METHOD OF DIFFERENTIATION

EXERCISE - 01

CHECK YOUR GRASP

5. $x^{\left(\frac{\ell+m}{(m-n)(n-\ell)}+\frac{m+n}{(n-\ell)(\ell-m)}+\frac{n+\ell}{(\ell-m)(m-n)}\right)} = x^{\left(\frac{\ell^2-m^2+m^2-n^2+n^2-\ell^2}{(m-n)(n-\ell)(\ell-m)}\right)}$

$= x^0 = 1 \qquad \therefore \quad \frac{d}{dx}(1) = 0$

7. $\cos^{-1}\left(\frac{1-y^2/x^2}{1+y^2/x^2}\right) = \log a$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{y}{x}\right) = \log\ a \quad \Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\log a}{2}$

$\Rightarrow \quad \frac{y}{x} = \tan\left(\frac{\log a}{2}\right)$

Now differentiating both sides, we get

$\frac{x\frac{dy}{dx}-y}{x^2} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$

8. $f(x) = (x-1)^{100}.\ (x-2)^{2.99}.\ (x-3)^{3.98}.....(x-100)^{100.1}$

Take log & than differentiate we get

Now $\frac{f'(x)}{f(x)} = \frac{1.100}{x-1}+\frac{2.99}{x-2}+\frac{3.98}{x-3}+....\frac{100.1}{x-100}$

$\frac{f'(101)}{f(101)} = 1 + 2 + 3\ .....\ + 100 = 5050$

$\therefore \quad \frac{f(101)}{f'(101)} = \frac{1}{5050}$

10. $y = \frac{x}{a+\frac{x}{b+y}} \Rightarrow y = \frac{x(b+y)}{ab+ay+x}$

$\Rightarrow \quad aby + ay^2 + xy = xb + xy$

$\Rightarrow \quad ab\frac{dy}{dx} + 2ay\frac{dy}{dx} = b \quad \Rightarrow \quad \frac{dy}{dx} = \frac{b}{ab+2ay}$

13. $x^2(1+y) = y^2(1+x) \quad \Rightarrow x^2 - y^2 + xy(x-y) = 0$

$\Rightarrow \quad (x-y)\ (x+y+xy) = 0$

Now $x \neq y$ [does not satisfy the given equation]

$\therefore \quad x + y + xy = 0 \Rightarrow y = \frac{-x}{1+x}$

$\therefore \quad \frac{dy}{dx} = \frac{-(1+x)+x}{(1+x)^2} = -\frac{1}{(1+x)^2}$

18. $g(x) = f^{-1}(x) \quad \Rightarrow \quad fog(x) = x$

$\Rightarrow \quad f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$

$\Rightarrow \quad g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(a)} = \frac{1+a^2}{a^{10}}$

22. $f'(x) = g(x)$ and $g'(x) = -f(x)$

Now $\frac{d}{dx}[f^2(x) + g^2(x)] = 2f(x)\ f'(x) + 2g(x)\ g'(x)$

$= 2f(x)g(x) - 2g(x)f\ (x) = 0$

$\therefore \quad f^2(x) + g^2(x) = \text{constant}$

$f^2(5) + g^2(5) = 4 + 4 = 8$

$\therefore \quad f^2(10) + g^2(10) = 8$

23. $f(x) = x^n$

$\therefore \quad f'(x) = nx^{n-1},\ f''(x) = n(n-1)x^{n-2}$

....

$f^{''''....... n\ times}(x) = n!$

Now $f(1) - \frac{f'(1)}{1!}+\frac{f''(1)}{2!} - .... \frac{f^{''''........n\ times}(1)}{n!}$

$= 1 - \frac{n}{1!}+\frac{n(n-1)}{2!} - .... \frac{n!}{n!} = (1-1)^n = 0$

28. $\lim_{x\to4}\frac{\sqrt{f(x)}-\sqrt{g(x)}}{\sqrt{x}-2} = \lim_{x\to4}\frac{\frac{1}{2}\frac{f'(x)}{\sqrt{f(x)}}-\frac{1}{2}\frac{g'(x)}{\sqrt{g(x)}}}{\frac{1}{2\sqrt{x}}}$

$= \frac{\frac{9}{\sqrt{2}}-\frac{6}{\sqrt{2}}}{\frac{1}{2}} = 3\sqrt{2}$

29. $y = e^{-x} \quad \& \quad y = e^{-x}\sin x$

$y' = -e^{-x}\ ...(i) \quad \& \quad y' = -e^{-x}(\sin x - \cos x)\ ...(ii)$

equating (i) & (ii)

$e^{-x}(1 - \sin x + \cos x) = 0$

$e^{-x} \neq 0 \quad \Rightarrow \quad 1 - \sin x + \cos x = 0$

$\Rightarrow \quad 2\cos^2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\Rightarrow \quad 2\cos\frac{x}{2}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right) = 0 \quad \Rightarrow x = \frac{\pi}{2}, \pi$

slope can be $-e^{-\pi/2}$ & $-e^{-\pi}$.

33. $f^{-1}(x) = g(x) \quad \Rightarrow \quad x = f(g(x))$

Differentiating both sides,

$1 = f'(g(x))\ g'(x) \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$

Now $f'(x) = 2x + 3$

So $g'(x) = \frac{1}{2g(x)+3} \quad \Rightarrow \quad g'(1) = \frac{1}{2(g(1))+3}$

$gof(x) = x \quad \Rightarrow \quad g'(f(x))\ f'(x) = 1$

$f(x) = 1$ at $x = 1$ & $f'(1) = 5$

$g'(1)f'(1) = 1 \quad \Rightarrow \quad g'(1) = 1/5$

EXERCISE - 02 BRAIN TEASERS

2. $y^2 = p(x) \Rightarrow 2y\frac{dy}{dx} = p'(x)$

$\Rightarrow 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = p''(x)$

$\Rightarrow 2y\frac{d^2y}{dx^2} + 2\left(\frac{p'(x)}{2y}\right)^2 = p''(x)$

$\Rightarrow 4y^3\frac{d^2y}{dx^2} + (p'(x))^2 = 2y^2p''(x)$

$\Rightarrow 4y^3\frac{d^2y}{dx^2} = 2p(x)p''(x) - (p'(x))^2$

$\Rightarrow 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$

$= \frac{1}{2}[2p'(x)p''(x)+2p(x)p'''(x) - 2p'(x)p''(x)]$

$\Rightarrow 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = p(x)p'''(x)$

3. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{dx/dy}\right) = \frac{d}{dy}\left(\frac{1}{dx/dy}\right)\frac{dy}{dx}$

$= -\frac{1}{(dx/dy)^2}\cdot\frac{d^2x}{dy^2}\cdot\frac{dy}{dx}$

Now put the value of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in

$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

On solving we get $\frac{d^2x}{dy^2} - y\left(\frac{dx}{dy}\right)^2 = 0$

11. $\sqrt{y+x} + \sqrt{y-x} = c$

$\frac{1}{2\sqrt{y+x}}\left(\frac{dy}{dx}+1\right) + \frac{1}{2\sqrt{y-x}}\left(\frac{dy}{dx}-1\right) = 0$

$\frac{dy}{dx}\left(\frac{1}{\sqrt{y+x}}+\frac{1}{\sqrt{y-x}}\right) = \frac{1}{\sqrt{y-x}} - \frac{1}{\sqrt{y+x}}$

$\frac{dy}{dx} = \frac{\sqrt{y+x}-\sqrt{y-x}}{\sqrt{y+x}+\sqrt{y-x}}$

$= \frac{x}{y+\sqrt{y^2-x^2}} = \frac{y-\sqrt{y^2-x^2}}{x} = \frac{2x}{c^2}$

(By rationalizing Nr. or Dr.)

14. $\ell = \lim_{x\to 0^+} x^m (\ell n\ x)^n$

$= \lim_{x\to 0}\frac{(\ell nx)^n}{(1/x)^m}$ $\left[\text{as } x \to 0 \left(\frac{\infty}{\infty}\right)\right]$

$\ell = \lim_{x\to 0}\frac{n(\ell nx)^{n-1}1/x}{-mx^{-m-1}}$ (applying L'Hopital's rule)

$= \lim_{x\to 0}\frac{n(\ell nx)^{n-1}}{-m(1/x)^m}$

Again differentiating (n – 1) times

$\ell = \lim_{x\to 0}\frac{n!}{(-1)^n m^n\left(\frac{1}{x^m}\right)}$

$\ell = 0$

15. $\lim_{x\to 0}\frac{\log\cos x}{\log\cos\frac{x}{2}}\cdot\frac{\log\left|\sin\frac{x}{2}\right|}{\log|\sin x|}$

$= \lim_{x\to 0}\frac{\log\cos x}{\log\cos\frac{x}{2}}\ \lim_{x\to 0}\frac{\log\left|\sin\frac{x}{2}\right|}{\log|\sin x|}$

$= \lim_{x\to 0}\frac{\tan x}{\tan\frac{x}{2}}\ \lim_{x\to 0}\frac{\cot\frac{x}{2}}{\cot x} = \lim_{x\to 0}\frac{\tan^2 x}{\tan^2\frac{x}{2}} = 4$

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

True and False :

1. $\frac{u(x)}{v(x)} = 7 \Rightarrow u(x) = 7v(x)$

$\frac{u'(x)}{v'(x)} = 7 = p$ & $q = 0$ so $\frac{p+q}{p-q} = 1$

Match the Column :

2. (A) $f'(x) = 3x^2 + 1 \Rightarrow f'(x^2 + 1) = 3(x^2 + 1)^2 + 1$

$f'(x^2 + 1)$ at $x = 0$ is 4

(B) $f(x) = \log_{x^2}\log(x) = \frac{1}{2}\log_x(\log x) = \frac{1}{2}\frac{\log(\log x)}{\log x}$

$f'(x) = \frac{1}{2}\left(\frac{\log x.\frac{1}{\log x}.\frac{1}{x} - \frac{\log(\log(x))}{x}}{(\log x)^2}\right)$

$= \frac{1}{2}\left(\frac{1-\log.(\log(x))}{x(\log x)^2}\right) \Rightarrow f'(e^e) = 0$

(C) $y = \ell n \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$$\frac{dy}{dx} = \frac{1}{2}\frac{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{1}{\cos x}$$

$= \sec x$

Hence $p = 0$

(D) $f(x) = |x^3 - x^2 + x - 1| \sin x$

$f(x) = |(x^2 + 1)(x - 1)| \sin x$

$= (x^2 + 1)(x - 1) \sin x$ when $x \geq 1$

$= -(x^2 + 1)(x - 1) \sin x$ when $x < 1$

Now $28f(f(\pi)) = 0$

$\therefore$ At $x = 0$

$f'(x) = -[2x(x-1) \sin x + (x^2 + 1)$
$\sin x + (x^2 + 1)(x - 1) \cos x]$

$4f'(0) = 4$

Assertion and Reason :

1. **Hint** : Statement I : f(x) is constant function
Statement II : It is true

Comprehension # 1 :

$f(x + y) - f(x) = f(y) - 1 + 2xy$

$\Rightarrow f(0 + 0) - f(0) = f(0) - 1 + 2(0)(0) \Rightarrow f(0) = 1$

and $f'(0) = 1$ (given)

Also $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left[\frac{f(h)-1}{h}+\frac{2xh}{h}\right]$

$= \lim_{h\to 0}\frac{f(h)-f(0)}{h} + 2x = f'(0) + 2x$

$f'(x) = 1 + 2x$

Integrate it

$f(x) = x^2 + x + c$

$f(x) = x^2 + x + 1$ $[f(0) = 1 \Rightarrow c = 1]$

1. $\ell n(x^2 + x + 1) \rightarrow$ Domain R

2. $y = \log_{3/4}(x^2 + x + 1)$

Now $x^2 + x + 1 \geq \frac{3}{4}$

hence range is $(-\infty, 1]$

3. $g(0) = \frac{g(0)+g(0)}{k} \Rightarrow 2g(0) = kg(0)$

$\Rightarrow g(0) = 0$ (as $k \neq 2$)

$$g'(x) = \lim_{h\to 0}\frac{g\left(\frac{x+h}{1}\right)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{g(x)+g(h)}{1}-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(h)-g(0)}{h} = g'(0) = \lambda$$

$g(x) = \lambda x + c$

$\Rightarrow g(x) = \lambda x$ $[g(0) = 0]$

Now $x^2 + x + 1 = \lambda x \Rightarrow x^2 + (1 - \lambda)x + 1 = 0$

For concident pt. $D = 0$

$(1 - \lambda)^2 - 4 = 0$

$\Rightarrow \lambda = 3, -1$

Comprehension # 3 :

1. $\text{LHD} = \lim_{h\to 0^-}\frac{f(-a+h)-f(-a)}{h}$

$= \lim_{h\to 0^-}\frac{-f(a-h)+f(a)}{h} = \lim_{h\to 0^-}\frac{f(a-h)-f(a)}{-h}$

2. $\text{RHD} = \lim_{h\to 0^-}\frac{f'(a)-f'(a-h)}{h} = \frac{f'(a)+f'(h-a)}{h}$

Since derivative of even function is odd & vice versa.

3. $\lim_{h\to 0}\frac{f(-x)-f(-x-h)}{h} = \lim_{h\to 0}\frac{f(-x-h)-f(-x)}{-h}$

$= f'(-x)$ ... (i)

and $\lim_{h\to 0}\frac{f(x)-f(x-h)}{-h} = -f'(x)$ ... (ii)

from (i) and (ii) f'(x) is odd function and hence f(x) is even function.

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

5. $f_1(x) = e^{f_0(x)} = e^x$

$f_2(x) = e^{f_1(x)} = e^{e^x}$

$f_3(x) = e^{e^{e^x}}$

similarly $f_n(x) = e^{e^{e^{\cdots\cdots(n-1)\text{times}(x)}}}$

Now $\frac{d}{dx}[f_n(x)] = e^{f_{n-1}(x)}.\frac{d}{dx}(e^{f_{n-1}(x)})$

On differentiating it completely we get

$\frac{d}{dx}[f_n(x)] = e^{f_{n-1}(x)}.e^{f_{(n-2)}(x)}.e^{f_{(n-3)}(x)}.......e^{f_0(x)}$

$= f_n(x) \,.\, f_{(n-2)}x.............f_1(x)$

6. $y = \frac{x^2}{2}+\frac{1}{2}x\sqrt{x^2+1} + \ell n\sqrt{x+\sqrt{x^2+1}}$

$2y = x^2 + x\sqrt{x^2+1}+\ell n\left(x+\sqrt{x^2+1}\right)$ .....(i)

$$2y' = 2x + \sqrt{x^2+1} + \frac{2x^2}{2\sqrt{x^2+1}}+\frac{1+\frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$$

$$2y' = 2x + \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}$$

$= 2x + \sqrt{x^2+1} + \sqrt{x^2+1}$

$y' = \left(x + \sqrt{x^2+1}\right)$

Put the value of $x + \sqrt{x^2+1}$ in (i)

$2y = xy' + \ell n\, y'$

11. $\sqrt{x^2+y^2} = e^{\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)}$

$\Rightarrow \quad \frac{1}{2}\ell n\,(x^2 + y^2) = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)$

$\Rightarrow \quad \frac{1}{2}\left(\frac{2x+2yy'}{x^2+y^2}\right) = \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\left[\frac{\sqrt{x^2+y^2}\,y' - \frac{y(2x+2yy')}{2\sqrt{x^2+y^2}}}{x^2+y^2}\right]$

$\Rightarrow \quad x + yy' = \frac{\sqrt{x^2+y^2}}{x}\left[\frac{(x^2+y^2)y' - y(x+yy')}{\sqrt{x^2+y^2}}\right]$

$\Rightarrow \quad x + yy' = xy' - y$

Again on differentiation we get

$y'' = \frac{2(x^2+y^2)}{(x-y)^3}$

12. $f(x) = x^2 - 4x - 3$ & $f(x) = 9$

For $x = 6, -2$

$\Rightarrow \quad x = 6 \qquad (x > 2)$

Now $y = f(x) \Rightarrow f^{-1}(y) = x$

$\Rightarrow \quad g(y) = x$

$\Rightarrow \quad g'(y) = \frac{dx}{dy} = \frac{1}{2x-4} = \frac{1}{8}$

14. $\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \tan\theta\sqrt{x^2+4}$

$\frac{dy}{d\theta} = n\sec^{n-1}\theta \,.\, \sec\theta \,.\, \tan\theta + n\cos^{n-1}\theta\,(-\sin\theta)$

$= n\tan\theta\,[\sec^n\theta + \cos^n\theta] = n\tan\theta\sqrt{y^2+4}$

$\therefore \quad \left(\frac{dy}{dx}\right)^2 = n^2\frac{(y^2+4)}{(x^2+4)}$

17. $\lim\limits_{x\to 0}\frac{(a+b\cos x)x - c\sin x}{x^5} = 1$

Now using L' hospital rule

$\lim\limits_{x\to 0}\frac{-bx\sin x + (a+b\cos x) - c\cos x}{5x^4}$

For limit to exist $a + b - c = 0$ ....(i)

Again $\lim\limits_{x\to 0}\frac{-b(\sin x + x\cos x) - b\sin x + c\sin x}{20\,x^3}$

$= \lim\limits_{x\to 0}\frac{-b(\cos x + \cos x - x\sin x) - b\cos x + c\cos x}{60\,x^2}$

$= \lim\limits_{x\to 0}\frac{(-3b+c)\cos x + xb\sin x}{60x^2}$

For limit to exist $-3b + c = 0$ ....(ii)

$\Rightarrow \quad \lim\limits_{x\to 0}\frac{xb\sin x}{60x^2} = 1$

$\Rightarrow \quad b = 60,\ c = 180,\ a = 120$

21. $\lim\limits_{x\to 0}\frac{\log|\tan 2x|}{\log|\tan x|} = \lim\limits_{x\to 0} 2\left(\frac{\sec^2 2x}{\sec^2 x}\cdot\frac{\tan x}{\tan 2x}\right) = 1$

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. $x = \frac{1}{z} \Rightarrow \frac{dx}{dz} = -\frac{1}{z^2}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}(-z^2)$

$\frac{d}{dx}(y') = \frac{dy'}{dz}\cdot\frac{dz}{dx} + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$

$\frac{d^2y}{dx^2} = 2z^3\cdot\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$

$\Rightarrow \quad \frac{d^2f}{dx^2} = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$

3. $z = \ell n\left(\tan\frac{x}{2}\right)$

$\frac{dz}{dx} = \frac{1}{\sin x}$ Now $\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{dy}{dx}\sin x$

$\frac{d^2y}{dz^2} = \frac{d}{dx}\left(\frac{dy}{dx}\sin x\right)\cdot\frac{dx}{dz}$

$\frac{d^2y}{dz^2} = \left(\sin x\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx}\right)\sin x$ ....(i)

$\frac{d^2y}{dz^2} = \sin^2 x\frac{d^2y}{dx^2} + \sin x\cos x\frac{dy}{dx}$

$\text{cosec}^2 x\frac{d^2y}{dz^2} = \frac{d^2y}{dx^2} + \cot x\frac{dy}{dx}$

Now put value from given equation

$\text{cosec}^2 x\frac{d^2y}{dz^2} + 4y\,\text{cosec}^2 x = 0$

$\frac{d^2y}{dz^2} + 4y = 0$

4. $f(2x) = f'(x)f''(x)$

Let the degree of 'f' be n.

Comparing highest power on both sides

$$n = n - 1 + n - 2 \Rightarrow n = 3$$

Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$

$f(2x) = f'(x)f''(x)$

$\therefore \quad (8a_0x^3 + 4a_1x^2 + 2a_2x + a_3)$

$= (3a_0x^2 + 2a_1x + a_2)(6a_0x + 2a_1)$

Comparing coefficient of x^3

$$8a_0 = 18a_0^2 \Rightarrow a_0 = \frac{4}{9}$$

Rest all are zero

$$\therefore \quad f(x) = \frac{4}{9}x^3$$

5. $$\begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + s_1X & t_1X + tX_1 \\ X_2 & sX_2 + 2s_1X_1 + Xs_2 & tX_2 + 2X_1t_1 + Xt_2 \end{vmatrix}$$

$[C_2 \to C_2 - sC_1,\ C_3 \to C_3 - tC_1]$

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X \begin{vmatrix} Xs_1 & Xt_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$(R_2 \to R_2 - 2X_1R_1)$

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

7. $f(x) = c(x - \alpha)^2$

$$F(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Now $F(\alpha) = 0 \quad \therefore \ (x - \alpha)$ is root of $F(x)$.

$$F'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Now $F'(\alpha) = 0 \Rightarrow x - \alpha$ is a factor of $F'(x)$. So $(x - \alpha)$ must be repeated at least two times in $F(x)$.

$\Rightarrow \quad F(x)$ is divisible by $f(x)$.

11. $$f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1} = \lim_{x\to 0}\frac{e^{2x}-1-2x}{x(e^{2x}-1)}$$

$$= \lim_{x\to 0}\frac{2e^{2x}-2}{e^{2x}-1+x(2e^{2x})} \text{ (by L–hopital rule)}$$

$$= \lim_{x\to 0}\frac{4e^{2x}}{2e^{2x}+2e^{2x}+4e^{2x}.x} = 1 \quad \Rightarrow f(0) = 1$$

$$f'(0^+) = \lim_{h\to 0}\frac{\frac{1}{0+h} - \frac{2}{e^{2h}-1} - 1}{h}$$

$$= \lim_{h\to 0}\frac{e^{2h}-1-2h-h(e^{2h}-1)}{h^2(e^{2h}-1)}$$

$$= \lim_{h\to 0}\frac{1+2h+\frac{4h^2}{2!}\ldots\ldots-1-2h-h(e^{2h}-1)}{h^2(e^{2h}-1)}$$

$$= \lim_{h\to 0}\frac{(2h^2+\frac{8h^3}{3!}+\ldots.)-h(2h+\frac{(2h)^2}{2!}+\frac{8h^3}{3!}\ldots)}{h^2(2h+\frac{(2h)^2}{2!}+\ldots.)} = -\frac{1}{3}$$

Similarly $f'(0^-) = -\frac{1}{3}$.

3. $y = \log_y x \qquad y = \dfrac{\log x}{\log y}$

$y \quad \log y = \log x$

$\dfrac{dy}{dx}\log y + y \quad \dfrac{1}{y}\ \dfrac{dy}{dx} = \dfrac{1}{x}$

$\dfrac{dy}{dx}(\log y + 1) = \dfrac{1}{x} \qquad \boxed{\dfrac{dy}{dx} = \dfrac{1}{x(1+\log y)}}$

4. $x = 3\cos\theta - 2\cos^3\theta \qquad y = 3\sin\theta - 2\sin^3\theta$

$\dfrac{dx}{d\theta} = -3\sin\theta + 6\cos^2\theta \quad \sin\theta$

$\dfrac{dy}{d\theta} = 3\cos\theta - 6\sin^2\theta\cos\theta$

$\dfrac{dy}{dx} = \dfrac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$

$= \dfrac{\cos\theta - 2\sin^2\theta\cos\theta}{-\sin\theta + 2\cos^2\theta\sin\theta} = \cot\theta$

6. $f(x) = x^n$

$f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} - \dfrac{f'''(1)}{3!} \ \dots \ \dfrac{(-1)^x f^n(1)}{n!}$

put $n = 1$

$f(x) = x \quad$ for series $\ f(1) - \dfrac{f'(1)}{1} = 0$

$f(1) = 1 \qquad$ put $n = 2$

$f(x) = x^2 \qquad f'(x) = 2x \quad f''(x) = 2$

$f(1) = 1$

so series $= f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} = 1 - 2 + \dfrac{2}{2} = 0$

Put $x = 3 \qquad f(x) = x^3$

$f'(x) = 3x^2 \qquad f''(x) = 6x$

$f(1) = 1 \qquad f'''(x) = 6$

series $f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} - \dfrac{f'''(1)}{3!}$

$= 1 - 3 + \dfrac{6}{2} - \dfrac{6}{6} = 0$

7. $f(x)$ is a polynomial function

$f(x) = ax^2 + bx + c \qquad f(1) = f(-1)$

$a + b + c = a - b + c \qquad b = 0$

a, b, c in A.P. $\quad b = \dfrac{a+c}{2} \qquad \boxed{a = -c}$

$f(x) = ax^2 + bx + c \qquad f'(x) = 2ax + b$

$f'(a) = 2a^2 + b \qquad f'(c) = 2ac + b$

$f'(b) = 2ab + b \qquad$ then $f'(a),\ f'(b),\ f'(c)$

$f'(b) = 0 \qquad f'(a) = 2a^2 \qquad f'(c) = -2a^2$

so that $\boxed{f'(a), f'(b), f'(c) \text{ are in A.P.}}$

8. $x = e^{y + e^{y + \dots\infty}} \qquad x > 0 \qquad \dfrac{dy}{dx} = ?$

$x = e^{y+x} \qquad 1 = e^{y+x}\left(1 + \dfrac{dy}{dx}\right)$

$\dfrac{1}{x} = 1 + \dfrac{dy}{dx} \qquad \dfrac{1-x}{x} = \dfrac{dy}{dx}$

10. $\left(x^x\right)^2 - 2\cot y\, x^x - 1 = 0$

$x^x = \dfrac{2\cot y \pm \sqrt{4\cot^2 y + 4}}{2} \qquad \left\{\begin{array}{l}\text{at } x = 1, \\ 1 = \cot y + \operatorname{cosec} y \\ \Rightarrow y = \dfrac{\pi}{2}\end{array}\right.$

$= \cot y \pm \operatorname{cosec} y$

$x^x = \cot y + \operatorname{cosec} y$

diff. w.r.t. to x

$x^x(1 + \log x) = [-\operatorname{cosec}^2 y - \operatorname{cosec} y \cot y]\dfrac{dy}{dx}$

$1 = -\operatorname{cosec} y\,[\operatorname{cosec} y + \cot y]\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = -1$

11. $g(x) = [f(2f(x) + 2)]^2$

$g'(x) = 2f\,(2f(x) + 2)\ f'\,(2f(x) + 2)\ 2f'(x)$

Put $x = 0$

$g'(0) = 2f(2f(0) + 2)\ f'(2f(0) + 2)\ 2f'(0)$

$= 2f(2(-1) + 2)\ f'(2(-1) + 2)\ 2f'(0)$

$= 2f(0)\ f'(0)\ 2f'(0)$

$= 4(-1)\ (1)\ (1) = -4$

12. $\dfrac{d}{dy}\left(\left(\dfrac{dy}{dx}\right)^{-1}\right) = \dfrac{d}{dy}\left(\left(\dfrac{dy}{dx}\right)^{-1}\right).\dfrac{dx}{dy}$

$= -\left(\dfrac{dy}{dx}\right)^{-2}.\dfrac{d^2x}{dx^2}.\dfrac{dx}{dy} = -\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

13. $y = \sec(\tan^{-1} x) = \sqrt{1 + x^2}$

$\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 + x^2}} \quad \Rightarrow \quad \left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{1}{\sqrt{2}}$

2. (b) Let $P(x) = ax^2 + bx + c$

$P(0) = 0 \Rightarrow c = 0$

$P(1) = 1 \Rightarrow a + b = 1$

$\therefore \quad P(x) = (1 - a)x^2 + ax$

$P'(x) = 2(1 - a)x + a > 0$

put $x = 0, \quad a > 0$

$x = 1, \quad a < 2$

$S = \{(1 - a)x^2 + ax \;;\; 0 < a < 2\}$.

5. (a) $g(x + 1) = \log(f(x + 1)) = \log x + \log f(x)$

$\Rightarrow g(x + 1) = \log x + g(x)$

$\Rightarrow g(x + 1) - g(x) = \log x$

$\Rightarrow g'(x + 1) - g'(x) = \dfrac{1}{x}$

$\Rightarrow g''(x + 1) - g''(x) = -\dfrac{1}{x^2}$

$\Rightarrow g''\left(1 + \dfrac{1}{2}\right) - g''\left(\dfrac{1}{2}\right) = -4$

$\Rightarrow g''\left(2 + \dfrac{1}{2}\right) - g''\left(1 + \dfrac{1}{2}\right) = -\dfrac{4}{9}$

....

$g''\left(N + \dfrac{1}{2}\right) - g''\left(N - \dfrac{1}{2}\right) = \dfrac{-4}{(2N - 1)^2}$

By adding

Hence $g''\left(N + \dfrac{1}{2}\right) - g''\left(\dfrac{1}{2}\right)$

$= -4\left(1 + \dfrac{1}{9} + \ldots\ldots + \dfrac{1}{(2N - 1)^2}\right)$

(b) $\lim\limits_{x\to 0} \dfrac{[g(x)\cos x - g(0)]}{\sin x}$

$= \lim\limits_{x\to 0} \dfrac{g'(x)\cos x - g(x)\sin x}{\cos x} = 0$

Now $f(x) = g(x)\sin x$

$f'(x) = g'(x)\sin x + g(x)\cos x$

$\therefore f'(0) = 0$

$f''(x) = g''(x)\sin x + g'(x)\cos x - g(x)\sin x + g'(x)\cos x$

$f''(0) = 0 \qquad \therefore$

Given limit $= f''(0)$ & also $f'(0) = g(0)$

So S(I) & S(II) both are correct but S(II) is not correct explaination of S(I)

6. $f(x) = x^3 + e^{x/2}$, $g(x) = f^{-1}(x)$

$\Rightarrow g'(f(x)).f'(x) = 1$

Put $f(x) = 1 \Rightarrow x^3 + e^{x/2} = 1$

$\Rightarrow x = 0$

$\Rightarrow g'(1).f'(0) = 1$, $f'(x) = 3x^2 + e^{x/2}.\dfrac{1}{2}$

$\Rightarrow g'(1) = 2$

7. Let $f(\theta) = \sin\alpha$ where $\alpha = \tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)$

$\Rightarrow \quad \tan\alpha = \dfrac{\sin\theta}{\sqrt{\cos 2\theta}}$

$\Rightarrow \quad \sin\alpha = \dfrac{\sin\theta}{\cos\theta} = \tan\theta \quad \left(\because \theta \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)\right)$

$\Rightarrow \quad f(\theta) = \tan\theta$

$\Rightarrow \quad \dfrac{d(f(\theta))}{d(\tan\theta)} = 1$