3} \ge abc$ (AM $\ge$ GM)

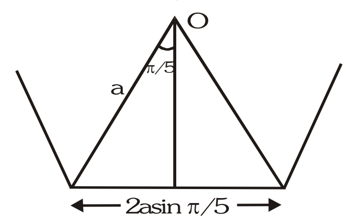
$$\frac{a^3+b^3+c^3}{abc} \ge 3$$

$$\Rightarrow 2\Delta\,.\left(\frac{a^3+b^3+c^3}{abc}\right) \ge 6\Delta.$$

(D) $$\Sigma p_1^{-2} = \frac{\Sigma a^2}{4\Delta^2}$$

Assertion & Reason :

2. Perimeter $= 10a \sin\frac{\pi}{5}$



For n sided polygon, perimeter $= \left(2a\sin\frac{\pi}{n}\right)$ n

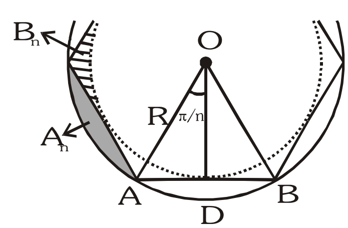
Hence statement II is false

4. $$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

AM $\ge$ HM

$$\frac{r_1+r_2+r_3}{3} \ge \frac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}} \Rightarrow \frac{r_1+r_2+r_3}{r} \ge 9.$$

Comprehension # 1 :



In Δ OAD

$$OD = R\cos\frac{\pi}{n},\ AD = R\sin\frac{\pi}{n}$$

A_n = Area of circle (circumscribing polygon)
– Area of polygon

$$A_n = \pi R^2 - \frac{n}{2}R^2 \sin\left(\frac{2\pi}{n}\right)$$

B_n = Area of polygon - Area of circle (Inscribed)

$$B_n = \frac{n}{2}R^2 \sin\left(\frac{2\pi}{n}\right) - \pi R^2 \cos^2\left(\frac{\pi}{n}\right)$$

1. If n = 6 then

$$A_n = \pi R^2 - \frac{3\sqrt{3}}{2}R^2$$

2. If n = 4 then value of

$$B_n = 2R^2 - \frac{\pi R^2}{2} = R^2\left(\frac{4-\pi}{2}\right)$$

3. $$\frac{A_n}{B_n} = \frac{\pi - \frac{n}{2}\sin\frac{2\pi}{n}}{\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) - \pi.\cos^2\frac{\pi}{n}}$$

put $\pi = n\theta$

we gets $\dfrac{2\theta - \sin 2\theta}{\sin 2\theta - 2\theta\cos^2\theta}$

$$= \frac{\theta - \sin\theta.\cos\theta}{\sin\theta.\cos\theta - \theta.\cos^2\theta} = \frac{\theta - \sin\theta.\cos\theta}{\cos\theta(\sin\theta - \theta\cos\theta)}$$

EXERCISE - 04[A] CONCEPTUAL SUBJECTIVE EXERCISE

2. LHS = $2R(\sum \sin A \cos B \cos C]$

$\{\because \sin(A + B + C)$

$= \sum\sin A \cos B \cos C - \sin A \sin B \sin C = 0$

$\Rightarrow \sum\sin A \cos B \cos C = \sin A \sin B \sin C\}$

LHS = $2R \sin A \sin B \sin C$

$$= 2R\left(\frac{a}{2R}\right)\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) = \frac{abc}{4} \times \frac{1}{R} \times \frac{1}{R} = \frac{\Delta}{R}$$

4. $$\text{LHS} = \frac{abc}{s}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{s}\frac{s\,\Delta}{abc} = \Delta$$

5. $B = 3C \Rightarrow \sin B = \sin 3C$

$$\Rightarrow \frac{\sin B}{\sin C} = 3 - 4\sin^2 C = \frac{b}{c}$$

$3 - 4\sin^2 C = \dfrac{b}{c}$

$\Rightarrow 3 - 4(1 - \cos^2 C) = \dfrac{b}{c}$

$\Rightarrow 4\cos^2 C = \dfrac{b+c}{c}$

$\Rightarrow \cos C = \sqrt{\dfrac{b+c}{4c}}$

$A + B + C = \pi$

$A = \pi - 4C$

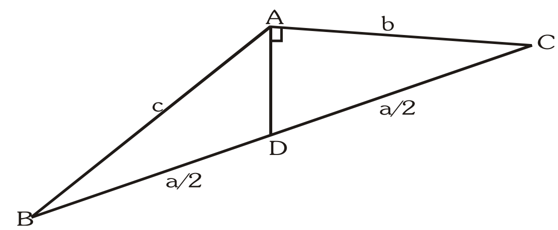
$\dfrac{A}{2} = \dfrac{\pi}{2} - 2C$

$\sin\dfrac{A}{2} = \cos 2C$

$= 2\cos^2 C - 1$

$= \dfrac{2(b+c)}{4c} - 1$

$\sin\dfrac{A}{2} = \dfrac{b-c}{2c}$

6. 

In ΔACD

$$\cos C = \frac{2b}{a}$$

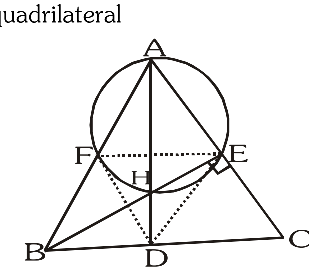
$$\Rightarrow \frac{a^2+b^2-c^2}{2ab} = \frac{2b}{a} \Rightarrow a^2 - c^2 = 3b^2$$

$$\cos A . \cos C = \frac{b^2+c^2-a^2}{2bc} \quad \frac{2b}{a}$$

$$= \frac{a^2-c^2+3c^2-3a^2}{3ac} = \frac{2(c^2-a^2)}{3ac}$$

8. $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{s-a+s-b+s-c}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

11. $\square$ AFHE is a cyclic quadrilateral



AH = 2R . cos A

R_1 = R . cos A

R_2 = R. cos B

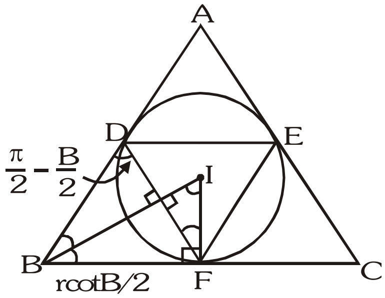
R_3 = R. cos C

Substituting values

ΣR_1 = R(cos A + cos B + cos C)

$$= R\left(4.\sin\frac{A}{2}\sin\frac{B}{2}.\sin\frac{C}{2} + 1\right) = R + r.$$

12.



Applying sine rule in Δ DBF

$$\frac{r.\cot B/2}{\cos B/2} = \frac{DF}{\sin B}$$

$$2r\cos\frac{B}{2} = DF$$

Similarly

$$2r\cos\frac{C}{2} = EF \text{ and } 2r\cos\frac{A}{2} = DE$$

$$\angle BDF = \angle BFD = \frac{\pi - B}{2} \Rightarrow \angle DFI = \frac{B}{2}$$

Similarly $\angle IFE = \dfrac{C}{2}$

$$\therefore \angle DFE = \frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$$

Similarly

$$\angle DEF = \frac{\pi}{2} - \frac{C}{2} \quad \& \quad \angle EDF = \frac{\pi}{2} - \frac{C}{2}$$

$$\text{Ar}(\Delta DEF) = 2r^2\cos\frac{B}{2}.\cos\frac{C}{2}.\cos\frac{A}{2}$$

$$= 2r^2\sqrt{\frac{(s)(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}\sqrt{\frac{(s)(s-a)}{bc}}$$

$$= \frac{2rs\,\Delta}{abc} = \frac{r^2 s}{2R}.$$

EXERCISE - 04[B] BRAIN STORMING SUBJECTIVE EXERCISE

1. To prove that $\rightarrow \dfrac{\sin^2 A/2}{\sin 2A}, \dfrac{\sin^2 B/2}{\sin 2B}, \dfrac{\sin^2 C/2}{\sin 2C}$

are in H.P.

or $\dfrac{\sin 2A}{\sin^2 A/2}, \dfrac{\sin 2B}{\sin^2 B/2}, \dfrac{\sin 2C}{\sin^2 C/2}$ are in A.P.

Now $\dfrac{\sin 2A}{\sin^2 A/2} = \dfrac{2\sin A.\cos A}{\sin^2 A/2} = 4\cot A/2\cos A$

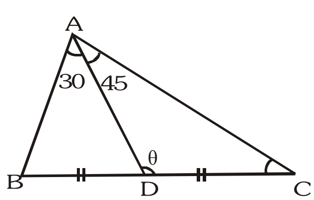
put $\cos A = 1 - 2\sin^2 A/2$

we get $\dfrac{\sin 2A}{\sin^2 A/2} = 4\left(\cot\dfrac{A}{2} - \sin A\right)$

then,

use half angle formulae to prove terms are in A.P.

4.



Applying m - n theorem we get

$2\cot\theta = \cot 30 - \cot 45$

$\Rightarrow \tan\theta = \sqrt{3} + 1$ ...(i)

Now $\sin C = \sin\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$

using (i), we get $\sin C = \dfrac{\sqrt{3}+2}{\sqrt{2}\sqrt{5+2\sqrt{3}}}$

Now applying sine law in Δ ADC we get,

$$\frac{AD}{\sin C} = \frac{DC}{\sin 45^\circ}$$

$$\therefore DC = \frac{1}{\sqrt{11-6\sqrt{3}}}\ \frac{\sqrt{2}\sqrt{5+2\sqrt{3}}}{\sqrt{3}+2}.\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{5+2\sqrt{3}}}{\sqrt{11-6\sqrt{3}}\sqrt{7+4\sqrt{3}}} = 1$$

Hence, DC = BD = 1 so BC = 2

5. $$\frac{r_1}{bc} = \frac{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{(2R\sin B)(2R\sin C)}$$

$$= \frac{4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{4R^2\left(4\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)}$$

$$= \frac{1}{4R}\frac{\sin^2\frac{A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\sin^2\frac{A}{2}}{r}$$

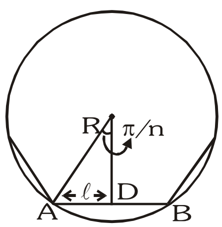
So, $$\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab} = \frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{r}$$

$$= \frac{\frac{1}{2}(3-(\cos A+\cos B+\cos C))}{r}$$

$$= \frac{\left[3-\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\right]}{2r}$$

$$= \frac{\left[2-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]}{2r} = \frac{\left[2-\frac{r}{R}\right]}{2r} = \frac{1}{r}-\frac{1}{2R}$$

10.



$\ell = R \,.\, \sin \pi/n = AD$

$$A_1 = \frac{n}{2}R^2 \sin\frac{2\pi}{n}$$

$$B_1 = nR^2 \tan\frac{\pi}{n}$$

Replacing n by 2n we get

$$A_2 = nR^2 \sin\frac{\pi}{n}$$

$$B_2 = 2nR^2 \tan\frac{\pi}{2n}$$

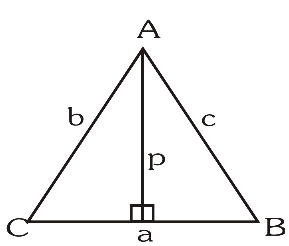
$$A_1B_1 = n^2R^4\sin^2\frac{\pi}{n} = A_2^{\,2}$$

Now,

$$\frac{1}{A_2}+\frac{1}{B_1} = \frac{1}{nR^2.\sin\pi/n}+\frac{\cos\pi/n}{nR^2.\sin\pi/n}$$

$$= \frac{1}{nR^2.\tan\pi/2n} = \frac{2}{B_2}.$$

12.



$\frac{b}{c} = r$ $(r < 1)$ [Given]

Now $\Delta = \frac{1}{2}\ ap = \frac{1}{2}bc \sin A$

$$\Rightarrow p = \frac{bc}{a}\sin A = \left(\frac{ar}{1-r^2}\right)\left(\frac{1-r^2}{ar}\right)\frac{bc}{a}\sin A$$

$$= \frac{ar}{(1-r^2)}\left(\frac{1-b^2/c^2}{a.\frac{b}{c}}\right)\frac{bc}{a}\sin A$$

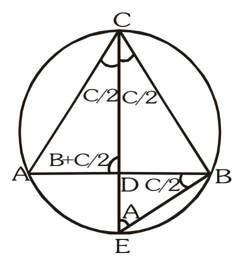
$$= \frac{ar}{1-r^2}\left(\frac{c^2-b^2}{abc}\right)\frac{bc\sin A}{a} = \frac{ar}{1-r^2}\left(\frac{c^2-b^2}{a^2}\right)\sin A$$

$$= \frac{ar}{1-r^2}\left(\frac{\sin^2 C-\sin^2 B}{\sin^2 A}\right)\sin A$$

$$= \frac{ar}{1-r^2}(\sin(C-B))$$

$$p \le \frac{ar}{1-r^2} \qquad [\because \sin(C-B) \le 1]$$

13.



Applying sine law in ΔBDE

$$\frac{\sin\frac{C}{2}}{DE} = \frac{\sin A}{\frac{ac}{a+b}} \quad ...(i) \qquad \left(BD = \frac{ac}{a+b}\right)$$

sine law in ΔCBE

$$\frac{\sin(B+C/2)}{CE} = \frac{\sin A}{a} \quad ...(ii)$$

$\Rightarrow$ divide (i) & (ii)

$$\frac{\sin\frac{C}{2}}{\sin\left(B+\frac{C}{2}\right)}\ \frac{CE}{DE} = \frac{a+b}{ac}.\,a$$

$$\frac{CE}{DE} = \frac{\frac{a+b}{c}.2\sin\left(B+\frac{C}{2}\right).\cos\frac{C}{2}}{\sin C}$$

$$= \frac{a+b}{c}.\frac{\sin(B+C)+\sin B}{\sin C}$$

$$= \frac{a+b}{c}.\frac{\sin A+\sin B}{\sin C}$$

$$\frac{CE}{DE} = \frac{(a+b)^2}{c^2}.$$

1. To prove that $16\Delta^2 \le 2s\,.\,abc$

$\Rightarrow (s-a)(s-b)(s-c) \le \dfrac{abc}{8}$

Let $\begin{cases} s-a = x \Rightarrow x+y=c \\ s-b = y \Rightarrow y+z=a \\ s-c = z \Rightarrow z+x=b \end{cases}$

Applying AM $\ge$ GM

$x + y \ge 2\sqrt{xy}$ ....(i)

$y + z \ge 2\sqrt{yz}$ ....(ii)

$z + x \ge 2\sqrt{zx}$ ....(iii)

Multiplying (i), (ii) & (iii)

$(x+y)(y+z)(z+x) \ge 8(x\,y\,z)$

$\dfrac{abc}{8} \ge (s-a)(s-b)(s-c).$

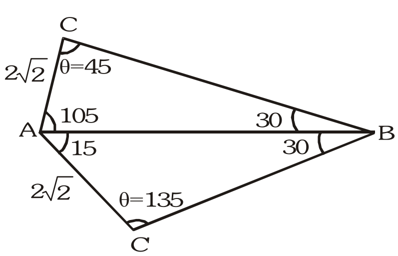
3. Hint : put $I_n = \dfrac{n}{2}.\sin\left(\dfrac{2\pi}{n}\right)$

& put $O_n = n\tan\left(\dfrac{\pi}{n}\right)$.

7. Applying sine-law in ΔABC

$\dfrac{4}{\sin\theta} = \dfrac{2\sqrt{2}}{\sin 30^\circ} \Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \theta = 45\,,\ 135$



Area of ΔABC – Area $\Delta AC'B$

$= \dfrac{1}{2}\quad 2\sqrt{2}\quad 4\ (\sin 105 - \sin 15)$

$= 4\sqrt{2}\quad 2\cos 60\quad \sin 45$

$= 4\sqrt{2}\quad 2\quad \dfrac{1}{2}\quad \dfrac{1}{\sqrt{2}} = 4.$

8.(b) $\Delta = 15\sqrt{3}$

$\angle ACB > 90^\circ$

$r^2 = ?$

$\Delta = \dfrac{1}{2}\,10.6.\sin C = 15\sqrt{3}$

$\sin C = \dfrac{\sqrt{3}}{2}$

$C = 120$

$\cos C = \dfrac{100+36-c^2}{2.10.6}$

$c = 14$

$r = \dfrac{\Delta}{s}$

$s = \dfrac{a+b+c}{2} = \dfrac{10+6+14}{2} = 15$

$\therefore\ r = \sqrt{3} \Rightarrow r^2 = 3$

(c) $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$\dfrac{\sqrt{3}}{2} = \dfrac{(x^2+x+1)^2+(x^2-1)^2-(2x+1)^2}{2(x^2+x+1)(x^2-1)}$

$\Rightarrow x^4(2-\sqrt{3})+x^3(2-\sqrt{3})-3x^2-x(2-\sqrt{3})+(\sqrt{3}+1)=0$

$\Rightarrow (x^2-1)[(2-\sqrt{3})x^2+(2-\sqrt{3})x-(\sqrt{3}+1)]=0$

Now $\begin{cases} x^2+x+1+x^2-1 > 2x+1 \\ x^2+x+1+2x+1 > x^2-1 \\ 2x+1+x^2-1 > x^2+x+1 \end{cases}$ ($\because$ sum of two sides is greater than third side)

$\Rightarrow x > 1 \Rightarrow x = 1+\sqrt{3}$

Alternate :

$\tan\dfrac{\pi}{12} = \sqrt{\dfrac{(s-b)(s-a)}{s(s-c)}}$

$s = x^2 + \dfrac{3x+1}{2}$,

$s-b = \dfrac{3(x+1)}{2}, s-a = \dfrac{x-1}{2}, s-c = x^2 - \dfrac{(x+1)}{2}$

$\tan\dfrac{\pi}{12} = \sqrt{\dfrac{\frac{3}{2}(x+1)\frac{(x-1)}{2}}{\left(x^2+\frac{3x+1}{2}\right)\left(x^2-\frac{x+1}{2}\right)}}$

Simiplying

$2-\sqrt{3} = \dfrac{\sqrt{3}}{2x+1}$

$x = \sqrt{3}+1$

9. $\dfrac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \dfrac{(1-\cos P)}{(1+\cos P)}$

$= \tan^2\dfrac{A}{2} = \dfrac{\Delta^2}{s^2(s-a)^2} = \dfrac{((s-b)(s-c))^2}{\Delta^2}$

$s = 4$

$= \left(\dfrac{\left(4-\frac{7}{2}\right)\left(4-\frac{5}{2}\right)}{\Delta}\right)^2 = \left(\dfrac{3}{4\Delta}\right)^2$